Lecture 2: *Greedy Approximation Algorithms*

Advanced Algorithms

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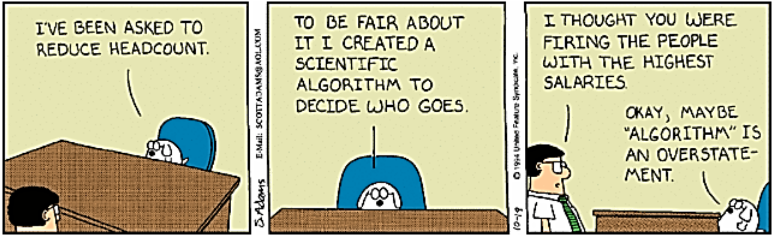
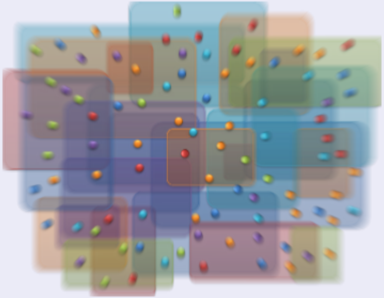
Being Greedy may be Good



- The simplest algorithm is to make the most improvement as much as possible in each step

⋘ Set Cover ⋙



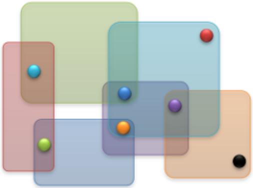
Set Cover Problem

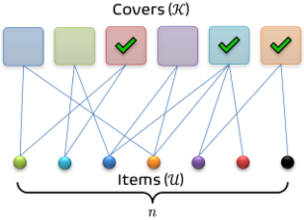
- Let the objective function of a NP-hard minimization problem be Cost$(\cdot)$

Definition (SetCover)

- Consider a set $\mathcal{U}$ with n items and a family of subsets (called covers) $\mathcal{K} \subseteq 2^{\mathcal{U}}$
- Each cover has a non-negative cost: Cost(S) for $S \in \mathcal{K}$
- Select the a subset of covers $\tilde{\mathcal{K}} \subseteq \mathcal{K}$ such that
  - Minimizing the total cost:
  $$\min_{\tilde{\mathcal{K}} \subseteq \mathcal{K}} \sum_{S \in \tilde{\mathcal{K}}} \mathsf{Cost}(S)$$
  - Subject to the constraint of covering all items:
  $$\bigcup_{S \in \tilde{\mathcal{K}}} S = \mathcal{U}$$

Set Cover Problem





	Covers	Items
Sensor Networks	Sensors	Targets of Interest
Logistics	Service Depots	Clients
Cloud Computing	Cloud Servers	Users
Testing	Tests	Properties

Example: Antivirus Scanner

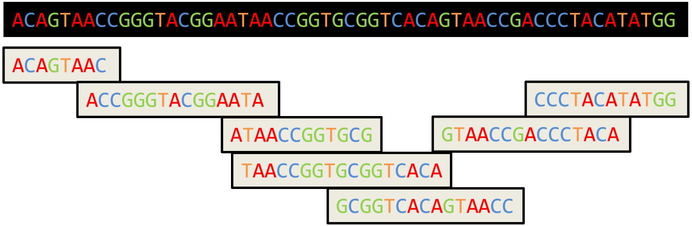
Example (Antivirus Scanner)

- Some special features can be detected in the boot sector of a computer, if a computer virus is present
- Let the items of SetCover be the known boot sector viruses ($\sim$150 at the time)
- Let each cover be a three-byte sequence in the boot sector, if viruses are present in a computer ($\sim$21,000 such sequences)
- Each cover contains all the boot sector viruses that have the corresponding three-byte sequence detected in the boot sector
- Goal: Find a minimum number of such sequences ($\ll$150) that are useful for an Antivirus scanner

Example: DNA Sequencing

Example (DNA Sequencing (Shortest Superstring))

- DNA sequencing is also a set cover problem
- When sequencing DNA, it is not achieved by one sequential operation
- Instead, many short segments of DNA sequences can be identified
- Need to reconstruct the original long DNA sequence from these short identified segments
- Goal: Find the shortest sequence that is a superstring of all short identified segments

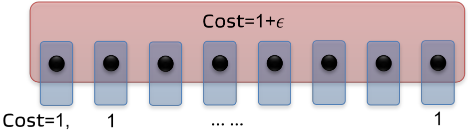


Set Cover

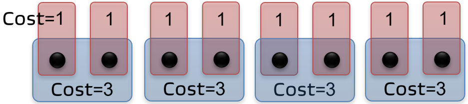
- What is a good strategy to solve SetCover?
- Intuitive approach is to use a greedy algorithm
- Find the most improvement in each step. But how?
  - Select the cover with the lowest cost?
  - Select the cover with the most uncovered items?
  - Select the cover with the lowest cost per uncovered item?
- Which one of them works?
- If not, why doesn't it work?

How bad is a Greedy Algorithm?

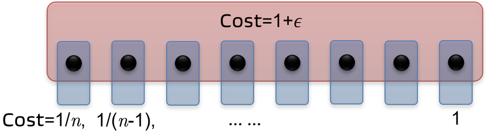
- Select the cover with the lowest cost?



- Select the cover with the most uncovered items?



- Select the cover with the lowest cost per uncovered item?



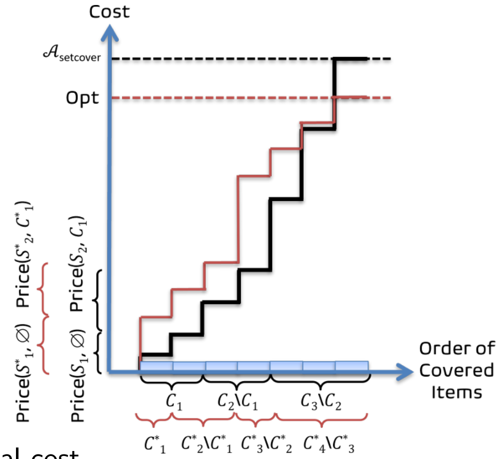
Better Greedy Algorithm for SetCover

- Let the set of items that are already covered before the k-th step be $\mathcal{C}_k$
- Define the price for each cover $S \in \mathcal{K}$ by $\text{Price}(S, \mathcal{C}_k) \triangleq \frac{\text{Cost}(S)}{|S \backslash \mathcal{C}_k|}$

Algorithm $\mathcal{A}_{\text{setcover}}$

- $\tilde{\mathcal{K}} \leftarrow \varnothing;\ \mathcal{C}_1 \leftarrow \varnothing;\ k \leftarrow 1$
- While $\mathcal{C}_k \neq \mathcal{U}$
  - Find $S \in \mathcal{K}$ with the least $\text{Price}(S, \mathcal{C}_k)$
  - $\tilde{\mathcal{K}} \leftarrow \tilde{\mathcal{K}} \cup \{S\}$
  - $\mathcal{C}_{k+1} \leftarrow \mathcal{C}_k \cup S$
  - $k \leftarrow k + 1$
- Return $\tilde{\mathcal{K}}$

Better Greedy Algorithm for SetCover



- Let Opt be the optimal cost
- Let S_k be the selected cover by $\mathcal{A}_{\text{setcover}}$ at the k-th step
- Let S^*_k be the selected cover by the optimal solution at the k-th step

Better Greedy Algorithm for SetCover

Theorem

The approximation ratio of $\mathcal{A}_{\text{setcover}}$ is $O(\log(n))$

Proof:

- The number of items being not covered at the k-th step is $n - |\mathcal{C}_k|$
- We show that $\text{Price}(S_k, \mathcal{C}_k) \leq \frac{\text{Opt}}{n-|\mathcal{C}_k|}$
  - $\mathcal{A}_{\text{setcover}}$ always selects the least-price cover (i.e. the least cost per uncovered item)
  - Let Opt_k be the optimal cost on $\mathcal{U}\backslash\mathcal{C}_k$ and the corresponding optimal covers be $\mathcal{K}_k^*$
  - The price of S_k must be lower than the overall price of Opt_k

$$\text{Price}(S_k, \mathcal{C}_k) = \frac{\text{Cost}(S_k)}{|S_k\backslash\mathcal{C}_k|} \leq \frac{\text{Opt}_k}{n - |\mathcal{C}_k|}$$

Better Greedy Algorithm for SetCover

Proof (Cont.):

- Why $\text{Price}(S_k, \mathcal{C}_k) \leq \frac{\text{Opt}_k}{n - |\mathcal{C}_k|}$?
  - Since $\text{Opt}_k = \sum_{S \in \mathcal{K}_k^*} |S \backslash \mathcal{C}_k| \cdot \text{Price}(S, \mathcal{C}_k)$, there always exists $S \in \mathcal{K}_k^*$, such that

$$\text{Price}(S, \mathcal{C}_k) \leq \frac{\text{Opt}_k}{\sum_{S \in \mathcal{K}_k^*} |S \backslash \mathcal{C}_k|}$$

  - Note that $\sum_{S \in \mathcal{K}_k^*} |S \backslash \mathcal{C}_k| \geq n - |\mathcal{C}_k|$. Hence, we obtain $\text{Price}(S_k, \mathcal{C}_k) \leq \frac{\text{Opt}_k}{n - |\mathcal{C}_k|}$
- Because $\text{Opt}_k \leq \text{Opt}$,

$$\text{Cost}(S_k) = \text{Price}(S_k, \mathcal{C}_k) \cdot (|\mathcal{C}_{k+1}| - |\mathcal{C}_k|) \leq \frac{|\mathcal{C}_{k+1}| - |\mathcal{C}_k|}{n - |\mathcal{C}_k|} \text{Opt}_k \leq \frac{|\mathcal{C}_{k+1}| - |\mathcal{C}_k|}{n - |\mathcal{C}_k|} \text{Opt}$$

Better Greedy Algorithm for SetCover

Proof (Cont.):

- Because $\text{Cost}(S_k) \leq \frac{|\mathcal{C}_{k+1}|-|\mathcal{C}_k|}{n-|\mathcal{C}_k|}\text{Opt}$,

$$\begin{aligned}
\text{Cost}(\mathcal{A}_{\text{setcover}}) &= \sum_{k=1}^{|\tilde{\mathcal{K}}|} \text{Cost}(S_k) \\
&\leq \text{Opt} \cdot \sum_{k=1}^{|\tilde{\mathcal{K}}|} \frac{|\mathcal{C}_{k+1}| - |\mathcal{C}_k|}{n - |\mathcal{C}_k|} \\
&\leq \text{Opt} \cdot \sum_{i=1}^{n} \frac{1}{i} \quad \big(\text{since } n - |\mathcal{C}_k| \geq n - i + 1, \text{ if the } i\text{-th item} \in \mathcal{C}_{k+1} \backslash \mathcal{C}_k\big) \\
&= \text{Opt} \cdot \text{O}(\log(n))
\end{aligned}$$

- Note that the sum $\sum_{i=1}^{n} \frac{1}{i}$ is called the harmonic number
- Therefore, the approximation ratio is $\alpha_n(\mathcal{A}_{\text{setcover}}) = \text{O}(\log(n))$

Is Greedy Algorithm Bad?

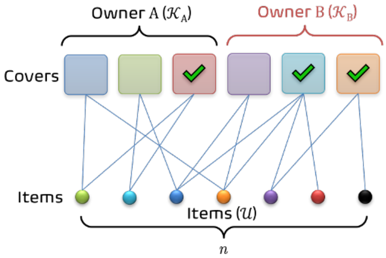
Theorem

For any polynomial-time algorithm $\mathcal{A}$, the approximation ratio is $\alpha_n(\mathcal{A}) = \Omega(\log(n))$ for SetCover, *unless* P $=$ NP

- Namely, $\alpha_n(\mathcal{A}) = \mathrm{O}(\log(n))$ is the best to achieve, unless P $=$ NP
- Hence, the greedy algorithm is already the best in terms of order magnitude
- SetCover does not admit any polynomial-time algorithm with a constant approximation ratio
- The proof uses PCP theorem, one of the most of fundamental theorems in complexity theory

Balanced Cover Ownership

- Suppose that the covers $\mathcal{K}$ belong to different owners
- We are required to balance the costs of covers among the owners



Example (Traffic Measurement Problem)

- The items are links in a network
- A cover is a path in the networks (comprising of some links)
- Several agents send probe packets along the selected paths for measurement in parallel
- Goal: Select a subset of paths that traverse all the intended links while keeping the maximum number of paths of any agent to be the minimum

Balanced Cover Ownership

Definition (BalSetCover)

- Consider a set $\mathcal{U}$ of n and covers $\mathcal{K} \subseteq 2^{\mathcal{U}}$ with $\text{Cost}(S)$ for $S \in \mathcal{K}$
- There are m owners, who split the covers by $\{\mathcal{K}_1, ..., \mathcal{K}_m\}$, such that $\bigcup_{i=1}^m \mathcal{K}_i = \mathcal{K}$ and $\mathcal{K}_i \cap \mathcal{K}_j = \varnothing$ for $i \neq j$
- Select the a subset of covers $\tilde{\mathcal{K}} \subseteq \mathcal{K}$ such that
  - Minimizing the maximum total cost among the owners:

$$\min_{\tilde{\mathcal{K}} \subseteq \mathcal{K}} \left(\max_{i \in \{1,...,m\}} \sum_{S \in \tilde{\mathcal{K}} \cap \mathcal{K}_i} \text{Cost}(S) \right)$$

  - Subject to the constraint of covering all items:

$$\bigcup_{S \in \tilde{\mathcal{K}}} S = \mathcal{U}$$

Greedy Algorithm for Balanced Set Cover

Algorithm $\mathcal{A}_{\text{bsetcover}}$

- $\tilde{\mathcal{C}} \leftarrow \varnothing$
- $\tilde{\mathcal{K}} \leftarrow \varnothing$
- While $\tilde{\mathcal{C}} \neq \mathcal{U}$
  - For each owner $i \in \{1, ..., m\}$
    - Find $S \in \mathcal{K}_i$ with the least Price$(S, \mathcal{C})$
    - $\tilde{\mathcal{K}} \leftarrow \tilde{\mathcal{K}} \cup \{S\}$
    - $\tilde{\mathcal{C}} \leftarrow \tilde{\mathcal{C}} \cup S$
- Return $\tilde{\mathcal{K}}$

- Loop through each owner to pick the least-price cover at each step
- Terminate when all items are covered, like the normal set cover problem

Greedy Algorithm for Balanced Set Cover

Theorem

The competitive ratio of $\mathcal{A}_{\text{bsetcover}}$ is $O(m \log(n))$

Proof:

- Consider a sub-problem by focusing on a particular owner i
- Let $\mathcal{U}_i = \cup_{S \in \mathcal{K}_i} S$
- Run $\mathcal{A}_{\text{setcover}}$ over $(\mathcal{U}_i, \mathcal{K}_i)$
- At each step the price of selected cover in $\mathcal{A}_{\text{setcover}}$ cannot be lower than $\mathcal{A}_{\text{bsetcover}}$ for the corresponding owner i
- Hence, the approximation ratio is at most $\mathrm{O}(\log(n))$ for each owner, and $\mathrm{O}(m \log(n))$ for all m owners

Application: Survivable Network

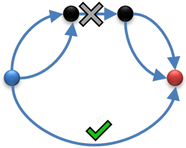
- Some problems may not appear as set cover problem
- But we can transform them into set cover problem

Definition (Survivable Network)

- Given a graph $\mathcal{G}$, there are some pairs of sources and destinations
- There are a set paths $\mathcal{P}$ connecting every pair of source and destination
- Find a minimal subset $\tilde{\mathcal{P}} \subseteq \mathcal{P}$, such that there exists at least one path in $\tilde{\mathcal{P}}$ for each pair of source and destination that can survive, even though any one link in $\mathcal{G}$ fails

Basic idea:

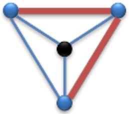
- Items are all the links in $\mathcal{G}$
- Covers are the paths in $\mathcal{P}$
- A path P covers link e, if P does not traverse e



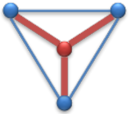
Steiner Tree

Definition

- Given a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- A subset of nodes $\mathcal{R} \subseteq \mathcal{V}$ are called *terminals*
- Goal: Connect the terminals using the minimum network in $\mathcal{G}$
  - Possibly using vertices not in $\mathcal{R}$ that called Steiner nodes
- Two versions:
  - *Edge-weighted*: Minimum number of links (or weighted total cost)
  - *Node-weighted*: Minimum number of vertices (or weighted total cost)



Minimum Node-Weighted Steiner Tree

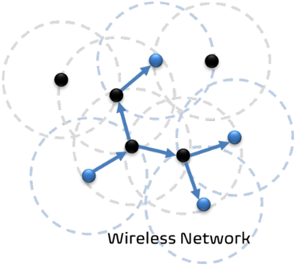


Minimum Edge-Weighted Steiner Tree

Example: Wireless Networks

Example (Wireless Networks)

- In a wireless network, a node can reach all the nodes within its transmission range
- Terminals want to communicate with each other, even though they are outside the transmission range of each other
- Our goal is to find a minimum number of relay nodes to relay all data among the set of terminals by a node-weighted Steiner tree



Inapproximability

Theorem

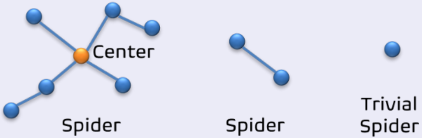
Node-weighted Steiner tree problem is inapproximable with an approximation ratio $\Omega(\log |\mathcal{R}|)$

Proof:

- Reduce any set cover problem to a node-weighted Steiner tree problem
- The solution of set cover problem has one-to-one correspondence to the solution node-weighted Steiner tree problem
- Given SetCover with $\mathcal{U}$ and $\mathcal{K}$, we create a graph $\mathcal{G}$ such that
  - The set of terminals $\mathcal{R}$ is $\mathcal{U}$
  - Each cover $S \in \mathcal{K}$ is a non-terminal vertex in $\mathcal{G}$
  - Add a link between the cover $S \in \mathcal{K}$ to all covered items by S
  - Connect all non-terminal vertices in $\mathcal{G}$ by a complete graph
- The cost of an optimal solution in SetCover is the same as an optimal node-weighted Steiner tree in $\mathcal{G}$

Steiner Tree: Spider Decomposition

Definition (Spider in a Graph)



- *Spider*:
  - A tree with at most one vertex of degree larger than two
- *Foot of Spider*:
  - Center of spider (when three or more leave nodes) or one of the leave nodes
- *Non-trivial Spider*:
  - A spider with at least two leave nodes
- *Spider Decomposition*:
  - Disjoint union of non-trivial spiders whose feet contains all the terminals in $\mathcal{R}$

Steiner Tree: Spider Decomposition

Lemma

Given a connected graph $\mathcal{G}$ and a subset of vertices $\mathcal{R}$ (where $|\mathcal{R}| \geq 2$), $\mathcal{G}$ always contains a spider decomposition of $\mathcal{R}$.

Implications:

- Node-weighted Steiner tree problem can be transformed into set cover problem, through spider decomposition
  - A spider is like a cover
  - The feet of a spider are like the items of a cover
- Goal of constructing a Node-weighted Steiner tree becomes to select the least-price spiders to connect all the terminals

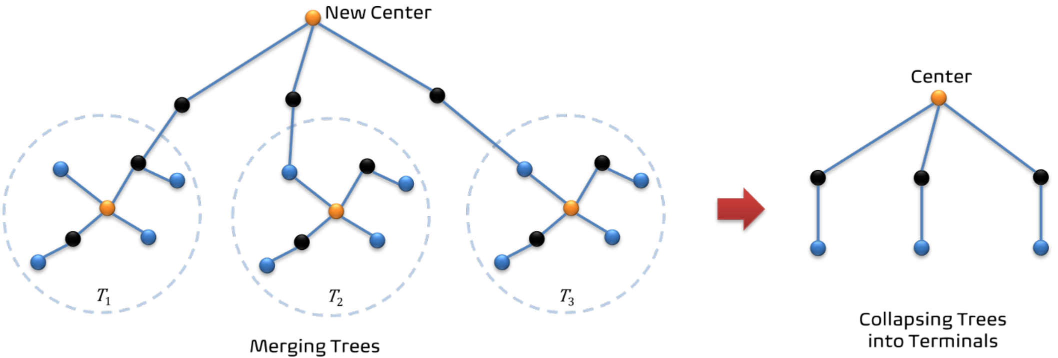
Greedy Algorithm for Steiner Tree

- A spider is like merging a set of trees $\{T_1, ...T_m\}$
- Define Price$(v, \{T_1, ..., T_m\}) \triangleq \frac{\text{cost of } v \text{ + total distance cost to } \{T_1,...,T_m\}}{m}$

Algorithm $\mathcal{A}_{\text{nwsteiner}}$

- $\tilde{\mathcal{C}} \leftarrow \varnothing$; $\tilde{\mathcal{K}} \leftarrow \varnothing$
- Trees $\leftarrow \big\{\{v\} \mid v \in \mathcal{R}\big\}$
- While $\tilde{\mathcal{C}} \neq \mathcal{R}$
  - Find $v \in \mathcal{V} \backslash \tilde{\mathcal{K}}$ and $\tilde{\mathcal{T}} \subseteq$ Trees with the least Price$(v, \tilde{\mathcal{T}})$ *//Find the least-price spider*
  - $\tilde{\mathcal{K}} \leftarrow \tilde{\mathcal{K}} \cup \{v\}$ *//To form a new spider with center at v*
  - $\tilde{\mathcal{C}} \leftarrow \big\{t \mid t \in \mathcal{R} \cap \tilde{\mathcal{T}}\big\}$ *//Count the covered terminals by the new spider*
  - Trees $\leftarrow$ Trees$\backslash \tilde{\mathcal{T}} \cup \{$Merging $\tilde{\mathcal{T}}$ as single tree by a spider at $v\}$ *//Collapse trees into a terminal*
- Return $\tilde{\mathcal{K}}$

Merging Trees in Greedy Algorithm for Steiner Tree



Theorem

The competitive ratio of $\mathcal{A}_{\text{nwsteiner}}$ is $O(\log(|\mathcal{R}|))$

Proof:

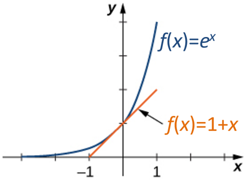
- Let the number of trees at the k-th step be $\phi_k \triangleq |\text{Trees}_k|$
- Let the number of trees merged at the k-th step be $m_k = \phi_{k-1} - \phi_k + 1$
- Let C_k be the total cost of adding the spider at the k-th step by $\mathcal{A}_{\text{nwsteiner}}$
- Since $\frac{C_k}{m_k} = \text{Price}(v_k, \tilde{\mathcal{T}}_k) \leq \frac{\text{Opt}}{\phi_{k-1}}$, we obtain

$$\frac{C_k \cdot \phi_{k-1}}{\text{Opt}} \leq m_k = \phi_{k-1} - \phi_k + 1 \leq 2(\phi_{k-1} - \phi_k) \quad (\text{since } \phi_{k-1} > \phi_k) \tag{1}$$

$$\phi_k \leq \phi_{k-1}\Big(1 - \frac{C_k}{2 \cdot \text{Opt}}\Big) \overset{\text{telescoping}}{\Longrightarrow} \phi_k \leq \phi_0 \prod_{j=1}^{k} \Big(1 - \frac{C_j}{2 \cdot \text{Opt}}\Big) \tag{2}$$

Greedy Algorithm for Steiner Tree

Proof (Cont.):



- Noting that $1 + x \leq e^x$,

$$\frac{\phi_k}{\phi_0} \leq \prod_{j=1}^{k} \left(1 - \frac{C_j}{2 \cdot \mathsf{Opt}}\right) \leq \prod_{j=1}^{k} e^{-\frac{C_j}{2 \cdot \mathsf{Opt}}} = e^{\frac{-\sum_{j=1}^{k} C_j}{2 \cdot \mathsf{Opt}}} \tag{3}$$

$$\Rightarrow \sum_{j=1}^{k} C_j \leq 2 \ln\left(\frac{\phi_0}{\phi_k}\right) \cdot \mathsf{Opt} \Rightarrow \mathsf{Cost}(\mathcal{A}_{\mathsf{nwsteiner}}) \leq 2 \ln(|\mathcal{R}|) \cdot \mathsf{Opt} \tag{4}$$

References

Reference Materials

- Approximation Algorithms (V. Vazirani), Springer
  - Chapter 2
- Design of Approximation Algorithms (Williamson, Shmoys), Cambridge University Press
  - Chapter 1

Recommended Materials

- *A Nearly Best-Possible Approximation Algorithm for Node-Weighted Steiner Tree* (P. Klein and R. Ravi), Journal of Algorithms, 19, pp104-115, 1995
- *An O(log n)-approximation for the Set Cover Problem with Set Ownership* (M. Gonen and Y. Shavitt), Information Processing Letters, Vol. 109 (3), Jan 2009