A Local System for Intuitionistic Logic

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Abstract. This paper presents systems for first-order intuitionistic logic and several of its extensions in which all the propositional rules are *local*, in the sense that, in applying the rules of the system, one needs only a fixed amount of information about the logical expressions involved. The main source of non-locality is the contraction rules. We show that the contraction rules can be restricted to the atomic ones, provided we employ *deep-inference*, i.e., to allow rules to apply anywhere inside logical expressions. We further show that the use of deep inference allows for modular extensions of intuitionistic logic to Dummett's intermediate logic LC, Gödel logic and classical logic. We present the systems in the calculus of structures, a proof theoretic formalism which supports deep-inference. Cut elimination for these systems are proved indirectly by simulating the cut-free sequent systems, or the hypersequent systems in the cases of Dummett's LC and Gödel logic, in the cut free systems in the calculus of structures.
Keywords: proof theory, intuitionistic logic, intermediate logics, deep inference, calculus of structures, locality.

1 Introduction

This paper presents systems for intuitionistic logic and its extensions, which are properly included in classical logic, in which all the propositional rules are *local*, in the sense of [4]. That is, in applying the rules of the system, one needs only a fixed amount of information about the logical expressions involved. For example, the usual contraction rule in sequent calculus, i.e.,

$$cL\,\frac{B, B, \Gamma \vdash C}{B, \Gamma \vdash C}$$

is non-local, since in order to apply the rule one has to check that two formulae are syntactically equal, and since B can be arbitrary formula, the "cost" of this checking varies with the size of B. Other examples include the (non-atomic) identity and cut rules, and the promotion rule in linear logic [11]. In [7], it is shown that it is possible to give a system for classical logic in which all the rules are local. This means in particular that the contraction, weakening, the cut and the identity rules are restricted to atomic forms. As it is shown in [5], this is difficult to achieve without some form of *deep inference*, i.e., to allow rules to apply anywhere inside logical expressions. The classical system in [7], called SKS, is presented in *the calculus of structures* [13], a formalism which allows deep inference in a way which preserves interesting proof theoretical notions and properties. We shall use the same formalism to present the intuitionistic systems to follow.

Deep inference and locality have been shown to allow for finer analyses on proofs, in particular, proofs in the deep-inference presentation of classical logic, i.e., the system SKS, have been shown to admit non-trivial categorical [18] and geometric interpretations [14]. Classical logic is among a number of logical systems that have been presented in the calculus of structures, e.g., non-commutative extension of linear logic [13], linear logic [21] and modal logics [20]. In these systems, the above notion of locality has been consistently exhibited. However, the logical systems in the calculus of structures studied so far have been those which are symmetric, in the sense that they have involutive negations and can be presented in one-sided sequent systems. The work presented in this paper is an attempt to find "good" presentations of asymmetric (two-sided) sequent systems in the calculus of structures, where locality is one important criteria. This will hopefully lead to further categorical or geometric models for two-sided sequent proofs for intuitionistic and intermediate logics. Another advantage of adopting deep inference is that it allows for a modular presentations of several extensions of intuitionistic logic, e.g., intermediate logics and classical logic: different logical systems can be obtained by adding rules which are derived straightforwardly from the axiomatic definitions of the extended systems. Our work can hopefully serve as a basis to give a uniform presentation for various intermediate logics.

We adopt the presentation of intuitionistic logic in the calculus of structures using positive and negative contexts, due to Kai Bruennler [6] and Phillipe de Groote[1]. Negative context corresponds to the left-hand side of a sequent and positive context corresponds to the right-hand side. In this presentation, rules are divided into negative rules, which apply under negative context, naturally, and positive rules which apply under positive context. Note that however since applying a rule would require checking for negative/positive context, the rules formalized this way are no longer local in the sense of [4]. But we can still achieve a weaker form of locality, that is, all rules that duplicate structures can be restricted to atomic forms. This system is then refined to a fully local one by exploiting the fact that all rule schemes in the system preserve polarities (see Section 6).

In Brünnler's intuitionistic system [6], it seems difficult, if not impossible, to reduce contraction to its atomic form. This is partly due to the fact that the contraction rule in this system (as it is the case with most sequent systems for intuitionistic logic) is *asymmetric*, i.e., contraction is allowed on the left (or negative context) but not on the right (positive context), while reducing contraction to its atomic form seems to require a symmetric contraction. The solution proposed here for reducing contraction to atomic is inspired by the multiple-conclusion intuitionistic system in sequent calculus [9, 23]. In this system, contraction and weakening are allowed on both sides of the sequent. The

[1] The author thanks Lutz Strassburger for pointing out the contribution of de Groote.

asymmetry of intuitionistic logic is captured by the implication rule:

$$\supset R \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B, \Delta}$$

One can see for instance that the classical theorem of excluded middle, i.e., $A \vee (A \supset \bot)$, is not provable. In the calculus of structures, this is reflected by the absence of certain "logical rules" under disjunctive context (see Section 2).

There exist numerous systems for intuitionistic and intermediate logics in the literature. These systems can be roughly divided into two categories: systems which are designed with decidability and proof search in mind, e.g., contraction-free sequent systems [16, 10], and those which are mainly concerned with generality of the formalisms, such as *labelled deduction* systems [3], hypersequents [1] and display calculi [12]. Our work is more in the latter category. In terms of expressivity, the calculus of structures is certainly at least as expressive as the non-standard sequent systems (display, hypersequents, and labelled systems), as one can simulate these systems inside cut-free systems in the calculus of structures. A common feature to these extended sequent systems is that they all employ some sort of structural extensions to sequents in order to capture various extensions of intuitionistic logic. In contrast, in the calculus of structures, there is no additional structural elements added to the proof system: one simply introduces more rules to get the extended logics. Moreover, these extended rules are derived straightforwardly from their axiomatic formulations (i.e., in Hilbert's systems). However, one of the drawbacks of the formulation of deep inference systems in our work is that we currently have no "internal" proof of cut-elimination. Our cut-elimination proof is indirect via translations to other systems, notably, sequent and hypersequent systems. Methodology for proof search in deep inference systems is not yet fully developed, although there is some work in this direction [17].

The rest of the paper is organized as follows. In Section 2, we present an intuitionistic system with the general (non-local) contraction rules, called SISgq. This is then followed by the soundness and completeness proof of SISgq with respect to a multiple-conclusion sequent system for intuitionistic logic and the cut elimination proof in Section 3. Section 4 shows how to extend SISg to cover Dummett's LC, Gödel logic and classical logic. Cut elimination for LC and Gödel logic are proved indirectly by simulating the corresponding hypersequent systems for these logics [1, 2]. In Section 5, the system SISg and its extensions are refined to systems in which the contraction rules are restricted to their atomic forms, but with additional *medial rules*. In Section 6 we show how to remove the context dependency in the propositional rules in all of the above logical systems, resulting in purely local systems for the propositional fragments, by introducing polarities into logical expressions. Section 7 discusses future work. Detailed proofs of the lemmas and the theorems in this paper can be found in the apendix.

2 An intuitionistic system in the calculus of structures

Inference rules in the calculus of structures can be seen as rewrite rules on formulae, i.e., the rules are of the form

$$\rho \frac{F\{B\}}{F\{C\}}$$

where ρ is the name of the rule, $F\{\}$ is a formula-context and B and C are formulae. Basically, any sound implication $B \supset C$ can be turned into a rule. The question is of course whether doing so would result in a good proof theory. The design philosophy of the calculus of structures has been centered around the concept of *interaction* and *symmetry* in inference rules. Just as the left and right rules in sequent calculus and introduction and elimination rules in natural deduction, a rule in the calculus of structures always has its dual, called its *co-rule*, which is obtained from the rule by taking the contrapositive of the implication defining the rule. The concept of interaction replaces the notion of identity and cut in sequent calculus. In classical logic [4], the interaction rules are (using the standard notation for classical formulae)

$$\mathsf{i}\!\downarrow \frac{S\{\top\}}{S\{A \vee \neg A\}} \qquad \mathsf{i}\!\uparrow \frac{S\{A \wedge \neg A\}}{S\{\bot\}}$$

In intuitionistic logic, we shall have a slightly different notation for the interaction rules, but the idea is essentially the same: the $\mathsf{i}\!\downarrow$-rule creates a dual pair of formulas (reading the rule top-down) while the $\mathsf{i}\!\uparrow$ rule destructs them.

In formulating the rules in the calculus of structures, one encounters certain rules which correspond to some logical equivalences in the logic being formalized. Some of the trivial equivalences, e.g., commutativity and associativity of conjunction, are more appropriately represented as equations rather than rules. We thus consider formulae modulo these equivalences. In the terms of the calculus of structures, these equivalent classes of formulae are referred to as *structures*. We shall be concerned with the following language of structures

$$S := p(t) \mid \mathsf{t} \mid \mathsf{f} \mid \langle S; S\rangle \mid [S, S] \mid (S, S) \mid \forall x R \mid \exists x R$$

where p is a predicate symbol, t is a term and the rest correspond to true, false, implication, disjunction, conjunction, universal and existential quantifications. For simplicity of presentation, we consider only unary predicates, but generalization to predicates of arbitrary arities is straightforward.

Note that we opt to use the above bracketing notations instead of the more traditional ones of connectives to simplify the presentation of the inference rules and derivations. Structures are ranged over by R, T, U, V, W and atomic structures are ranged over by a, b, c, d. A *structure context*, or context for short, is a structure with a hole, denoted by $S\{\ \}$. Given a structure R and a context $S\{\ \}$, we write $S\{R\}$ to denote the structure that results from replacing the hole $\{\ \}$ in $S\{\ \}$ with R. In presenting a structure R in a context $S\{\ \}$, we often omit

Units:	$[\mathsf{t},\mathsf{t}] = \mathsf{t} \quad (\mathsf{f},\mathsf{f}) = \mathsf{f} \qquad \langle \mathsf{f};\mathsf{t}\rangle = \mathsf{t} \qquad \langle \mathsf{f};\mathsf{f}\rangle = \mathsf{t}$
	$[\mathsf{f},R] = R \qquad (\mathsf{t},R) = R \quad \langle \mathsf{t};R\rangle = R$
Associativity:	$[R,[T,U]] = [[R,T],U] \qquad (R,(T,U)) = ((R,T),U)$
Commutativity:	$[R,T] = [T,R] \qquad (R,T) = (T,R)$
Currying:	$\langle (R,T);U\rangle = \langle R;\langle T;U\rangle\rangle$
Quantifiers:	$\forall x.R = \exists x.R = R,$ if x is not free in R.
	$\forall x.R = \forall y.R[y/x], \exists x.R = \exists y.R[y/x],$ y is not free in $\forall x.R$.
Congruence:	$S\{R\} = S\{T\},$ if $R = T$.

Fig. 1. Syntactic equality of structures

$$
\mathsf{i}{\downarrow}\frac{S^+\{\mathsf{t}\}}{S^+\langle R;R\rangle} \qquad
\mathsf{cr}{\downarrow}\frac{S^+[R,R]}{S^+\{R\}} \qquad
\mathsf{cl}{\downarrow}\frac{S^-(R,R)}{S^-\{R\}} \qquad
\mathsf{wr}{\downarrow}\frac{S^+\{\mathsf{f}\}}{S^+\{R\}} \qquad
\mathsf{wl}{\downarrow}\frac{S^-\{\mathsf{t}\}}{S^-\{R\}}
$$

$$
\mathsf{s}{\downarrow}\frac{S^+([R,T],U)}{S^+[R,(T,U)]} \qquad
\mathsf{sc}{\downarrow}\frac{S^+(\langle R;T\rangle,U)}{S^+\langle R;(T,U)\rangle} \qquad
\mathsf{sd}{\downarrow}\frac{S^+(\langle R;T\rangle,\langle U;V\rangle)}{S^+\langle [R,U];[T,V]\rangle}
$$

$$
\mathsf{sid}{\downarrow}\frac{S^+[\langle R;T\rangle,U]}{S^+\langle R;[T,U]\rangle} \qquad
\mathsf{sic}{\downarrow}\frac{S^+(R,\langle T;U\rangle)}{S^+\langle (R;T\rangle;U\rangle} \qquad
\mathsf{sac}{\downarrow}\frac{S^+(\forall xR,\forall xT)}{S^+\{\forall x(R,T)\}}
$$

$$
\mathsf{sa}{\downarrow}\frac{S^+\{\forall x\langle R;T\rangle\}}{S^+\langle R;\forall xT\rangle} \qquad
\mathsf{se}{\downarrow}\frac{S^+\{\forall x\langle R;T\rangle\}}{S^+\langle \exists xR;T\rangle} \qquad
\mathsf{nr}{\downarrow}\frac{S^+\{R[t/x]\}}{S^+\{\exists xR\}} \qquad
\mathsf{nl}{\downarrow}\frac{S^-\{R[t/x]\}}{S^-\{\forall xR\}}
$$

Fig. 2. System ISgq: an intuitionistic system in the calculus of structures. The rules $\mathsf{sa}{\downarrow}$ and $\mathsf{se}{\downarrow}$ have the provisos that x is not free in R and T, respectively.

the curly braces surrounding the R, if R is composed with a binary relation, e.g., we shall write $S[U,V]$ instead of $S\{[U,V]\}$. Structures are considered modulo the syntactic equivalence given in Figure 1. Note that we assume the domain of the quantification is non-empty. This is reflected in the equations concerning quantifiers.

We distinguish between *positive contexts* and *negative contexts*. Positive and negative contexts are defined inductively as follows.

1. $\{\ \}$ is a positive context,
2. if $S\{\ \}$ is a positive context then $(S\{\ \},R)$, $(R,S\{\ \})$, $[S\{\ \},R]$, $[R,S\{\ \}]$, $\forall x\{\ \}$, $\exists x\{\ \}$ and $\langle R;S\{\ \}\rangle$ are positive contexts, otherwise they are negative contexts,
3. if $S\{\ \}$ is a positive context then $\langle S\{\ \};R\rangle$ is a negative context, otherwise it is a positive context.

Given a positive context $S\{\ \}$, we often write it as $S^+\{\ \}$ to emphasize that it is a positive context. Similarly we write $S^-\{\ \}$ to emphasize that $S\{\ \}$ is a negative context.

The inference rules for the general system (non-local) for intuitionistic logic is given in Figure 2. We refer to this system as ISgq. As we have noted previously,

$$\mathsf{i}\uparrow \frac{S^-\langle R; R\rangle}{S^-\{\mathsf{t}\}} \qquad \mathsf{cr}\uparrow \frac{S^-\{R\}}{S^-[R,R]} \qquad \mathsf{cl}\uparrow \frac{S^+\{R\}}{S^+(R,R)} \qquad \mathsf{wr}\uparrow \frac{S^-\{R\}}{S^-\{\mathsf{f}\}} \qquad \mathsf{wl}\uparrow \frac{S^+\{R\}}{S^+\{\mathsf{t}\}}$$

$$\mathsf{s}\uparrow \frac{S^-[R,(T,U)]}{S^-([R,T],U)} \qquad \mathsf{sc}\uparrow \frac{S^-\langle R;(T,U)\rangle}{S^-(\langle R;T\rangle,U)} \qquad \mathsf{sd}\uparrow \frac{S^-\langle [R,U];[T,V]\rangle}{S^-(\langle R;T\rangle,\langle U;V\rangle)}$$

$$\mathsf{sid}\uparrow \frac{S^-\langle R;[T,U]\rangle}{S^-[\langle R;T\rangle,U]} \qquad \mathsf{sic}\uparrow \frac{S^-\langle\langle R;T\rangle;U\rangle}{S^-(R,\langle T;U\rangle)} \qquad \mathsf{sac}\uparrow \frac{S^-\{\forall x(R,T)\}}{S^-(\forall xR,\forall xT)}$$

$$\mathsf{sa}\uparrow \frac{S^-\langle R;\forall xT\rangle}{S^-\{\forall x\langle R;T\rangle\}} \qquad \mathsf{se}\uparrow \frac{S^-\langle \exists xR;T\rangle}{S^-\{\forall x\langle R;T\rangle\}} \qquad \mathsf{nr}\uparrow \frac{S^-\{\exists xR\}}{S^-\{R[t/x]\}} \qquad \mathsf{nl}\uparrow \frac{S^+\{\forall xR\}}{S^+\{R[t/x]\}}$$

Fig. 3. System cISgq: the dual of ISgq.

each rule in the calculus of structures has its co-rule. In the case of ISgq, the co-rule of a rule ρ is obtained from ρ by exchanging the premise with the conclusion and reversing the condition on the context of the rule (i.e., positive to negative and vice versa). The name of a rule is usually suffixed with an up or a down arrow, and its co-rule has the same name but with the arrow reversed. We use the term *up-rules* to denote rules with up-arrow in their names and *down-rules* if their names contain down-arrow. The rule i$\downarrow$ corresponds to the identity rule in sequent calculus. Its co-rule, i$\uparrow$ (see Figure 3), corresponds to cut. Together they are referred to as the *interaction rules*. The rules cl$\downarrow$ and cr$\downarrow$ are the contraction left and right rules, and wl$\downarrow$ and wr$\downarrow$ are the weakening left and right rules. The rules prefixed with the letter s are the *switch* rules, using the terminology of [13]. The notation $[t/x]$ in the nr$\downarrow$ and nl$\downarrow$ rules denotes capture-avoiding substitutions.

Notice that if we take the dual of the rules of ISgq, we obtain another, "dual" system of intuitionistic logic. This system, called cISgq, is shown in Figure 3. Each of the systems ISgq and cISgq is incomplete in its own, since either cut or identity is missing. The fully symmetric system for intuitionistic logic is thus obtained by combining the two, and is referred to as SISgq. SISgq naturally corresponds to first-order LJ and either one of ISgq or cISgq corresponds to the cut-free fragment of first-order LJ. Note that either system can be chosen to represent the cut-free LJ; it is just a matter of convention that we fix our choice to ISgq. We refer to the propositional fragment of SISgq (ISgq) as SISg (respectively, ISg).

Definition 1. *A* derivation Δ *in a system in the calculus of structures is a finite chain of instances of inference rules in the system. A derivation can consist of just one structure. The topmost structure in a derivation is called the* premise *of the derivation, and the structure at the bottom is called its* conclusion. *A* proof Π *in the calculus of structures is a derivation whose premise is* t. *A rule* ρ *is* derivable *in a system* $\mathscr{S}$ *if* $\rho \notin \mathscr{S}$ *and for every instance of* $\rho\,\frac{T}{R}$ *there is a* *derivation with premise R and conclusion T in $\mathscr{S}$. Two systems are* equivalent *if they have the same set of provable structures.*

Logical systems formalized in the calculus of structrues enjoy the so-called *top-down symmetry*. For symmetric systems (with involutive negation), this means that from any derivation Δ we can obtain another derivation Δ' with the premise and the conclusion exchanged and dualized, and with the up rules replacing their down versions and vice versa. In our setting, instead of dualizing the structures, we dualize the context, i.e., changing positive to negative and vice versa.

Proposition 2. *Let Δ be a derivation from T to R in* SISgq*. Then for every negative context $S^{-}\{\ \}$, there exists a derivation Δ' from $S^{-}\{R\}$ to $S^{-}\{T\}$ in* SISgq*, where Δ' is obtained from Δ by exchanging every rule for its dual (i.e., up-rules to down-rules and vice versa).*

3 Soundness, completeness and cut elimination

We shall now prove that the system SISgq is sound and complete with respect to intuitionistic logic and that it has cut-elimination. The notion of cut-elimination in the calculus of structures is more general than that of sequent calculus, that is, not only the cut rule (the $\mathsf{i}\uparrow$) is admissible, but the entire up-rules are also admissible. We prove the soundness and completeness of SISgq with respect to a multiple-conclusion sequent system for intuitionistic logic [9]. We refer to this system as LJm. Its rules are those of Gentzen's LK, except for the right introduction rules for universal quantifier and implication:

$$\supset R\,\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B, \Psi} \qquad \forall R\,\frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall x A, \Psi}$$

where y in the $\forall R$ rule is not free in the lower sequent. Cut-elimination for SISgq is obtained indirectly via the cut-elimination theorem in sequent calculus, by observing that all the rules in LJm, except the cut, are derivable in ISgq, i.e., the fragment of SISgq without the up-rules.

The formulae of LJm are given by the following grammar:

$$F ::= p(t) \mid \top \mid \bot \mid F \supset F \mid F \vee F \mid F \wedge F \mid \forall x F \mid \exists x F.$$

As in structures, p here denotes a unary predicate, and the rest of the constants correspond to true, false, implication, disjunction, conjunction, universal and existential quantifiers.

Definition 3. *The functions $\underline{\ .\ }_{\mathsf{s}}$ and $\underline{\ .\ }_{\mathsf{J}}$ given below transform formulae in LJm into structures and vice versa:*

$\underline{\top}_{\mathsf{s}} = \mathsf{t}$	$\underline{\bot}_{\mathsf{s}} = \mathsf{f}$	$\underline{\mathsf{t}}_{\mathsf{J}} = \top$	$\underline{\mathsf{f}}_{\mathsf{J}} = \bot$
$\underline{p(t)}_{\mathsf{s}} = p(t)$	$\underline{A \wedge B}_{\mathsf{s}} = (\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}})$	$\underline{p(t)}_{\mathsf{J}} = p(t)$	$\underline{(R, T)}_{\mathsf{J}} = \underline{R}_{\mathsf{J}} \wedge \underline{R}_{\mathsf{J}}$
$\underline{A \vee B}_{\mathsf{s}} = [\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}}]$	$\underline{A \supset B}_{\mathsf{s}} = \langle \underline{A}_{\mathsf{s}}; \underline{B}_{\mathsf{s}} \rangle$	$\underline{[R, T]}_{\mathsf{J}} = \underline{R}_{\mathsf{J}} \vee \underline{T}_{\mathsf{J}}$	$\underline{\langle R; T \rangle}_{\mathsf{J}} = \underline{R}_{\mathsf{J}} \supset \underline{T}_{\mathsf{J}}$
$\underline{\forall x A}_{\mathsf{s}} = \forall x \underline{A}_{\mathsf{s}}$	$\underline{\exists x A}_{\mathsf{s}} = \exists x \underline{A}_{\mathsf{s}}$	$\underline{\forall x R}_{\mathsf{J}} = \forall x \underline{R}_{\mathsf{J}}$	$\underline{\exists x R}_{\mathsf{J}} = \exists x \underline{R}_{\mathsf{J}}$

$$
\supset R\,\dfrac{\wedge L\,\dfrac{\wedge R\,\dfrac{\supset L\,\dfrac{id\,\dfrac{}{R \vdash R} \qquad id\,\dfrac{}{T \vdash T}}{R \supset T, R \vdash T} \qquad id\,\dfrac{}{U \vdash U}}{R \supset T, U, R \vdash T \wedge U}}{(R \supset T) \wedge U, R \vdash T \wedge U}}{(R \supset T) \wedge U \vdash R \supset (T \wedge U)}
\qquad\qquad
\begin{array}{rc}
 & (\langle \Gamma_1; [A, \Psi_2]\rangle, \langle (B, \Gamma_2); \Psi_1\rangle) \\ \hline
\mathsf{sc}\downarrow & \langle \Gamma_1; ([A, \Psi_2], \langle (B, \Gamma_2); \Psi_1\rangle)\rangle \\ \hline
= & \langle \Gamma_1; (\langle \Gamma_2; \langle B; \Psi_1\rangle\rangle, [A, \Psi_2])\rangle \\ \hline
\mathsf{sc}\downarrow & \langle \Gamma_1; \langle \Gamma_2; (\langle B; \Psi_1\rangle, [A, \Psi_2])\rangle\rangle \\ \hline
= & \langle (\Gamma_1, \Gamma_2); ([A, \Psi_2], \langle B; \Psi_1\rangle)\rangle \\ \hline
\mathsf{s}\downarrow & \langle (\Gamma_1, \Gamma_2); [(A, \langle B; \Psi_1\rangle), \Psi_2]\rangle \\ \hline
\mathsf{sic}\downarrow & \langle (\Gamma_1, \Gamma_2); [\langle\langle A; B\rangle; \Psi_1\rangle, \Psi_2]\rangle \\ \hline
\mathsf{sid}\downarrow & \langle (\Gamma_1, \Gamma_2); \langle\langle A; B\rangle; [\Psi_1, \Psi_2]\rangle\rangle \\ \hline
= & \langle (\Gamma_1, \Gamma_2, \langle A; B\rangle); [\Psi_1, \Psi_2]\rangle
\end{array}
$$

Fig. 4. A correspondence between ISgq and LJm.

The function $\underline{\ .\ }_{\mathsf{s}}$ *is generalized to sequents as follows:*

$$\underline{A_1, \ldots, A_n \vdash B}_{\mathsf{s}} = \forall x_1 \ldots \forall x_n \langle (\underline{A_1}_{\mathsf{s}}, \ldots, \underline{A_n}_{\mathsf{s}}); \underline{B}_{\mathsf{s}}\rangle$$

where $x_1, \ldots, x_n$ *are the eigenvariables of the sequent, and empty conjunction is interpreted as the constant* $\mathtt{t}$*.*

The key to proving soundness is to show that each instance of a rule in SISgq corresponds to an implication in LJm and that equivalent structures map to logically equivalent formulas. For instance, the soundness of the $\mathsf{sc}\downarrow$ rule is demonstrated by the left derivation in Figure 4.

Theorem 4. *For every structure* R*,* R *is provable in* SISgq *if and only if* $\underline{R}_{\mathsf{J}}$ *is provable in* LJm*.*

To prove completeness, and cut-elimination, we show that ISgq can simulate all the sequent rules of LJm. For instance, we show in the right derivation in Figure 4 a simulation of the left introduction rule for implication:

$$\supset L\,\dfrac{\Gamma_1 \vdash A, \Psi_1 \quad B, \Gamma_2 \vdash \Psi_2}{\Gamma_1, \Gamma_2, A \supset B \vdash \Psi_1, \Psi_2}$$

Notice that the branching in the rule is mapped to the conjunctive structural relation in ISgq.

Theorem 5. *For every structure* R*,* R *is provable in* SISgq *if and only if it is provable in* ISgq*.*

4 Intermediate and classical logics

We now consider three extensions of intuitionistic logic: Dummett's LC [8], Gödel logic [22] and classical logic. Dummett's LC is obtained by adding the following axiom $A \supset B \vee B \supset A$ to the propositional fragment of intuitionistic logic. Gödel logic is obtained by adding to LC the axiom $\forall x(A \vee B) \supset \forall x A \vee B$, where x is not free in B. Classical logic is obtained, obviously, by dropping the restriction on the contexts in the introduction rules for implication and universal quantifiers. We discuss each of these extension in the following.

4.1 Dummett's *LC*

Dummett's LC can be formalized in the calculus of structures by adding the following rules to SISg (i.e., the propositional fragment of SISgq).

$$\mathsf{com}\!\downarrow \frac{S^{+}(\langle R; T\rangle, \langle U; V\rangle)}{S^{+}[\langle R; V\rangle, \langle U; T\rangle]} \qquad \mathsf{com}\!\uparrow \frac{S^{-}[\langle R; V\rangle, \langle U; T\rangle]}{S^{-}(\langle R; T\rangle, \langle U; V\rangle)}$$

These rules are called the *communication* rules, and are inspired by the corresponding rules in the hypersequent formulation of LC [1, 2]. With the $\mathsf{com}\!\downarrow$ rule, we can derive the axiom $A \supset B \vee B \supset A$ as follows:

$$\mathsf{com}\!\downarrow \frac{\mathsf{i}\!\downarrow \frac{\mathsf{i}\!\downarrow \frac{= \frac{\mathsf{t}}{(\mathsf{t}, \mathsf{t})}}{(\mathsf{t}, \langle B; B\rangle)}}{(\langle A; A\rangle, \langle B; B\rangle)}}{[\langle A; B\rangle, \langle B; A\rangle]}$$

We refer to the system SISg extended with both rules as SCSg. We call the down fragment of SCSg, i.e., ISg plus the $\mathsf{com}\!\downarrow$ rule, the system CSg. As we will see, it is enough to consider CSg since it is equivalent to SCSg.

Both $\mathsf{com}\!\downarrow$ and $\mathsf{com}\!\uparrow$ rules correspond to the formula

$$(R \supset T) \wedge (U \supset V) \supset (R \supset V) \vee (U \supset T).$$

This formula can be easily shown to be provable from the following three formulas:

1. $(T \supset V) \vee (V \supset T)$,
2. $(R \supset T) \wedge (T \supset V) \supset (R \supset V)$,
3. $(U \supset V) \wedge (V \supset T) \supset (U \supset T)$.

The first formula is an axiom of LC, the second and the third are intuitionistic theorems. Therefore the $\mathsf{com}\!\downarrow$ and $\mathsf{com}\!\uparrow$ rules are sound with respect to LC. The completeness proof of CSg (and SCSg) is more involved; it uses a translation from a hypersequent system for LC to CSg. We state the result here and refer the interested reader to the apendix for the detailed proofs.

Theorem 6. *For every structure R, R is provable in CSg if and only if $\underline{R}_{\lrcorner}$ is provable in LC.*

4.2 Gödel logic

Gödel logic is obtained by adding $\mathsf{com}\downarrow$, $\mathsf{com}\uparrow$ and the following rules to SISgq:

$$\mathsf{g}\downarrow \frac{S^{+}\{\forall x[R,T]\}}{S^{+}[\forall x R, T]} \qquad \mathsf{g}\uparrow \frac{S^{-}[\forall x R, T]}{S^{-}\{\forall x[R,T]\}}$$

We refer to the formulation of this logic as SGSg. The down fragment, i.e., CSg plus the $\mathsf{g}\downarrow$ rule, is referred to as GSg.

The rules $\mathsf{g}\downarrow$ and $\mathsf{g}\uparrow$ are obviously sound since they correspond directly to the axiom $\forall x(R \vee T) \supset \forall x R \vee T$. To prove completeness and cut-elimination, we encode a hypersequent system for Gödel logic, i.e., the system HIF [2] (also known as first-order intuitionistic fuzzy logic) in GSg. The details of the encoding can be found in the appendix.

Theorem 7. *For every structure R, R is provable in GSg if and only if $\underline{R}_{\lrcorner}$ is provable in HIF.*

4.3 Classical logic

Classical logic is obtained by adding $\mathsf{g}\downarrow$, $\mathsf{g}\uparrow$ and the following rules

$$\mathsf{ci}\downarrow \frac{S^{+}\langle R; [T,U]\rangle}{S^{+}[\langle R;T\rangle, U]} \qquad \mathsf{ci}\uparrow \frac{S^{-}[\langle R;T\rangle, U]}{S^{-}\langle R; [T,U]\rangle}$$

to SISgq. These rules allow one to simulate the right-introduction rules for implication and universal quantifier in LK:

$$\supset R\, \frac{\Gamma, A \vdash B, \Psi}{\Gamma \vdash A \supset B, \Psi} \qquad \forall R\, \frac{\Gamma \vdash A[y/x], \Psi}{\Gamma \vdash \forall x A, \Psi}$$

More precisely, these rules are derived as follows:

$$\mathsf{ci}\downarrow \dfrac{= \dfrac{\langle (\Gamma, A); [B, \Psi]\rangle}{\langle \Gamma; \langle A; [B,\Psi]\rangle\rangle}}{\langle \Gamma; [\langle A; B\rangle, \Psi]\rangle} \qquad = \dfrac{\mathsf{g}\downarrow \dfrac{\mathsf{sa}\downarrow \dfrac{\forall y \langle \Gamma; [A[y/x], \Psi]\rangle}{\langle \Gamma; \forall y [A[y/x], \Psi]\rangle}}{\langle \Gamma; [\forall y. A[y/x], \Psi]\rangle}}{\langle \Gamma; [\forall x A, \Psi]\rangle}$$

We refer to the system SISgq extended with $\mathsf{ci}\downarrow$, $\mathsf{ci}\uparrow$, $\mathsf{g}\downarrow$ and $\mathsf{g}\uparrow$ as $\mathsf{SKS}_{\mathsf{2g}}$. The down fragment, i.e., ISgq extended with $\mathsf{ci}\downarrow$ and $\mathsf{g}\downarrow$, is referred to as $\mathsf{KS}_{\mathsf{2g}}$.

Theorem 8. *For every structure R, R is provable in $\mathsf{KS}_{\mathsf{2g}}$ if and only if $\underline{R}_{\lrcorner}$ is provable in LK.*

5 Atomic contraction

We shall now refine the system SISgq and its extensions to systems in which the interaction rules (i.e., the $\mathsf{i}\!\downarrow$ and $\mathsf{i}\!\uparrow$ rules), contraction and weakening are restricted to atomic propositions. The transformations required to reduce the interaction, weakening and contraction rules to their atomic forms are independent of the particular extensions to SISgq, so without loss of generality we shall work only with the system SISgq in this section. The main challenge in reducing contraction to its atomic form is in finding the right *medial rules*, just like those in SKS [4]. They are basically some form of distributivity among the connectives. In order to reduce contraction in negative context to atomic ones, it is crucial that we allow contraction on positive context as well. This is due to the reversal of polarity introduced by the implication connective.

The atomic versions of the interaction, contraction and weakening rules are as follows:

$$\mathsf{ai}\!\downarrow\frac{S^+\{\mathsf{t}\}}{S^+\langle a;a\rangle}\quad \mathsf{acr}\!\downarrow\frac{S^+[a,a]}{S^+\{a\}}\quad \mathsf{acl}\!\downarrow\frac{S^-(a,a)}{S^-\{a\}}\quad \mathsf{awr}\!\downarrow\frac{S^+\{\mathsf{f}\}}{S^+\{a\}}\quad \mathsf{awl}\!\downarrow\frac{S^-\{\mathsf{t}\}}{S^-\{a\}}$$

and their respective duals, obtained by exchanging the premise and the conclusion, with the polarity of the context reversed. Here we denote with a an atomic formula.

The medial rules for intuitionistic logic are given in Figure 5. The classical medial rule m of SKS [7] splits into two rules: the (right) medial rule mr and the (left) medial rule ml. This is because we have contraction on both the positive and negative contexts. Notice that mr and ml are dual to each other, that is, mr is the up-version of ml and vice versa. There are extra medial rules that deal with implication and quantifiers. All those rules are derivable from the contraction and weakening rules in ISgq, and hence their soundness follows from the soundness of ISgq. By taking the duals of the medial rules in Figure 5, we obtain the *co-medial* rules, which by symmetry, are needed to reduce the *co-contraction* (i.e., the up-version of the contraction rules) to atomic. The co-medial rules are denoted by the same name but with the arrows reversed.

The general interaction rules $\mathsf{i}\!\downarrow$ and the weakening rule $\mathsf{wr}\!\downarrow$ can be shown reducible to their atomic versions, and the contraction rule $\mathsf{cr}\!\downarrow$ can be reduced to the atomic one with the medial rules. We illustrate here a step in the reduction of the contraction rule; more details can be found in the appendix. Consider for instance, contractions on an implication structure, on both the positive and negative context:

$$\mathsf{cr}\!\downarrow\frac{S^+[\langle R;T\rangle,\langle R;T\rangle]}{S^+\langle R;T\rangle}\qquad \mathsf{cl}\!\downarrow\frac{S^-(\langle R;T\rangle,\langle R;T\rangle)}{S^-\langle R;T\rangle}$$

$$\text{ml}\ \frac{S^-([R,T],[U,V])}{S^-[(R,U),(T,V)]} \qquad \text{mr}\ \frac{S^+[(R,U),(T,V)]}{S^+([R,T],[U,V])}$$

$$\text{mil}\downarrow\ \frac{S^-(\langle R;U\rangle,\langle T;V\rangle)}{S^-\langle [R,T];(U,V)\rangle} \qquad \text{mir}\downarrow\ \frac{S^+[\langle R;U\rangle,\langle T;V\rangle]}{S^+\langle (R,T);[U,V]\rangle}$$

$$\text{mal}\downarrow\ \frac{S^-(\forall xR,\forall xT)}{S^-\{\forall x(R,T)\}} \qquad \text{mar}\downarrow\ \frac{S^+[\forall xR,\forall xT]}{S^+\{\forall x[R,T]\}}$$

$$\text{mel}\downarrow\ \frac{S^-(\exists xR,\exists xT)}{S^-\{\exists x(R,T)\}} \qquad \text{mer}\downarrow\ \frac{S^+[\exists xR,\exists xT]}{S^+\{\exists x[R,T]\}}$$

Fig. 5. The medial rules for reducing contraction to atomic.

These instances of contractions can be replaced by the following derivations:

$$\text{cl}\downarrow\ \dfrac{\text{cr}\downarrow\ \dfrac{\text{mir}\downarrow\ \dfrac{S^+[\langle R;T\rangle,\langle R;T\rangle]}{S^+\langle (R,R);[T,T]\rangle}}{S^+\langle (R,R);T\rangle}}{S^+\langle R;T\rangle} \qquad \text{cr}\downarrow\ \dfrac{\text{cl}\downarrow\ \dfrac{\text{mil}\downarrow\ \dfrac{S^-[\langle R;T\rangle,\langle R;T\rangle]}{S^-\langle [R,R];(T,T)\rangle}}{S^-\langle [R,R];T\rangle}}{S^-\langle R;T\rangle}$$

Notice that in the above derivations, contractions are applied to a subformula of the original formula. Repeating this process, we eventually end up with contractions on atomic formulas only.

Definition 9. *System* ISaq *is obtained from* ISgq *by replacing the interaction rule* i↓ *with* ai↓*, the weakening rules* wr↓ *and* wl↓ *with* awr↓ *and* awl↓*, the contraction rules* cr↓ *and* cl↓ *with* acr↓*,* acl↓ *and the medial rules in Figure 5. System* SISaq *is obtained by adding to* ISaq *its own dual rules. The propositional fragment of* SISaq *and* ISaq *are referred to as* SISa *and* ISa*, respectively.*

Theorem 10. *The systems* SISgq *and* SISaq *are equivalent.*

Theorem 11. *The systems* SISaq *and* ISaq *are equivalent.*

6 A local system for propositional intuitionistic logic

The rules in both SISgq and SISaq are non-local since in order to apply the rules, one has to check whether the redex is in a positive or negative context. However, if one carefully observes the rules, one notices a certain conservation of *polarities* in the rules. That is to say there is never the case where a structure in a positive context is moved to a negative context and vice versa. For example, in the rule sc↓ in Figure 2, the substructures R, T, U and V have the same polarities in both the premise and the conclusion of the rule. That is R is in negative context in both premise and conclusion, T is in positive context, and so on. This observation leads to the following idea: When proving a structure,

Units:	$[\mathsf{t}^+,\mathsf{t}^+]^+ = \mathsf{t}^+ \quad (\mathsf{f}^+,\mathsf{f}^+)^+ = \mathsf{f}^+ \quad \langle \mathsf{f}^-;\mathsf{t}^+\rangle^+ = \mathsf{t}^+ \quad \langle \mathsf{f}^-;\mathsf{f}^+\rangle^+ = \mathsf{t}^+$ $[\mathsf{f}^+,R^+]^+ = R^+ \quad (\mathsf{t}^+,R^+)^+ = R^+ \quad \langle \mathsf{t}^-;R^+\rangle^+ = R^+$
Associativity:	$[R,[T,U]^+]^+ = [[R,T]^+,U]^+ \qquad (R,(T,U)^+)^+ = ((R,T)^+,U)^+$
Commutativity:	$[R,T]^+ = [T,R]^+ \qquad (R,T)^+ = (T,R)^+$
Currying:	$\langle (R,T)^-;U\rangle^+ = \langle R;\langle T;U\rangle^+\rangle^+$
Orthogonality:	$\overline{R} = \overline{T}$, if $R = T$.
Congruence:	$S\{R\} = S\{T\}$, if $R = T$.

Fig. 6. Syntactic equality for polarized structures

we first label each substructure with either a '+' or a '−' depending on whether the substructure is in a positive or a negative context respectively. Each time a structure is modified by a rule, the premise of the rule is relabelled consistently, that is, substructures are labelled depending on which context they reside in. The polarity-preserving property of the rules guarantees that there is no need of relabelling of substructures which are not affected by the rule. For the $\mathsf{sc}\!\downarrow$ rule, the labelled version would be:

$$\mathsf{sc}\!\downarrow \frac{S(\langle R;T\rangle^+,U)^+}{S\langle R;(T,U)^+\rangle^+}$$

This modified rule of $\mathsf{sc}\!\downarrow$ is local since we need only to check for polarity of three substructures in the rule, instead of checking the entire context. We shall give a fully local system for the propositional fragment of ISaq by introducing polarities into structures.

Definition 12. Polarized structures *are expressions generated from the following grammar:*

$$\begin{aligned} S &::= P \mid N \\ P &::= a^+ \mid \mathsf{t}^+ \mid \mathsf{f}^+ \mid (P,P)^+ \mid [P,P]^+ \mid \langle N;P\rangle^+ \\ N &::= a^- \mid \mathsf{t}^- \mid \mathsf{f}^- \mid (N,N)^- \mid [N,N]^- \mid \langle P;N\rangle^- \end{aligned}$$

A positive polarized structure*, or positive structure for short, is a polarized structure labelled with '+', and a* negative polarized structure*, or negative structure, is a polarized structure labelled with '−'. Positive structures are often denoted by R^+ and negative structures by R^-. The* orthogonal of a structure R*, denoted by $\overline{R}$, is the structure obtained from R by exchanging the labels '+' with '−' and vice versa. A* polarized context *is a polarized structure with a hole $\{\ \}$. Given a polarized context $S\{\ \}$ and a polarized structure R, the placement of R in $S\{\ \}$, i.e., $S\{R\}$, is allowed only if doing so results in a well-formed polarized structure. Polarized structures are considered modulo the equality in Figure 6.*

The propositional intuitionistic system with polarized structures is given in Figure 7. We refer to this system as ISp. Each polarized rule has a dual version which is obtained by exchanging the premise and the conclusion and exchanging

$$\mathsf{ai}\!\downarrow \frac{S\{\mathsf{t}^+\}}{S\langle a^-; a^+\rangle^+} \quad \mathsf{acr}\!\downarrow \frac{S[a^+, a^+]^+}{S\{a^+\}} \quad \mathsf{acl}\!\downarrow \frac{S(a^-, a^-)^-}{S\{a^-\}} \quad \mathsf{awr}\!\downarrow \frac{S\{\mathsf{f}^+\}}{S\{a^+\}} \quad \mathsf{wl}\!\downarrow \frac{S\{\mathsf{t}^-\}}{S\{a^-\}}$$

$$\mathsf{s}\!\downarrow \frac{S([R,T]^+, U)^+}{S[R,(T,U)^+]^+} \qquad \mathsf{sc}\!\downarrow \frac{S(\langle R;T\rangle^+, U)^+}{S\langle R;(T,U)^+\rangle^+} \qquad \mathsf{sd}\!\downarrow \frac{S(\langle R;T\rangle^+, \langle U;V\rangle^+)^+}{S\langle [R,U]^-; [T,V]^+\rangle^+}$$

$$\mathsf{sid}\!\downarrow \frac{S[\langle R;T\rangle^+, U]^+}{S\langle R;[T,U]^+\rangle^+} \qquad \mathsf{sic}\!\downarrow \frac{S(R, \langle T;U\rangle^+)^+}{S\langle\langle R;T\rangle^+; U\rangle^+}$$

$$\mathsf{ml}\, \frac{S([R,T]^-, [U,V]^-)^-}{S[(R,U)^-, (T,V)^-]^-} \qquad \mathsf{mr}\, \frac{S[(R,U)^+, (T,V)^+]^+}{S([R,T]^+, [U,V]^+)^+}$$

$$\mathsf{mil}\!\downarrow \frac{S(\langle R;U\rangle^-, \langle T;V\rangle^-)^-}{S\langle [R,T]^+; (U,V)^-\rangle^-} \qquad \mathsf{mir}\!\downarrow \frac{S[\langle R;U\rangle^+, \langle T;V\rangle^+]^+}{S\langle (R,T)^-; [U,V]^+\rangle^+}$$

Fig. 7. System ISp.

the polarities. The system obtained by adding ISp to its own duals is is referred to as SISp. Both the inference rules and the structural equality are derived straightforwardly from the inference rules and structural equality of SISa, that is, by giving appropriate labels to the structures. Care has to be taken to ensure that the rules and the equality between polarized structures preserve polarity. We shall now proceed to prove formally that SISp, SISa, ISp and ISa are all equivalent in terms of provability.

The notion of derivations in SISp is the same as that in SISa. The notion of proof is slightly different.

Definition 13. *A* proof *of a polarized structure R in $\mathscr{S}$ is a derivation in $\mathscr{S}$ with premise* t^+ *and conclusion R.*

By this definition, it is obvious that all provable polarized structures are positive structures since all rules in SISp preserve polarities.

The key idea to proving the correspondence between SISp and the propositional fragment of SISaq is the following: the polarity of any substructure R in $S\{R\}$ should determine the polarity of the context. In particular, positive structures R and $S\{R\}$ are translated to some structures T and $S'\{T\}$ such that $S'\{\ \}$ corresponds to $S\{\ \}$ and T corresponds to R, and most importantly, $S'\{\ \}$ is a positive context. In this way, rules that apply to positive substructures in SISp translate to the same rules that apply under positive context in SISa, and a simple observation on the inference rules of SISp and SISa shows that they co-incide. The same observation holds for negative structures and negative contexts. In the following theorems, we denote with $\underline{R}_{\mathsf{s}}$, where R is a polarized structure, the structure obtained from R by dropping all the polarity signs.

Theorem 14. *For every polarized structure R, R is provable in* ISp *if and only if $\underline{R}_{\mathsf{s}}$ is provable in* ISa*.*

7 Future work

Properties of proofs and derivations in the systems SISgq and its extensions remain to be studied. An immediate future works would be to find direct proofs (i.e., without the detour through sequent calculus or hypersequent) of cut-elimination. It would also be interesting to investigate various substructural logics that arise from either restricting or extending the base system SISgq. For instance, it would be interesting to see what sort of logic we get from dropping the atomic contraction rules but keeping the medial rules. Another open problem is to come up with a fully local first-order intuitionistic system. The rules which instantiate quantifiers, i.e., $\mathsf{nr}\downarrow$ and $\mathsf{nl}\downarrow$, involve substitutions which are non-local. This can probably be made local by giving rules which effectively "implement" explicit substitutions. On the more general problem of formalizing asymmetric systems, it would be intereting to see if the methodology presented here can be generalized to formalize non-standard asymmetric systems such as Bunched Logic [19]. Some preliminary result in this direction can be found in [15]. The current work focusses mainly on the proof theoretic aspects. It would be interesting to see if the analyses on the deep inference systems, in particular the notions of locality and atomicity, will be useful for implementing proof search for these logics.

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$$id\,\frac{}{A \vdash A} \qquad cut\,\frac{\Gamma_1 \vdash \Psi_1, A \quad A, \Gamma_2 \vdash \Psi_2}{\Gamma_1, \Gamma_2 \vdash \Psi_1, \Psi_2} \qquad \bot\,\frac{}{\bot \vdash} \qquad \top\,\frac{}{\vdash \top}$$

$$cL\,\frac{A, A, \Gamma \vdash \Psi}{A, \Gamma \vdash \Psi} \qquad cR\,\frac{\Gamma \vdash \Psi, A, A}{\Gamma \vdash \Psi, A} \qquad wL\,\frac{\Gamma \vdash \Psi}{A, \Gamma \vdash \Psi} \qquad wR\,\frac{\Gamma \vdash \Psi}{\Gamma \vdash \Psi, A}$$

$$\wedge L\,\frac{A, B, \Gamma \vdash \Psi}{A \wedge B, \Gamma \vdash \Psi} \qquad \wedge R\,\frac{\Gamma_1 \vdash A, \Psi_1 \quad \Gamma_2 \vdash B, \Psi_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Psi_1, \Psi_2}$$

$$\vee L\,\frac{A, \Gamma_1 \vdash \Psi_1 \quad B, \Gamma_2 \vdash \Psi_2}{A \vee B, \Gamma_1, \Gamma_2 \vdash \Psi_1, \Psi_2} \qquad \vee R\,\frac{\Gamma \vdash A, B, \Psi}{\Gamma \vdash A \vee B, \Psi}$$

$$\supset L\,\frac{\Gamma_1 \vdash A, \Psi_1 \quad B, \Gamma_2 \vdash \Psi_2}{A \supset B, \Gamma_1, \Gamma_2 \vdash \Psi_1, \Psi_2} \qquad \supset R\,\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B, \Psi}$$

$$\forall L\,\frac{A[t/x], \Gamma \vdash \Psi}{\forall x A, \Gamma \vdash \Psi} \qquad \forall R\,\frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall x A, \Psi}$$

$$\exists L\,\frac{A[y/x], \Gamma \vdash \Psi}{\exists x A, \Gamma \vdash \Psi} \qquad \exists R\,\frac{\Gamma \vdash A[t/x], \Psi}{\Gamma \vdash \exists x A, \Psi}$$

Fig. 8. A multiple-conclusion sequent system for intuitionistic logic. The rules $\forall R$ and $\exists L$ have the proviso that y is not free in the lower sequent.

Appendix A. Proofs for Section 3

In the following, we shall use the notation

$$\begin{array}{c} T \\ \Delta \| \mathscr{S} \\ R \end{array} \quad \text{and} \quad \begin{array}{c} \Pi \| \mathscr{S} \\ R \end{array}$$

to denote a derivation Δ with premise R and conclusion T and a proof Π of R in the system $\mathscr{S}$.

Soundness and completeness of SISgq are proved with respect to the sequent system LJm given in Figure 8.

In the following proofs, we shall make explicit the application of structural equivalence in derivations. This is done by introducing a rule for structural equivalence:

$$=\frac{S\{T\}}{S\{R\}}$$

with the proviso that $R = T$. In the following derivations, ρ^n denotes n-applications of the rule ρ.

Lemma 15. *If $S_1\{\ \}$ is a positive context and $S_2\{\ \}$ is a positive (negative) context, then $S_1\{S_2\{\ \}\}$ is a positive (respectively, negative) context. If $S_1\{\ \}$ is a negative context and $S_2\{\ \}$ is a positive (negative) context, then $S_1\{S_2\{\ \}\}$ is a negative (respectively, positive) context.*

Proof. By structural induction on $S_1\{\ \}$ and the definition of positive/negative context. $\square$

In the following, we use $\equiv$ to denote logical equivalence, i.e., $A \equiv B$ abbreviates $A \supset B \wedge B \supset A$.

Lemma 16. *For any structure R, it holds that $R = \underline{\underline{R}_{\lrcorner}}_{\mathsf{s}}$.*

Proof. By structural induction on R, Definition 3 and the definition of structure equivalence in Figure 1. $\square$

Lemma 17. *Let R and T be two structures such that $\underline{R}_{\lrcorner} \supset \underline{T}_{\lrcorner}$ is provable in LJm and let $S\{\ \}$ be a context. If $S\{\ \}$ is a positive context then $\underline{S\{R\}}_{\lrcorner} \supset \underline{S\{T\}}_{\lrcorner}$ is provable in LJm. Otherwise, if $S\{\ \}$ is a negative context then $\underline{S\{T\}}_{\lrcorner} \supset \underline{S\{R\}}_{\lrcorner}$ is provable in LJm.*

Proof. By structure induction on $S\{\ \}$ and the definition of positive/negative context.
Base case: $S = \{\ \}$, follows immediately from the assumption.
Inductive cases: We show some cases here where $S\{\ \}$ is a positive context; others can be proved in a similar way.

1. $S = (U, S'\{\ \})$: we have

$$\underline{S\{R\}}_{\lrcorner} = \underline{U}_{\lrcorner} \wedge \underline{S'\{R\}}_{\lrcorner} \quad \text{and} \quad \underline{S\{T\}}_{\lrcorner} = \underline{U}_{\lrcorner} \wedge \underline{S'\{T\}}_{\lrcorner}.$$

   By induction hypothesis: $\underline{S'\{R\}}_{\lrcorner} \supset \underline{S'\{T\}}_{\lrcorner}$ is provable in LJm, and hence the sequent $\underline{S'\{R\}}_{\lrcorner} \vdash \underline{S'\{T\}}_{\lrcorner}$ is provable in LJm as well. The proof for $\underline{S\{R\}}_{\lrcorner} \supset \underline{S\{T\}}_{\lrcorner}$ is constructed as follows:

$$\supset R; cL; \wedge L^2\ \dfrac{\wedge R\ \dfrac{id\ \dfrac{}{\underline{U}_{\lrcorner} \vdash \underline{U}_{\lrcorner}} \qquad \dfrac{\vdots}{\underline{S'\{R\}}_{\lrcorner} \vdash \underline{S'\{T\}}_{\lrcorner}}}{\underline{U}_{\lrcorner}, \underline{S'\{R\}}_{\lrcorner} \vdash \underline{U}_{\lrcorner} \wedge \underline{S'\{T\}}_{\lrcorner}}}{\vdash (\underline{U}_{\lrcorner} \wedge \underline{S'\{R\}}_{\lrcorner}) \supset (\underline{U}_{\lrcorner} \wedge \underline{S'\{T\}}_{\lrcorner})}$$

   Here we use ρ^n to denote n applications of the rule ρ.
2. $S = \langle S'\{\ \}; U\rangle$: by induction hypothesis we have a proof of $\underline{S'\{T\}}_{\lrcorner} \vdash \underline{S'\{R\}}_{\lrcorner}$. The proof for $\underline{S\{R\}}_{\lrcorner} \supset \underline{S\{T\}}_{\lrcorner}$ is constructed as follows:

$$\supset R^2\ \dfrac{\supset L\ \dfrac{\dfrac{\vdots}{\underline{S'\{T\}}_{\lrcorner} \vdash \underline{S'\{R\}}_{\lrcorner}} \qquad id\ \dfrac{}{\underline{U}_{\lrcorner} \vdash \underline{U}_{\lrcorner}}}{\underline{S'\{R\}}_{\lrcorner} \supset \underline{U}_{\lrcorner}, \underline{S'\{T\}}_{\lrcorner} \supset \underline{U}_{\lrcorner} \vdash \underline{U}_{\lrcorner}}}{\vdash (\underline{S'\{R\}}_{\lrcorner} \supset \underline{U}_{\lrcorner}) \supset (\underline{S'\{T\}}_{\lrcorner} \supset \underline{U}_{\lrcorner})}$$

3. $S = \forall x S'\{\ \}$: we have $\underline{\forall x S'\{R\}}_\lrcorner = \forall x \underline{S'\{R\}}_\lrcorner$ and $\underline{\forall x S'\{T\}}_\lrcorner = \forall x \underline{S'\{T\}}_\lrcorner$.

$$
\supset R\,\dfrac{\forall R\,\dfrac{\forall L\,\dfrac{\begin{array}{c}\bigtriangledown\\ \underline{S'\{T\}}_\lrcorner \vdash \underline{S'\{R\}}_\lrcorner\end{array}}{\forall x\underline{S'\{R\}}_\lrcorner \vdash \underline{S'\{T\}}_\lrcorner}}{\forall x\underline{S'\{R\}}_\lrcorner \vdash \forall x\underline{S'\{T\}}_\lrcorner}}{\vdash \forall x\underline{S'\{R\}}_\lrcorner \supset \forall x\underline{S'\{T\}}_\lrcorner}
$$

□

Lemma 18. *If $R = T$ then $\underline{R}_\lrcorner \equiv \underline{T}_\lrcorner$.*

Proof. It is sufficient to show the lemma holds for the elementary equality (i.e., the equality without the congruence axiom), which can easily be shown to be correspond to logical equivalence. The case for the congruence axiom follows straightforwardly from the elementary cases and Lemma 17. □

Lemma 19. *For any structures R and T, if T is derivable from R in* SISgq*, then $\underline{R}_\lrcorner \supset \underline{T}_\lrcorner$ is provable in LJm.*

Proof. We first show that each rule in SISgq is sound, i.e.,

$$
\rho\,\frac{S\{T\}}{S\{R\}} \quad \text{implies} \quad \underline{S\{R\}}_\lrcorner \supset \underline{S\{T\}}_\lrcorner.
$$

By Lemma 17, it is enough to show either $\underline{R}_\lrcorner \supset \underline{T}_\lrcorner$, i.e., if $S\{\ \}$ is positive, or $\underline{T}_\lrcorner \supset \underline{R}_\lrcorner$ if $S\{\ \}$ is negative. Further, for each dual pair of rules, we need only to show the soundness of the down rules, since the soundness of the up rule amounts to showing the same implication. We show here a couple of cases; the rest are not difficult to show and we leave them to the readers. For simplicity of presentation we omit the translation function $\underline{\ }_\lrcorner$ in the inference figures below.

sc↓:

$$
\supset R\,\dfrac{\wedge L\,\dfrac{\wedge R\,\dfrac{\supset L\,\dfrac{id\,\dfrac{}{R \vdash R} \quad id\,\dfrac{}{T \vdash T}}{R \supset T, R \vdash T} \quad id\,\dfrac{}{U \vdash U}}{R \supset T, U, R \vdash T \wedge U}}{(R \supset T) \wedge U, R \vdash T \wedge U}}{(R \supset T) \wedge U \vdash R \supset (T \wedge U)}
$$

$\mathsf{sd}{\downarrow}$:

$$
\supset\! R\;\dfrac{cL;\wedge R^2\;\dfrac{\vee L\;\dfrac{\supset\! L\;\dfrac{id\;\dfrac{}{R \vdash R} \qquad \vee R\;\dfrac{id\;\dfrac{}{T \vdash T}}{T \vdash T \vee V}}{R \supset T, R \vdash T \vee V} \qquad \supset\! L\;\dfrac{id\;\dfrac{}{U \vdash U} \qquad \vee R\;\dfrac{id\;\dfrac{}{V \vdash V}}{V \vdash T \vee V}}{U \supset V, U \vdash T \vee V}}{R \supset T, U \supset V, R \vee U \vdash T \vee V}}{(R \supset T) \wedge (U \supset V), R \vee U \vdash T \vee V}}{(R \supset T) \wedge (U \supset V) \vdash (R \vee U) \supset (T \vee V)}
$$

The general case where we have a derivation

$$
\rho\;\dfrac{\begin{array}{c} R \\ \Delta \Big\Vert \mathsf{SISgq} \\ T' \end{array}}{T}
$$

is proved by induction on the height of the derivation. That is, in the above derivation, by induction hypothesis we have $\underline{R}_{\lrcorner} \supset \underline{T'}_{\lrcorner}$. By the soundness of the rule ρ we have $\underline{T}_{\lrcorner}{}' \supset \underline{T}_{\lrcorner}$, and hence by cut we obtain the following

$$
\supset\! R\;\dfrac{cut\;\dfrac{\underline{R}_{\lrcorner} \vdash \underline{T'}_{\lrcorner} \qquad \underline{T'}_{\lrcorner} \vdash \underline{T}_{\lrcorner}}{\underline{R}_{\lrcorner} \vdash \underline{T}_{\lrcorner}}}{\vdash \underline{R}_{\lrcorner} \supset \underline{T}_{\lrcorner}}
$$

□

Lemma 20. *For any structure R, if R is provable in* SISgq *then $\underline{R}_{\lrcorner}$ is provable in LJm.*

Proof. A simple corollary of Lemma 19. □

Lemma 21. *For any formula A, if A is provable in LJm then $\underline{A}_{\mathsf{s}}$ is provable in* SISgq*.*

Proof. It is enough to show that every instances of the rules in LJm is derivable in SISgq, that is, for every rule

$$
\rho\;\dfrac{\Gamma_1 \vdash B_1 \quad \cdots \quad \Gamma_n \vdash B_n}{\Gamma \vdash B}
$$

there is a derivation

$$
\begin{array}{c}
(\forall \vec{x}\, \underline{\Gamma_1 \vdash B_1}_{\mathsf{s}}, \cdots, \forall \vec{x}\, \underline{\Gamma_1 \vdash B_1}_{\mathsf{s}}) \\
\Delta \Big\Vert \mathsf{SISgq} \\
\forall \vec{x}\, \underline{\Gamma \vdash B}_{\mathsf{s}}
\end{array}
$$

where $\vec{x}$ are the eigenvariables in the sequent and the empty conjunction denotes t. Since we can distribute universal quantifiers over conjunction using the $\mathsf{sac}\downarrow$ rule, i.e.,

$$\begin{array}{c}(\forall\vec{x}R_1,\cdots,\forall\vec{x}R_n)\\ \Big\|\\ \forall\vec{x}(R_1,\cdots,R_n)\end{array}$$

in the following derivations we keep the eigenvariables implicit. We show here some non-trivial cases.

cut:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\langle\Gamma_1;[\Psi_1,A]\rangle,\langle(\Gamma_2,A);\Psi_2\rangle)}{(\langle\Gamma_1;[\Psi_1,A]\rangle,\langle\Gamma_2;\langle A;\Psi_2\rangle\rangle)}=}{\langle\Gamma_1;([\Psi_1,A],\langle\Gamma_2;\langle A;\Psi_2\rangle\rangle)\rangle}\mathsf{sc}\downarrow}{\langle(\Gamma_1,\Gamma_2);([\Psi_1,A],\langle A;\Psi_2\rangle)\rangle}\mathsf{sc}\downarrow}{\langle(\Gamma_1,\Gamma_2);[\Psi_1,(A,\langle A;\Psi_2\rangle)]\rangle}\mathsf{s}\downarrow}{\langle(\Gamma_1,\Gamma_2);[\Psi_1,\langle\langle A;A\rangle;\Psi_2\rangle]\rangle}\mathsf{sic}\downarrow}{\langle(\Gamma_1,\Gamma_2);[\Psi_1,\Psi_2]\rangle}\mathsf{i}\uparrow$$

$\wedge R$:

$$\dfrac{\dfrac{\dfrac{\dfrac{(\langle\Gamma_1;[A,\Psi_1]\rangle,\langle\Gamma_2;[B,\Psi_2]\rangle)}{\langle\Gamma_1;([A,\Psi_1],\langle\Gamma_2;[B,\Psi_2]\rangle)\rangle}\mathsf{sc}\downarrow}{\langle(\Gamma_1,\Gamma_2);([A,\Psi_1],[B,\Psi_2])\rangle}\mathsf{sc}\downarrow}{\langle(\Gamma_1,\Gamma_2);[([A,\Psi_1],B),\Psi_2]\rangle}\mathsf{s}\downarrow}{\langle(\Gamma_1,\Gamma_2);[(A,B),\Psi_1,\Psi_2]\rangle}\mathsf{s}\downarrow$$

$\supset L$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\langle\Gamma_1;[A,\Psi_1]\rangle,\langle(\Gamma_2,B);\Psi_2\rangle)}{(\langle\Gamma_1;[A,\Psi_1]\rangle,\langle\Gamma_2;\langle B;\Psi_2\rangle\rangle)}=}{\langle\Gamma_1;([A,\Psi_1],\langle\Gamma_2;\langle B;\Psi_2\rangle\rangle)\rangle}\mathsf{sc}\downarrow}{\langle(\Gamma_1,\Gamma_2);([A,\Psi_1],\langle B;\Psi_2\rangle)\rangle}\mathsf{sc}\downarrow}{\langle(\Gamma_1,\Gamma_2);[\Psi_1,(A,\langle B;\Psi_2\rangle)]\rangle}\mathsf{s}\downarrow}{\langle(\Gamma_1,\Gamma_2);[\Psi_1,\langle\langle A;B\rangle;\Psi_2\rangle]\rangle}\mathsf{sic}\downarrow}{\langle(\langle A;B\rangle,\Gamma_1,\Gamma_2);[\Psi_1,\Psi_2]\rangle}\mathsf{sid}\downarrow$$

□

Lemma 22. *For any formula A, if A is cut-free provable in LJm then* $\underline{A}_{\mathsf{s}}$ *is provable in* ISgq.

Proof. The proof is using the same proof-transformation similar to those in the proof of Lemma 21. Observe that in the translation from LJm to SISgq we do not introduce any up-rules, except when translating the cut rule, and hence cut-free proofs in LJm translates to proofs in ISgq. □

Proof of Theorem 4. By Lemma 20, Lemma 16 and Lemma 21. □

Proof of Theorem 5 By Lemma 20, the formula $\underline{R}_{\lrcorner}$ is provable in LJm. By the cut elimination theorem of LJm, $\underline{R}_{\lrcorner}$ is cut free provable, and hence R is provable in ISgq by Lemma 22 and Lemma 16. □

Appendix B. Proofs for Section 4

To prove the cut-elimination theorems for SCSg and SGSg we use the existing cut-elimination for Gödel logic in the hypersequent system HIF [2]. Since LC is properly included in HIF, and the propositional fragment of HIF is convservative with respect to LC, it is enough to show that we can simulate the cut-free hypersequent proofs of HIF in SGSg.

$$id\,\frac{}{A \vdash A} \qquad \bot\,\frac{}{\bot \vdash} \qquad cut\,\frac{G \mid \Gamma \vdash A \quad G \mid A, \Gamma \vdash C}{G \mid \Gamma \vdash C}$$

$$wl\,\frac{G \mid \Gamma \vdash C}{G \mid \Gamma, A \vdash C} \qquad wr\,\frac{G \mid \Gamma \vdash}{G \mid \Gamma \vdash A} \qquad cl\,\frac{G \mid \Gamma, A, A \vdash C}{G \mid \Gamma, A \vdash C}$$

$$EW\,\frac{G}{G \mid \Gamma \vdash A} \qquad EC\,\frac{G \mid \Gamma \vdash A \mid \Gamma \vdash A}{G \mid \Gamma \vdash A}$$

$$com\,\frac{G \mid \Gamma_1, \Gamma_2 \vdash A \quad G \mid \Gamma_1, \Gamma_2 \vdash B}{G \mid \Gamma_1 \vdash A \mid \Gamma_2 \vdash B}$$

$$\supset l\,\frac{G \mid \Gamma \vdash A \quad G \mid B, \Gamma \vdash C}{G \mid \Gamma, A \supset B \vdash C} \qquad \supset r\,\frac{G \mid \Gamma, A \vdash B}{G \mid \Gamma \vdash A \supset B}$$

$$\wedge l\,\frac{G \mid \Gamma, A_i \vdash C}{G \mid \Gamma, A_1 \wedge A_2 \vdash C} \qquad \wedge r\,\frac{G \mid \Gamma \vdash A \quad G \mid \Gamma \vdash B}{G \mid \Gamma \vdash A \wedge B}$$

$$\vee l\,\frac{G \mid \Gamma, A \vdash C \quad G \mid \Gamma, B \vdash C}{G \mid \Gamma, A \vee B \vdash C} \qquad \vee r\,\frac{G \mid \Gamma \vdash A_i}{G \mid \Gamma \vdash A_1 \vee A_2}$$

$$\forall l\,\frac{G \mid A[t/x], \Gamma \vdash C}{G \mid \forall x A, \Gamma \vdash C} \qquad \forall r\,\frac{G \mid \Gamma \vdash A[y/x]}{G \mid \Gamma \vdash \forall x A}$$

$$\exists l\,\frac{G \mid A[y/x], \Gamma \vdash C}{G \mid \exists x A, \Gamma \vdash C} \qquad \exists r\,\frac{G \mid \Gamma \vdash A[t/x]}{G \mid \Gamma \vdash \exists x A}$$

Fig. 9. A hypersequent system for Gödel logic. The rules $\forall r$ and $\exists l$ have the proviso that y is not free in the lower hypersequent.

We first review the hypersequent system HIF. Hypersequents are finite multiset of sequents, written

$$\Gamma_1 \vdash C_1 \mid \Gamma_2 \vdash C_2 \mid \cdots \mid \Gamma_n \vdash C_n.$$

A hypersequent is encoded in the calculus of structures as simply a disjunction of implications. The above hypersequent is translated to the following structures:

$$\forall\vec{x}[\langle \underline{\Gamma_1}_{\mathsf{s}}; \underline{C_1}_{\mathsf{s}}\rangle, \cdots, \langle \underline{\Gamma_n}_{\mathsf{s}}; \underline{C_n}_{\mathsf{s}}\rangle]$$

where $\vec{x}$ are the eigenvariables of the hypersequent. Empty disjunction is translated to the constant f. The proof system for HIF is given in Figure 9.

First we prove the soundness of SCSg and SGSg.

Lemma 23. *If R is provable in* SCSg *then $\underline{R}_{\mathsf{J}}$ is provable in LC*

Proof. It is enough to show that the rule com↓ and com↑ are sound. The rest of the proof then follows the same structure as the soundness proof of SISgq. The soundness of com↓ and com↑ translates to the provability of the following formula in LC:

$$(R \supset T) \wedge (U \supset V) \supset (R \supset V) \vee (U \supset T).$$

This formula can be easily shown to be provable from the following three formulas:

1. $(T \supset V) \vee (V \supset T)$,
2. $(R \supset T) \wedge (T \supset V) \supset (R \supset V)$,
3. $(U \supset V) \wedge (V \supset T) \supset (U \supset T)$.

The first lemma is an axiom of LC, the second and the third are intuitionistic theorems. Therefore the com↓ and com↑ rules are sound with respect to LC. □

Lemma 24. *If R is provable in* SGSg *then $\underline{R}_{\mathsf{J}}$ is provable in HIF.*

Proof. Follows from the soundness of $SCSg$ and the fact that the rules g↓ and g↑ corresponds to the Gödel axiom $\forall x(R \vee T) \supset \forall x R \vee T$. □

Lemma 25. *If a hypersequent G is provable in HIF then its translation to structures $\underline{G}_{\mathsf{s}}$ is provable in* GSg.

Proof. We show that every rule in HIF can be simulated in GSg. Most cases are straightforward. We show the interesting cases involving the *com* rule and the $\forall r$ rule. The derivation of the *com* rule is as follows:

$$\mathsf{cl}{\downarrow}^2 \frac{= \frac{\mathsf{com}{\downarrow} \frac{= \frac{\mathsf{cr}{\downarrow} \frac{\mathsf{s}{\downarrow}^2 \frac{([G, \langle(\Gamma_1, \Gamma_2); A\rangle], [G, \langle(\Gamma_1, \Gamma_2); B\rangle])}{[G, G, (\langle(\Gamma_1, \Gamma_2); A\rangle, \langle(\Gamma_1, \Gamma_2); B\rangle)]}}{[G, (\langle(\Gamma_1, \Gamma_2); A\rangle, \langle(\Gamma_1, \Gamma_2); B\rangle)]}}{[G, (\langle\Gamma_1; \langle\Gamma_2; B\rangle\rangle, \langle\Gamma_2; \langle\Gamma_1; A\rangle\rangle)]}}{[G, \langle\Gamma_1; \langle\Gamma_1; A\rangle\rangle, \langle\Gamma_2; \langle\Gamma_2; B\rangle\rangle]}}{[G, \langle(\Gamma_1, \Gamma_1); A\rangle, \langle(\Gamma_2, \Gamma_2); B\rangle]}}{[G, \langle\Gamma_1; A\rangle, \langle\Gamma_2; B\rangle]}$$

and the derivation of $\forall r$ is

$$\mathsf{g}\downarrow \frac{\mathsf{sa}\downarrow \frac{= \frac{\forall y[G, \langle \Gamma; A[y/x] \rangle]}{[G, \forall y \langle \Gamma; A[y/x] \rangle]}}{[G, \langle \Gamma; \forall y A[y/x] \rangle]}}{[G, \langle \Gamma; \forall x A \rangle]}$$

$\square$

Proof of Theorem 6: Follows immediately from Lemma 23 and Lemma 25. $\square$

Proof of Theorem 7: Follows immediately from Lemma 24 and Lemma 25. $\square$

Proof of Theorem 8: The soundness of $\mathsf{SKS_{2g}}$ is straightforward. For the completness of the $\mathsf{KS_{2g}}$ with respect to cut-free LK, most of the cases are the same as in the completeness proof of lSgq with respect to LJm. The only differences between LK and LJm are in the right introduction rules for implication and universal quantifiers:

$$\supset R \frac{\Gamma, A \vdash B, \Psi}{\Gamma \vdash A \supset B, \Psi} \qquad \forall R \frac{\Gamma \vdash A[y/x], \Psi}{\Gamma \vdash \forall x A, \Psi}$$

These rules are simulated in $\mathsf{KS_{2g}}$ as follows:

$$\mathsf{ci}\downarrow \frac{= \frac{\langle (\Gamma, A); [B, \Psi] \rangle}{\langle \Gamma; \langle A; [B, \Psi] \rangle \rangle}}{\langle \Gamma; [\langle A; B \rangle, \Psi] \rangle} \qquad = \frac{\mathsf{g}\downarrow \frac{\mathsf{sa}\downarrow \frac{\forall y \langle \Gamma; [A[y/x], \Psi] \rangle}{\langle \Gamma; \forall y [A[y/x], \Psi] \rangle}}{\langle \Gamma; [\forall y. A[y/x], \Psi] \rangle}}{\langle \Gamma; [\forall x A, \Psi] \rangle}$$

$\square$

Appendix C. Proofs for Section 5

Lemma 26. *The interaction rule* $\mathsf{i}\downarrow$ *is derivable in* $\{\mathsf{ai}\downarrow, \mathsf{s}\downarrow, \mathsf{sc}\downarrow, \mathsf{sd}\downarrow, \mathsf{sid}\downarrow, \mathsf{sic}\downarrow, \mathsf{sa}\downarrow, \mathsf{se}\downarrow, \mathsf{nr}\downarrow, \mathsf{nl}\downarrow\}$.

Proof. Let $\mathscr{S} = \{\mathsf{ai}\downarrow, \mathsf{sc}\downarrow, \mathsf{sd}\downarrow, \mathsf{si}\downarrow, \mathsf{sid}\downarrow, \mathsf{sic}\downarrow, \mathsf{sa}\downarrow, \mathsf{se}\downarrow, \mathsf{nr}\downarrow, \mathsf{nl}\downarrow\}$: We show that for every R and positive context $S^+\{\ \}$ there is a derivation

$$\begin{array}{c} S^+\{\mathsf{t}\} \\ \Delta \| \mathscr{S} \\ S^+\langle R; R \rangle \end{array}$$

We do this by induction on R. The base cases where R is either atom or the constants t or f are trivial. The inductive cases are as follows.

1. If $R = [R_1, R_2]$ then Δ is constructed as follows:

$$
\begin{array}{c}
S^{+}\{\mathsf{t}\} \\
\Delta_2 \Big\Vert \mathscr{S} \\
= \dfrac{S^{+}\langle R_1; R_1\rangle}{S^{+}(\langle R_1; R_1\rangle, \mathsf{t})} \\
\Delta_1 \Big\Vert \mathscr{S} \\
\mathsf{sd}\!\downarrow \dfrac{S^{+}(\langle R_1; R_1\rangle, \langle R_2; R_2\rangle)}{S^{+}\langle [R_1, R_2]; [R_1, R_2]\rangle}
\end{array}
$$

where Δ_1 and Δ_2 are obtained from induction hypothesis.

2. If $R = (R_1, R_2)$ then Δ is constructed as follows:

$$
\begin{array}{c}
S^{+}\{\mathsf{t}\} \\
\Delta_1 \Big\Vert \mathscr{S} \\
= \dfrac{S^{+}\langle R_1; R_1\rangle}{S^{+}(\langle R_1; R_1\rangle, \mathsf{t})} \\
\Delta_2 \Big\Vert \mathscr{S} \\
= \dfrac{\mathsf{sc}\!\downarrow \dfrac{\mathsf{sc}\!\downarrow \dfrac{S^{+}(\langle R_1; R_1\rangle, \langle R_2; R_2\rangle)}{S^{+}\langle R_1; (R_1, \langle R_2; R_2\rangle)\rangle}}{S^{+}\langle R_1; \langle R_2; (R_1, R_2)\rangle\rangle}}{S^{+}\langle (R_1, R_2); (R_1, R_2)\rangle}
\end{array}
$$

where Δ_1 and Δ_2 are obtained from induction hypothesis.

3. If $R = \langle R_1; R_2\rangle$ then Δ is constructed as follows:

$$
\begin{array}{c}
S^{+}\{\mathsf{t}\} \\
\Delta_2 \Big\Vert \mathscr{S} \\
= \dfrac{S^{+}\langle R_1; R_1\rangle}{S^{+}(\langle R_1; R_1\rangle, \mathsf{t})} \\
\Delta_1 \Big\Vert \mathscr{S} \\
= \dfrac{\mathsf{sic}\!\downarrow \dfrac{\mathsf{sc}\!\downarrow \dfrac{S^{+}(\langle R_1; R_1\rangle, \langle R_2; R_2\rangle)}{S^{+}\langle R_1; (R_1, \langle R_2; R_2\rangle)\rangle}}{S^{+}\langle R_1; \langle\langle R_1; R_2\rangle; R_2\rangle\rangle}}{S^{+}\langle\langle R_1; R_2\rangle; \langle R_1; R_2\rangle\rangle}
\end{array}
$$

where Δ_1 and Δ_2 are obtained from induction hypothesis.

4. If $R = \forall x T$ then Δ is

$$
\begin{array}{rl}
= & \dfrac{S^{+}\{\mathsf{t}\}}{S^{+}\{\forall x \mathsf{t}\}} \\
 & \Vert \\
\mathsf{nl}\!\downarrow & \dfrac{S^{+}\langle \forall x \langle T; T\rangle\rangle}{S^{+}\langle \forall x \langle \forall x T; T\rangle\rangle} \\
\mathsf{sa}\!\downarrow & \dfrac{}{S^{+}\langle \forall x T; \forall x T\rangle}
\end{array}
$$

5. If $R = \exists x T$ then Δ is

$$
\begin{array}{rl}
= & \dfrac{S^{+}\{\mathsf{t}\}}{S^{+}\{\forall x \mathsf{t}\}} \\
 & \Vert \\
\mathsf{nr}\!\downarrow & \dfrac{S^{+}\langle \forall x \langle T; T\rangle\rangle}{S^{+}\langle \forall x \langle T; \exists T\rangle\rangle} \\
\mathsf{se}\!\downarrow & \dfrac{}{S^{+}\langle \exists x T; \exists x T\rangle}
\end{array}
$$

□

Lemma 27. *The interaction rule* $\mathsf{i}\!\uparrow$ *is derivable in* $\{\mathsf{ai}\!\uparrow, \mathsf{s}\!\uparrow, \mathsf{sc}\!\uparrow, \mathsf{sd}\!\uparrow, \mathsf{sid}\!\uparrow, \mathsf{sic}\!\uparrow, \mathsf{sa}\!\uparrow, \mathsf{se}\!\uparrow, \mathsf{nr}\!\uparrow, \mathsf{nl}\!\uparrow\}$.

Proof. Let $\mathscr{S}$ be the set $\{\mathsf{ai}\!\uparrow, \mathsf{sc}\!\uparrow, \mathsf{sd}\!\uparrow, \mathsf{sid}\!\uparrow, \mathsf{sic}\!\uparrow, \mathsf{sa}\!\uparrow, \mathsf{se}\!\uparrow, \mathsf{nr}\!\uparrow, \mathsf{nl}\!\uparrow\}$. Dual to the admissibility of $\mathsf{i}\!\downarrow$, we show that there is a derivation from $S^{-}\langle R; R\rangle$ to $S^{-}\{\mathsf{t}\}$ in $\{\mathsf{ai}\!\uparrow, \mathsf{sc}\!\uparrow, \mathsf{sd}\!\uparrow, \mathsf{si}\!\uparrow\}$. This follows from the admissibility of $\mathsf{i}\!\downarrow$ for $\{\mathsf{ai}\!\downarrow, \mathsf{sc}\!\downarrow, \mathsf{sd}\!\downarrow, \mathsf{si}\!\downarrow\}$ (Lemma 26) and Proposition 2, i.e., by dualizing the derivations obtained in the proof of Lemma 26. □

Lemma 28. *The rules* $\mathsf{wr}\!\downarrow$ *and* $\mathsf{wl}\!\downarrow$ *are derivable in* $\{\mathsf{awr}\!\downarrow, \mathsf{awl}\!\downarrow\}$.

Proof. For any structure R we show that the following hold:

(i) For every positive context $S^{+}\{\ \}$, there is a derivation from $S^{+}\{\mathsf{f}\}$ to $S^{+}\{R\}$ using only the rules in $\mathcal{S}$.

(ii) For every negative context $S^{-}\{\ \}$, there is a derivation from $S^{-}\{\mathsf{t}\}$ to $S^{-}\{R\}$ using only the rules in $\mathcal{S}$.

This is done by induction on the size of R. We show here the case for $R = \langle U; V\rangle$. Other cases are proved similarly, i.e., by induction hypothesis and structural equivalence.

(i) The derivation from $S^{+}\{\mathsf{f}\}$ to $S^{+}\{R\}$ is constructed as follows:

$$
\begin{array}{c}
= \dfrac{S^{+}\{\mathsf{f}\}}{S^{+}\langle \mathsf{t}; \mathsf{f}\rangle} \\
\Delta_2 \Vert \mathcal{S} \\
S^{+}\langle \mathsf{t}; V\rangle \\
\Delta_1 \Vert \mathcal{S} \\
S^{+}\langle U; V\rangle
\end{array}
$$

where the derivations Δ_1 and Δ_2 are obtained from induction hypothesis.
(*ii*) The derivation from $S^-\{\mathsf{t}\}$ to $S^-\{R\}$ is constructed as follows:

$$= \frac{S^-\{\mathsf{t}\}}{S^-\langle \mathsf{f}; \mathsf{t}\rangle} \quad \begin{array}{c} \Delta_2 \Big\| \mathcal{S} \\ S^-\langle \mathsf{f}; V\rangle \\ \Delta_1 \Big\| \mathcal{S} \\ S^-\langle U; V\rangle \end{array}$$

where Δ_1 and Δ_2 are obtained from induction hypothesis.

□

Lemma 29. *The rules* $\mathsf{wr}\uparrow$ *and* $\mathsf{wl}\uparrow$ *are derivable in* $\{\mathsf{awr}\uparrow, \mathsf{awl}\uparrow\}$.

Proof. Dual to the proof of Lemma 28. □

Lemma 30. *The rules* mr, ml, $\mathsf{mir}\downarrow$, $\mathsf{mil}\downarrow$, $\mathsf{mar}\downarrow$, $\mathsf{mal}\downarrow$, $\mathsf{mer}\downarrow$ *and* $\mathsf{mel}\downarrow$ *are derivable in* $\{\mathsf{cr}\downarrow, \mathsf{cl}\downarrow, \mathsf{wr}\downarrow, \mathsf{wl}\downarrow\}$.

Proof. We show that each medial rule can be derived using the contraction and weakening rules.

mr:

$$\mathsf{cr}\downarrow \frac{\mathsf{wr}\downarrow^4 \dfrac{= \dfrac{S^+[(R,U),(T,V)]}{S^+[([R,\mathsf{f}],[U,\mathsf{f}]),([\mathsf{f},T],[\mathsf{f},V])]}}{S^+[([R,T],[U,V]),([R,T],[U,V])]}}{S^+([R,T],[U,V])}$$

ml:

$$\mathsf{cl}\downarrow \frac{\mathsf{wl}\downarrow^4 \dfrac{= \dfrac{S^-([R,U],[T,V])}{S^-([(R,\mathsf{t}),(U,\mathsf{t})],[(\mathsf{t},T),(\mathsf{t},V)])}}{S^-([(R,T),(U,V)],[(R,T),(U,V)])}}{S^-[(R,T),(U,V)]}$$

$\mathsf{mir}\downarrow$:

$$\mathsf{cr}\downarrow \frac{\mathsf{wr}\downarrow^2;\mathsf{wl}\downarrow^2 \dfrac{= \dfrac{S^+[\langle R;U\rangle,\langle T;V\rangle]}{S^+[\langle (R,\mathsf{t});[U,\mathsf{f}]\rangle,\langle (\mathsf{t},T);[\mathsf{f},V]\rangle]}}{S^+[\langle (R,T);[U,V]\rangle,\langle (R,T);[U,V]\rangle]}}{S^+\langle (R,T);[U,V]\rangle}$$

$\mathsf{mil}\downarrow$:

$$\mathsf{cl}\downarrow \frac{\mathsf{wl}\downarrow^2;\mathsf{wr}\downarrow^2 \dfrac{= \dfrac{S^-(\langle R;U\rangle,\langle T;V\rangle)}{S^-(\langle [R,\mathsf{f}];(U,\mathsf{t})\rangle,\langle [\mathsf{f},T];(\mathsf{t},V)\rangle)}}{S^-(\langle [R,T];(U,V)\rangle,\langle [R,T];(U,V)\rangle)}}{S^-\langle [R,T];(U,V)\rangle}$$

$\mathsf{mal}\!\downarrow$:

$$
\mathsf{cl}\!\downarrow \dfrac{\mathsf{wl}\!\downarrow^2 \dfrac{= \dfrac{S^-(\forall x R, \forall x T)}{S^-(\forall x(R,\mathsf{t}), \forall x(\mathsf{t},T))}}{S^-(\forall x(R,T), \forall x(R,T))}}{S^-\{\forall x(R,T)\}}
$$

$\mathsf{mar}\!\downarrow$:

$$
\mathsf{cr}\!\downarrow \dfrac{\mathsf{wr}\!\downarrow^2 \dfrac{= \dfrac{S^+[\forall x R, \forall x T]}{S^+[\forall x[R,\mathsf{f}], \forall x[\mathsf{f},T]]}}{S^+[\forall x[R,T], \forall x[R,T]]}}{S^+\{\forall x[R,T]\}}
$$

$\mathsf{mel}\!\downarrow$:

$$
\mathsf{cl}\!\downarrow \dfrac{\mathsf{wl}\!\downarrow^2 \dfrac{= \dfrac{S^-(\exists x R, \exists x T)}{S^-(\exists x(R,\mathsf{t}), \exists x(\mathsf{t},T))}}{S^-(\exists x(R,T), \exists x(R,T))}}{S^-\{\exists x(R,T)\}}
$$

$\mathsf{mer}\!\downarrow$:

$$
\mathsf{cr}\!\downarrow \dfrac{\mathsf{wr}\!\downarrow^2 \dfrac{= \dfrac{S^+[\exists x R, \exists x T]}{S^+[\exists x[R,\mathsf{f}], \exists x[\mathsf{f},T]]}}{S^+[\exists x[R,T], \exists x[R,T]]}}{S^+\{\exists x[R,T]\}}
$$

□

Lemma 31. *The rules* $\mathsf{mir}\!\uparrow$, $\mathsf{mil}\!\uparrow$, $\mathsf{mar}\!\uparrow$, $\mathsf{mal}\!\uparrow$, $\mathsf{mer}\!\uparrow$ *and* $\mathsf{mel}\!\uparrow$ *are derivable in* $\{\mathsf{cr}\!\uparrow, \mathsf{cl}\!\uparrow, \mathsf{wr}\!\uparrow, \mathsf{wl}\!\uparrow\}$.

Proof. Dual to the proof of Lemma 30. □

$$
\begin{array}{c}
\mathsf{mir}\!\downarrow \dfrac{S^+[\langle U;V\rangle, \langle U;V\rangle]}{S^+\langle (U,U);[V,V]\rangle} \\
\Delta_2 \Big\| \mathscr{S} \\
S^+\langle (U,U);V\rangle \\
\Delta_1 \Big\| \mathscr{S} \\
S^+\langle U;V\rangle
\end{array}
\qquad\qquad
\begin{array}{c}
\mathsf{mil}\!\downarrow \dfrac{S^-(\langle U;V\rangle, \langle U;V\rangle)}{S^-\langle [U,U];(V,V)\rangle} \\
\Delta_2' \Big\| \mathscr{S} \\
S^-\langle [U,U];V\rangle \\
\Delta_1' \Big\| \mathscr{S} \\
S^-\langle U;V\rangle
\end{array}
$$

Fig. 10. Reducing contraction to atomic.

Lemma 32. *The contraction rules* $\mathsf{cr}{\downarrow}$ *and* $\mathsf{cl}{\downarrow}$ *are derivable in* $\{\mathsf{acr}{\downarrow}, \mathsf{acl}{\downarrow}, \mathsf{mr}, \mathsf{ml}, \mathsf{mir}{\downarrow}, \mathsf{mil}{\downarrow}, \mathsf{mar}{\downarrow}, \mathsf{mal}{\downarrow}, \mathsf{mer}{\downarrow}, \mathsf{mel}{\downarrow}\ \}$.

Proof. Let $\mathscr{S}$ be $\{\mathsf{acr}{\downarrow}, \mathsf{acl}{\downarrow}, \mathsf{mr}, \mathsf{ml}, \mathsf{mir}{\downarrow}, \mathsf{mil}{\downarrow}\}$. The proof reduces to proving the following statement: for every R the following hold:

(*i*) For every positive context $S^+\{\ \}$, there is a derivation from $S^+[R, R]$ to $S^+\{R\}$ using only the rules in $\mathscr{S}$.
(*ii*) For every negative context $S^-\{\ \}$, there is a derivation from $S^-(R, R)$ to $S^-\{R\}$ using only the rules in $\mathscr{S}$.

This is proved by induction on the size of R. We show here the interesting case where $R = \langle U; V\rangle$ for some non-units U and V (otherwise it is trivial). The derivation from $S^+[R, R]$ to $S^+\{R\}$ is given in the left derivation in Figure 10. Here, the derivations Δ_1 and Δ_2 are obtained from induction hypothesis on (*ii*) and (*i*), respectively. The derivation from $S^-(R, R)$ to $S^-\{R\}$ is given in the right derivation in the same figure, where Δ'_1 and Δ'_2 are obtained from induction hypothesis. □

Lemma 33. *The co-contraction rules* $\mathsf{cr}{\uparrow}$ *and* $\mathsf{cl}{\uparrow}$ *are derivable in* $\{\mathsf{acr}{\uparrow}, \mathsf{acl}{\uparrow}, \mathsf{mr}, \mathsf{ml}, \mathsf{mir}{\uparrow}, \mathsf{mil}{\uparrow}, \mathsf{mar}{\uparrow}, \mathsf{mal}{\uparrow}, \mathsf{mer}{\uparrow}, \mathsf{mel}{\uparrow}\ \}$.

Proof. Dual to the proof of Lemma 32. □

Proof of Theorem 10:
We show that every derivation in SISgq can be transformed to a derivation in SISaq and vice-versa. This follows straightforwardly from Lemma 26, Lemma 27, Lemma 28, Lemma 29, Lemma 30, Lemma 31, Lemma 32, and Lemma 33. □

Proof of Theorem 11:
By the observation that translation from ISgq (which is equivalent to SISgq by Theorem 5) back to SISaq (Lemma 26, Lemma 28, Lemma 30 and Lemma 32) uses only the rules in ISaq. □

Appendix D. Proofs for Section 6

Definition 34. *The function* $\underline{\ \ }_{\mathsf{S}}$ *translates polarized structures into structures by simply removing the polarities. The functions* $\underline{\ \ }_{\mathsf{P}}$ *and* $\underline{\ \ }_{\mathsf{N}}$ *translate structures to positive and negative structures, respectively. They are mutual-recursively defined as follows.*

$$
\begin{array}{ll}
\underline{a}_{\mathsf{P}} = a^+, & \underline{a}_{\mathsf{N}} = a^-, \\
\underline{\mathsf{t}}_{\mathsf{P}} = \mathsf{t}^+, \quad \underline{\mathsf{f}}_{\mathsf{P}} = \mathsf{f}^+, & \underline{\mathsf{t}}_{\mathsf{N}} = \mathsf{t}^-, \quad \underline{\mathsf{f}}_{\mathsf{N}} = \mathsf{f}^-, \\
\underline{(R, T)}_{\mathsf{P}} = (\underline{R}_{\mathsf{P}}, \underline{T}_{\mathsf{P}})^+, & \underline{(R, T)}_{\mathsf{N}} = (\underline{R}_{\mathsf{N}}, \underline{T}_{\mathsf{N}})^-, \\
\underline{(R, T)}_{\mathsf{P}} = (\underline{R}_{\mathsf{P}}, \underline{T}_{\mathsf{P}})^+, & \underline{(R, T)}_{\mathsf{N}} = (\underline{R}_{\mathsf{N}}, \underline{T}_{\mathsf{N}})^-, \\
\underline{\langle R; T\rangle}_{\mathsf{P}} = \langle \underline{R}_{\mathsf{N}}; \underline{T}_{\mathsf{P}}\rangle^+, & \underline{\langle R; T\rangle}_{\mathsf{N}} = \langle \underline{R}_{\mathsf{P}}; \underline{T}_{\mathsf{N}}\rangle^-.
\end{array}
$$

These functions are extended to (polarized) structure context in the obvious way, i.e., by adding these definitions to the above ones: $\underline{\{\ \}}_{\mathsf{S}} = \{\ \}$, $\underline{\{\ \}}_{\mathsf{P}} = \{\ \}$ *and* $\underline{\{\ \}}_{\mathsf{N}} = \{\ \}$.

Lemma 35. *Let R be a positive (negative) polarized structure. Then $\underline{R}_{\mathsf{s}\,\mathsf{P}} = R$ (respectively, $\underline{R}_{\mathsf{s}\,\mathsf{N}} = R$).*

Lemma 36. *Let $S\{\ \}$ be a polarized context and R a polarized structure.*

(i) *If R is a positive structure, then*
- (a) *if $S\{R\}$ is a positive structure then $\underline{S\{R\}}_{\mathsf{s}} = S'\{\underline{R}_{\mathsf{s}}\}$, for some positive context $S'\{\ \}$ such that $S'\{\ \} = \underline{S\{\ \}}_{\mathsf{s}}$,*
- (b) *if $S\{R\}$ is a negative structure then $\underline{S\{R\}}_{\mathsf{s}} = S'\{\underline{R}_{\mathsf{s}}\}$, for some negative context $S'\{\ \}$ such that $S'\{\ \} = \underline{S\{\ \}}_{\mathsf{s}}$.*

(ii) *If R is a negative structure, then*
- (a) *if $S\{R\}$ is a positive structure then $\underline{S\{R\}}_{\mathsf{s}} = S'\{\underline{R}_{\mathsf{s}}\}$, for some negative context $S'\{\ \}$ such that $S'\{\ \} = \underline{S\{\ \}}_{\mathsf{s}}$,*
- (b) *if $S\{R\}$ is a negative structure then $\underline{S\{R\}}_{\mathsf{s}} = S'\{\underline{R}_{\mathsf{s}}\}$, for some positive context $S'\{\ \}$ such that $S'\{\ \} = \underline{S\{\ \}}_{\mathsf{s}}$.*

Proof. The proof is by structural induction on $S\{\ \}$. We show the interesting cases where $S\{\ \} = \langle S_1\{\ \}; T\rangle^+$ and where $S\{\ \} = \langle S_1\{\ \}; T\rangle^-$.

($i.a$) $S\{\ \} = \langle S_1\{\ \}; T\rangle^+$ and R and $S\{R\}$ are positive structures. In this case, $S_1\{R\}$ is a negative structure and by induction hypothesis we have $\underline{S_1\{R\}}_{\mathsf{s}} = S'_1\{\underline{R}_{\mathsf{s}}\}$ for some negative context $S'_1\{\ \}$ such that $S'_1\{\ \} = \underline{S_1\{\ \}}_{\mathsf{s}}$. We therefore have

$$\underline{S\{R\}}_{\mathsf{s}} = \langle \underline{S_1\{R\}}_{\mathsf{s}}; \underline{T}_{\mathsf{s}}\rangle = \langle S'_1\{\underline{R}_{\mathsf{s}}\}; \underline{T}_{\mathsf{s}}\rangle$$

and $\underline{S\{\ \}}_{\mathsf{s}} = \langle S'_1\{\ \}; \underline{T}_{\mathsf{s}}\rangle$ is a positive context.

($i.b$) $S\{\ \} = \langle S_1\{\ \}; T\rangle^-$, R is a positive structure and $S\{R\}$ is a negative structure (and hence $S_1\{R\}$ a positive structure). By induction hypothesis we have $\underline{S_1\{R\}}_{\mathsf{s}} = S'_1\{\underline{R}_{\mathsf{s}}\}$ for some positive context $S'_1\{\ \}$. Hence

$$\underline{S\{R\}}_{\mathsf{s}} = \langle \underline{S_1\{R\}}_{\mathsf{s}}; \underline{T}_{\mathsf{s}}\rangle = \langle S'_1\{\underline{R}_{\mathsf{s}}\}; \underline{T}_{\mathsf{s}}\rangle$$

and $\underline{S\{\ \}}_{\mathsf{s}} = \langle S'_1\{\ \}; \underline{T}_{\mathsf{s}}\rangle$ is a negative context.

($ii.a$) $S\{\ \} = \langle S_1\{\ \}; T\rangle^+$, R is a negative structure and $S\{R\}$ is a positive structure. By induction hypothesis: $\underline{S_1\{R\}}_{\mathsf{s}} = S'_1\{\underline{R}_{\mathsf{s}}\}$ for some positive context $S'_1\{\ \}$, and hence

$$\underline{S\{R\}}_{\mathsf{s}} = \langle \underline{S_1\{R\}}_{\mathsf{s}}; \underline{T}_{\mathsf{s}}\rangle = \langle S'_1\{\underline{R}_{\mathsf{s}}\}; \underline{T}_{\mathsf{s}}\rangle$$

and $\underline{S\{\ \}}_{\mathsf{s}} = \langle S'_1\{\ \}; \underline{T}_{\mathsf{s}}\rangle$ is a negative context.

($ii.b$) $S\{\ \} = \langle S_1\{\ \}; T\rangle^-$ and R and $S\{R\}$ are negative structures. By induction hypothesis, $\underline{S\{R\}}_{\mathsf{s}} = S_1\{\underline{R}_{\mathsf{s}}\}$ for some negative context $S'_1\{\ \}$ and hence

$$\underline{S\{R\}}_{\mathsf{s}} = \langle \underline{S_1\{R\}}_{\mathsf{s}}; \underline{T}_{\mathsf{s}}\rangle = \langle S'_1\{\underline{R}_{\mathsf{s}}\}; \underline{T}_{\mathsf{s}}\rangle$$

and $\underline{S\{\ \}}_{\mathsf{s}} = \langle S'_1\{\ \}; \underline{T}_{\mathsf{s}}\rangle$ is a positive context.

□

Lemma 37. *Let $S\{\ \}$ be a context and let R be a structure.*

(i) *If $S\{\ \}$ is a positive context, then*
 (a) $\underline{S\{R\}}_{\mathsf{P}} = S'\{\underline{R}_{\mathsf{P}}\}$ *for some* $S'\{\ \}$ *such that* $S'\{\ \} = \underline{S\{\ \}}_{\mathsf{P}}$, *and*
 (b) $\underline{S\{R\}}_{\mathsf{N}} = S'\{\underline{R}_{\mathsf{N}}\}$ *for some* $S'\{\ \}$ *such that* $S'\{\ \} = \underline{S\{\ \}}_{\mathsf{N}}$.
(ii) *If $S\{\ \}$ is a negative context, then*
 (a) $\underline{S\{R\}}_{\mathsf{P}} = S'\{\underline{R}_{\mathsf{N}}\}$ *for some* $S'\{\ \}$ *such that* $S'\{\ \} = \underline{S\{\ \}}_{\mathsf{P}}$, *and*
 (b) $\underline{S\{R\}}_{\mathsf{N}} = S'\{\underline{R}_{\mathsf{P}}\}$ *for some* $S'\{\ \}$ *such that* $S'\{\ \} = \underline{S\{\ \}}_{\mathsf{N}}$.

Proof. By induction on $S\{\ \}$, we prove simultaneously $(i.a)$, $(i.b)$, $(ii.a)$ and $(ii.b)$. We consider the non-trivial case where there is a reversal in polarity.

(i) $S\{\ \} = \langle S_1\{\ \}; T\rangle$ is a positive context: Since $S_1\{\ \}$ is a negative context, by induction hypothesis we have

$$\underline{S_1\{R\}}_{\mathsf{P}} = S'_1\{\underline{R}_{\mathsf{N}}\},\ \text{where } S'_1\{\ \} = \underline{S_1\{\ \}}_{\mathsf{P}} \tag{1}$$

$$\underline{S_1\{R\}}_{\mathsf{N}} = S''_1\{\underline{R}_{\mathsf{P}}\},\ \text{where } S''_1\{\ \} = \underline{S_1\{\ \}}_{\mathsf{N}} \tag{2}$$

From (2), it follows that

$$\underline{S\{R\}}_{\mathsf{P}} = \langle \underline{S_1\{R\}}_{\mathsf{N}}; \underline{T}_{\mathsf{P}}\rangle^+ = \langle S''_1\{\underline{R}_{\mathsf{P}}\}; \underline{T}_{\mathsf{P}}\rangle^+ \quad \text{and} \quad \underline{S\{\ \}}_{\mathsf{P}} = \langle S''_1\{\ \}; \underline{T}_{\mathsf{P}}\rangle^+$$

from which $(i.a)$ follows trivially. Similarly, from (1), we prove $(i.b)$:

$$\underline{S\{R\}}_{\mathsf{N}} = \langle \underline{S_1\{R\}}_{\mathsf{P}}; \underline{T}_{\mathsf{P}}\rangle^- = \langle S'_1\{\underline{R}_{\mathsf{N}}\}; \underline{T}_{\mathsf{N}}\rangle^- \quad \text{and} \quad \underline{S\{\ \}}_{\mathsf{N}} = \langle S'_1\{\ \}; \underline{T}_{\mathsf{N}}\rangle^-.$$

(ii) $S\{\ \} = \langle S_1\{\ \}; T\rangle$ and $S\{\ \}$ is a negative context. By induction hypothesis we have

$$\underline{S_1\{R\}}_{\mathsf{P}} = S'_1\{\underline{R}_{\mathsf{P}}\}, \qquad S'_1\{\ \} = \underline{S_1\{\ \}}_{\mathsf{P}}, \quad \text{and}$$
$$\underline{S_1\{R\}}_{\mathsf{N}} = S''_1\{\underline{R}_{\mathsf{N}}\}, \qquad S''_1\{\ \} = \underline{S_1\{\ \}}_{\mathsf{N}}$$

and hence

$$\begin{array}{ll} (ii.a) & \underline{S\{R\}}_{\mathsf{P}} = \langle \underline{S_1\{R\}}_{\mathsf{N}}; \underline{T}_{\mathsf{P}}\rangle^+ = \langle S''_1\{\underline{R}_{\mathsf{N}}\}; \underline{T}_{\mathsf{P}}\rangle^+ \quad \text{and} \\ & \underline{S\{\ \}}_{\mathsf{P}} = \langle S''_1\{\ \}; \underline{T}_{\mathsf{P}}\rangle^+ \\ (ii.b) & \underline{S_1\{R\}}_{\mathsf{N}} = \langle \underline{S_1\{R\}}_{\mathsf{P}}; \underline{T}_{\mathsf{N}}\rangle^- = \langle S'_1\{\underline{R}_{\mathsf{P}}\}; \underline{T}_{\mathsf{N}}\rangle^- \quad \text{and} \\ & \underline{S\{\ \}}_{\mathsf{N}} = \langle S'_1\{\ \}; \underline{T}_{\mathsf{N}}\rangle^- \end{array}$$

□

Lemma 38. *Given any two positive structures R and T, there is a derivation Δ from R to T in* SISp *if and only if there is a derivation Δ' in from $\underline{R}_{\mathsf{S}}$ to $\underline{T}_{\mathsf{S}}$ in* SISaq *such that Δ and Δ' contain the same sequence of rules.*

Proof. It is enough to show that for every instance of a rule

$$\rho\,\frac{S\{V\}}{S\{U\}}$$

in SISp where $S\{U\}$ and $S\{V\}$ are positive structures, there is a corresponding instance of a rule under the same name in SISaq

$$\rho\,\frac{\underline{S\{V\}}_{\mathsf{S}}}{\underline{S\{U\}}_{\mathsf{S}}}$$

and vice versa. One direction, that is, from SISp to SISaq, is a consequence of Lemma 36. In this case, $\underline{S\{\ \}}_{\mathsf{S}}$ is a positive context if U is a positive structure, otherwise it is a negative context. Therefore it remains to show that the structures $\underline{U}_{\mathsf{S}}$ and $\underline{V}_{\mathsf{S}}$ are related by the rule ρ in SISaq. A simple observation of the rules in SISp and SISaq proves that this is trivially the case.

On the other direction, we show that if the structures $\underline{S\{U\}}_{\mathsf{S}} = S'\{U'\}$ and $\underline{S\{V\}}_{\mathsf{S}} = S'\{V'\}$ are related by some ρ' in SISaq, then the same rule relates $\underline{S'\{U'\}}_{\mathsf{P}}$ and $\underline{S'\{V'\}}_{\mathsf{P}}$ in SISp. We use Lemma 37 to establish the fact that structures nested under the positive (negative) context is translated to positive (negative) structures. That is, $\underline{S'\{U'\}}_{\mathsf{P}} = S''\{\underline{U'}_{\mathsf{P}}\}$ and $\underline{S'\{V'\}}_{\mathsf{P}} = S''\{\underline{V'}_{\mathsf{P}}\}$, if $S'\{\ \}$ is a positive context. Otherwise, $\underline{S'\{U'\}}_{\mathsf{P}} = S''\{\underline{U'}_{\mathsf{N}}\}$ and $\underline{S'\{V'\}}_{\mathsf{P}} = S''\{\underline{V'}_{\mathsf{N}}\}$. Note that we need to show that $S\{U\}$ is equivalent to $S''\{\underline{U'}_{\mathsf{P}}\}$. This is a consequence of Lemma 35:

$$S\{U\} = \underline{\underline{S\{U\}}_{\mathsf{S}}}_{\mathsf{P}} = \underline{S'\{U'\}}_{\mathsf{P}} = S''\{\underline{U'}_{\mathsf{P}}\}.$$

Again it is a simple matter to verifiy that the two structures $\underline{U}_{\mathsf{P}}{}'$ and $\underline{V}_{\mathsf{P}}{}'$ (respectively, $\underline{U}_{\mathsf{N}}{}'$ and $\underline{V}_{\mathsf{N}}{}'$) are related by the rule ρ' in SISp. □

Proof of Theorem 14. Follows immediately from Lemma 38. □