Lecture 12: *Algorithmic Game Theory & Price-of-Anarchy*

Advanced Algorithms

Sid Chi-Kin Chau

Australian National University

sid.chau@anu.edu.au

October 24, 2022

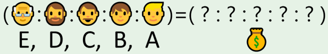
Pirates' Puzzle 🏴‍☠️

Pirates' Puzzle

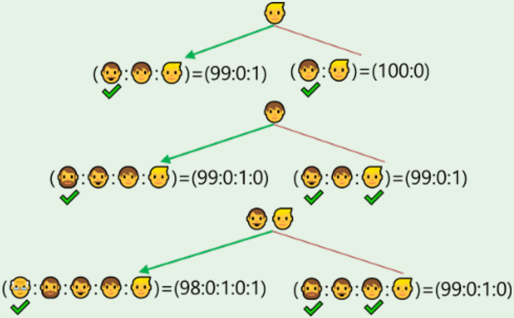
(👴:🧔:👨:👦:👱)=(? : ? : ? : ? : ?)
E, D, C, B, A 💰

- Five pirates of different ages have a treasure of 100 gold coins
- On their ship, they decide to split the coins using the following rules:
  1. The oldest pirate proposes how to share the coins, and ALL pirates (including the oldest) vote for or against it
  2. If 50% or more of the pirates vote for it, then the coins will be shared that way. Otherwise, the pirate proposing the scheme will be thrown out of the ship, and the process is repeated with the pirates who remain
  3. As pirates tend to be brutal, if a pirate would get the same number of coins if he voted for or against a proposal, he will vote against, so that the pirate who proposed the plan will be thrown out of the ship
- Assume all 5 pirates are rational, greedy, and do not wish to die. How to split the coins?

Pirates' Puzzle



- *Solution*: E proposes (98:0:1:0:1) split; E, C, A will vote for it
- Backward induction:



Everyday, we interact with each other non-cooperatively

- In a society, everyone competes for resources (e.g., roads, jobs, services, properties)

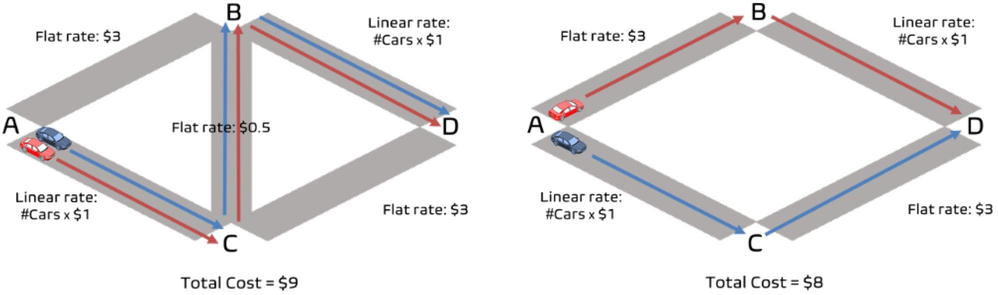
Selfish Routing 🚖

Example (Selfish Routing)

- Congestion in road traffic incurs certain costs (e.g., latency or congestion charges)
- Each driver selfishly chooses the route to minimize her own travel time (or tolls)
- The road network is in equilibrium if no driver can get to the destination in a lower cost by switching to a different route

- Different congestion costing models:
  - Flat rate model: Same congestion cost incurred to everyone regardless the congestion level
    - E.g., fast highways
  - Linear rate model: Congestion cost increases linearly as the congestion level (e.g., number of cars in the road)
    - E.g., capacitated city roads
- Road networks feature a mix of congestion costing models
- Changes in road networks may lead to unintuitive phenomena

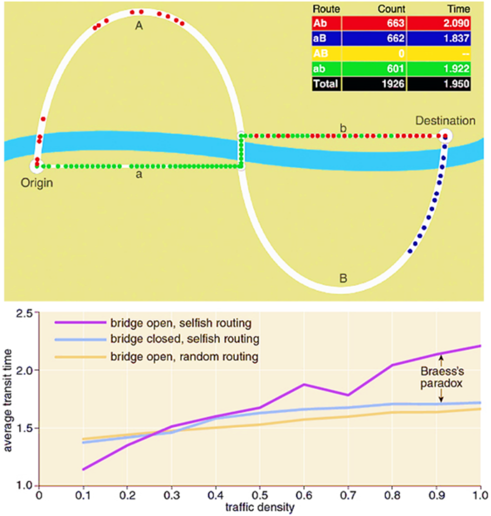
Braess Paradox



- Consider two cars with selfish routing
- Before road closure: The lowest-cost route for both cars is A $\rightarrow$ C $\rightarrow$ B $\rightarrow$ D, total cost $= \$2(2) + \$2(0.5) + \$2(2) = \$9$
- After road closure: Both cars travel in separate routes, total cost $= \$2(3) + \$2(1) = \$8$
- Braess paradox - closing a road benefits everyone !?

Braess Paradox in Real World

- Braess paradox is observed in real world
- Use real-world traffic simulation, considering traffic behavior and geographical pattern
  - *Playing in Traffic* (Brian Hayes) American Scientist, Jul-Aug, 2015 📖
  - Simulator: `http://bit-player.org/extras/traffic/`
- Other example: New Work City Times Square was converted to a pedestrian plaza back in 2009, closing it off to cars. Partially motivated by Braess paradox
  - *Beach Chairs in Times Square – How closing Broadway to cars could solve a century of traffic woes*, Slate, 16 Jun 2009 📖



Network Sharing Game

Definition (Network Sharing Game)

- There are n players sharing a set of resources (e.g., edges in a road network)
- Each player i uses a set of resources P_i (e.g., paths $s_i \to t_i$)
- A strategy profile is $\vec{P} \triangleq (P_i)_{i=1}^n$
- Each edge e has cost C_e
- Cost C_e is split among all players who use edge e
  - Let $n_e(\vec{P}) = |\{i \mid e \in P_i\}|$ be the number of players using edge e
  - Equal cost sharing of C_e: each player i pays $\frac{C_e}{n_e(\vec{P})}$

- Given strategy profile $\vec{P}$, $c_i(\vec{P})$ is the cost paid by player i for all edges $e \in P_i$
- Let social cost be $C(\vec{P}) = \sum_i c_i(\vec{P})$
- Note that $C(\vec{P}) = \sum_{e \in \mathcal{E}(\vec{P})} C_e$, where $\mathcal{E}(\vec{P}) = \{$edges used in $\vec{P}$ by at least one player$\}$

Congestion Game

Definition (Congestion Game)

- Similar to network sharing game
- There are n players sharing a set of resources (e.g., edges in a road network)
- But each edge e has a cost function $c_e(n_e)$ that is paid by each player of using e
  - Edge cost depends on n_e the number of players using e
  - E.g., linear cost function $c_e(n_e) = a_e \cdot n_e + b_e$

- Network sharing game: the more people use a resource, the less the cost to each player
- Congestion game: the more people on a road, the greater the congestion, the greater the delay (cost) to each player

Non-cooperative Game

Definition (Non-cooperative Game)

- Generalization of zero-sum game with n players
- Let S_i be the set of all possible strategies for player i
- Let $s_i \in S_i$ be the selected strategy of player i
- Let $s_{-i} = (s_j)_{j \neq i}$ be strategies of all the players except i
- Let $c_i(s_i, s_{-i})$ be player i's cost as a function of the strategies of all players
- Each player aims to minimize her cost in response to other players' strategies s_{-i}

(c_i, c_j)	s_i = Confess	s_i = Deny
s_j = Confess	(5,5)	(10,0)
s_j = Deny	(0,10)	(0,0)

Nash Equilibrium

Definition (Nash Equilibrium)

- Let $s = (s_i)_{i=1}^{n}$ be a strategy profile, a set consisting of one strategy for each player
- Strategy profile s^* is a *Nash equilibrium* if

$$c_i(s_i^*, s_{-i}^*) \leq c_i(s_i, s_{-i}^*), \text{for all } s_i \in S, s_i \neq s_i^*$$

Namely, a strategy profile is a Nash equilibrium if no player can do better by unilaterally changing her strategy

(c_i, c_j)	s_i = `Confess`	s_i = `Deny`
s_j = `Confess`	(5,5)	(10,0)
s_j = `Deny`	(0,10)	(0,0)

(c_i, c_j)	s_i = `Fight`	s_i = `Yield`
s_j = `Fight`	(10,10)	(3.5,0)
s_j = `Yield`	(0,3)	(1,1)

Nash Equilibrium

- Rationality is common knowledge in Nash equilibrium
  - All players are rational
  - All players know that all players are rational
  - All players know that all players know that all players are rational
  - continue ad infinitum ...
- Also, there may be multiple Nash equilibria, and no clear way of “choosing” one Nash equilibrium over another Nash equilibrium

(c_i, c_j)	$s_i =$ `Confess`	$s_i =$ `Deny`
$s_j =$ `Confess`	(5,5)	(10,0)
$s_j =$ `Deny`	(0,10)	(0,0)

(c_i, c_j)	$s_i =$ `Fight`	$s_i =$ `Yield`
$s_j =$ `Fight`	(10,10)	(3.5,0)
$s_j =$ `Yield`	(0,3)	(1,1)

Price of Anarchy & Price of Stability

- **Social Optimal**: The strategy profile that minimizes the social cost: $\min_s \sum_{i=1}^{n} c_i(s)$
- Questions of comparing social optimal vs Nash equilibrium
  - How does the Nash equilibrium compare to the social optimal, in the worst case and in the best case?
  - How bad it is for the players to play a Nash equilibrium compared to playing the best outcome (if they could coordinate)?
- Price of Anarchy (PoA): Ratio of the worst Nash equilibrium over the social optimal
- Price of Stability (PoS): Ratio of the best Nash equilibrium over the social optimal
  - Comparison to optimal is in similar vein to approximation ratio and competitive ratio

(c_i, c_j)	$s_i =$ Fight	$s_i =$ Yield
$s_j =$ Fight	(10,10)	(3.5,0)
$s_j =$ Yield	(0,3)	(1,1)

Social Optimal	2
Worst Nash Equilibrium	3.5
Best Nash Equilibrium	3
PoA	3.5/2
PoS	3/2

Network Sharing Game

Lemma

Define potential function:

$$\phi(\vec{P}) \triangleq \sum_{e} \sum_{k=1}^{n_e(\vec{P})} \frac{C_e}{k}$$

Strategy profile $\vec{P}^*$ that is a minimum of potential function $\phi(\vec{P}^*)$ is also a Nash equilibrium

Proof:

- Suppose player i switches from P_i' to P_i, while all other players' strategies P_{-i} unchanged
- The change in player i's cost will be

$$c_i(P_i', P_{-i}) - c_i(P_i, P_{-i}) = \sum_{e \in P_i'} \frac{C_e}{n_e(P_i', P_{-i})} - \sum_{e \in P_i} \frac{C_e}{n_e(P_i, P_{-i})} = \phi(P_i', P_{-i}) - \phi(P_i, P_{-i})$$

Network Sharing Game

Proof (Cont.):

- Hence, when each player minimizes her cost, the value of potential function also decreases

$$\Delta c_i = \Delta \phi$$

- Each player keeps switching strategy in response to other players' strategies, until no one can improve one's cost any more (i.e., reaching a Nash equilibrium)
- Therefore, the strategy profile $\vec{P}^*$ that is a minimum of potential function $\phi(\vec{P}^*)$ is a also Nash equilibrium

Network Sharing Game: Price of Stability

Theorem

The price of stability of Network Sharing Game is $\mathsf{O}\big(\log(n)\big)$

Proof:

- Note that

$$C(\vec{P}) = \sum_{e \in \mathcal{E}(\vec{P})} C_e \le \sum_{e \in \mathcal{E}(\vec{P})} \sum_{k=1}^{n_e(\vec{P})} \frac{C_e}{k} = \phi(\vec{P}) \le \sum_{e \in \mathcal{E}(\vec{P})} C_e \cdot \sum_{k=1}^{n} \frac{1}{k}$$

- Hence,

$$C(\vec{P}) \le \phi(\vec{P}) \le \mathsf{O}\big(\log(n)\big) \cdot C(\vec{P})$$

- Let Opt be the social optimal strategy profile and $\vec{P}^*$ be a Nash equilibrium

$$C(\vec{P}^*) \le \phi(\vec{P}^*) \le \phi(\mathsf{Opt}) \le \mathsf{O}\big(\log(n)\big) \cdot C(\mathsf{Opt})$$

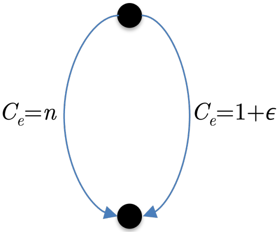
Network Sharing Game: Price of Anarchy

Theorem

The price of anarchy of Network Sharing Game is at most n

Proof:

- For example, consider a two-edge network with costs $(n, 1+\epsilon)$
  - All taking the n-edge: social cost $= n$
  - All taking the $(1+\epsilon)$-edge: social cost $= 1+\epsilon$ (social optimal)
- Suppose the social optimum is $(P_1, P_2, ..., P_n)$, in which the cost to player i is c_i
- Suppose a Nash equilibrium has cost $\hat{c}_i$ to player i
- Let c_i' be player i's cost if he switches to P_i from Nash equilibrium
- Note that $c_i' \geq \hat{c}_i$ (since $\hat{c}_i$ is Nash equilibrium) and $c_i' \leq nc_i$ (cannot be worse than the social optimum without sharing with anyone else)



Congestion Game: Price of Stability

Theorem

Consider linear cost function: $c_e(n_e) = a_e \cdot n_e + b_e$. The price of stability of Congestion Game is at most 2

Proof:

- Define potential function:

$$\phi(\vec{P}) \triangleq \sum_e \sum_{k=1}^{n_e(\vec{P})} c_e(k)$$

- Because of linear cost function: $c_e(n_e) = a_e \cdot n_e + b_e$

$$\sum_{k=1}^{n_e} c_e(k) = a_e \cdot \Big(\sum_{k=1}^{n_e} k \Big) + b_e \cdot n_e = n_e \Big(\frac{a_e \cdot (n_e + 1)}{2} + b_e \Big)$$

$$\geq \frac{n_e(a_e \cdot n_e + b_e)}{2} = \frac{n_e \cdot c_e(n_e)}{2}$$

Congestion Game: Price of Stability

Proof (Cont.):

- Hence,

$$\phi(\vec{P}) \geq \sum_{e} \frac{n_e \cdot c_e(n_e)}{2} = \frac{C(\vec{P})}{2}$$

- Note that

$$\phi(\vec{P}) = \sum_{e} \sum_{k=1}^{n_e(\vec{P})} c_e(k) \leq \sum_{e} n_e \cdot c_e(n_e) = C(\vec{P})$$

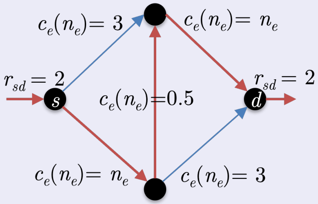
- Similar to Network Sharing Game, when player reduces cost, the potential function decreases: $\Delta c_i = \Delta \phi$
- Let Opt be the social optimal strategy profile

$$\frac{C(\vec{P}^*)}{2} \leq \phi(\vec{P}^*) \leq \phi(\mathsf{Opt}) \leq C(\mathsf{Opt})$$

Continuous Congestion Game

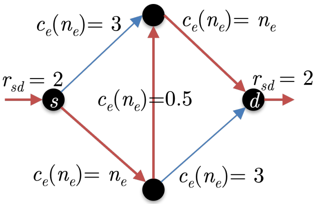
Definition (Continuous Congestion Game)

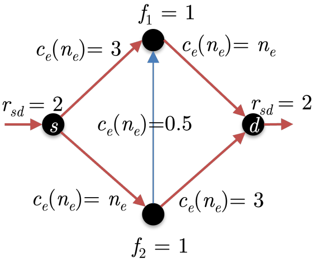
- Continuous congestion games are the limiting case, when the number of players $n \to \infty$
  - Each player is infinitesimally small; one player's change of path has negligible impact on others
- There are n types of players, where each type i is associated with a number r_i, representing the rate of traffic from one source to one destination
- Consider a network G; each type picks a set of paths $\{P\}$ from the source to the destination
- r_i is distributed fractionally over $\{P\}$
- An equilibrium is attained, when all paths for each type have the equal cost



Continuous Congestion Game

- Define a *flow*, f, in network as how r_i is distributed fractionally over $\{P\}$
- Let $C(f)$ be the social cost of flow f
- Let f^* be a Nash equilibrium flow in network G, and Opt be the social optima flow that minimizes $C(\text{Opt})$





Theorem

Consider linear cost function: $c_e(x_e) = a_e \cdot x_e + b_e$. The price of anarchy of Continuous Congestion Game is at most $\frac{4}{3}$, namely, $C(f^*) \leq \frac{4}{3} \cdot C(\text{Opt})$

Congestion Game

Theorem (Bounding Braess Paradox)

- *Consider a network G and an expanded network H with additional roads added to G*
- *Let $C_G(\cdot)$ and $C_H(\cdot)$ be the social costs with respect to G and H*
- *Let f^*_G and f^*_H be Nash equilibrium flows in G and H*
- *If the price of anarchy in H is at most α, then $C_H(f^*_H) \leq \alpha \cdot C_G(f^*_G)$ for any f^*_H*

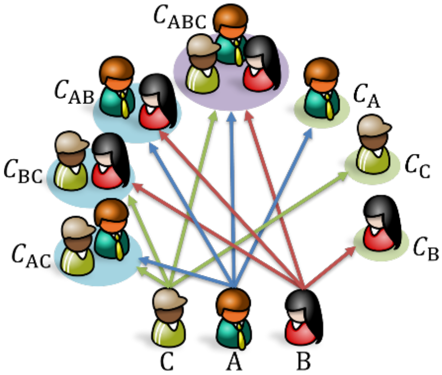
Proof:

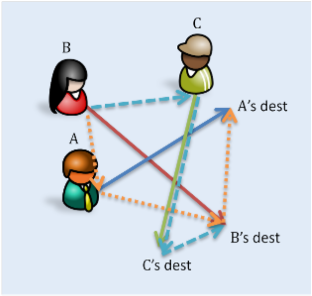
- Since network H has more roads than network G, we obtain

$$\begin{aligned} C_H(f^*_H) &\leq \alpha \cdot C_H(\mathsf{Opt}_H) \\ &\leq \alpha \cdot C_H(\mathsf{Opt}_G) = \alpha \cdot C_G(\mathsf{Opt}_G) \leq \alpha \cdot C_G(f^*_G) \end{aligned}$$

- This can bound the effect of Braess paradox, namely, Braess paradox cannot be worse than the price of anarchy

Coalition Formation





Ride-sharing

- Coalition formation by self-interested players to share cost
  - E.g., in ride-sharing, users form coalition to share a ride
- How coalition can be formed? How the cost can be divided among players?

Coalition Game

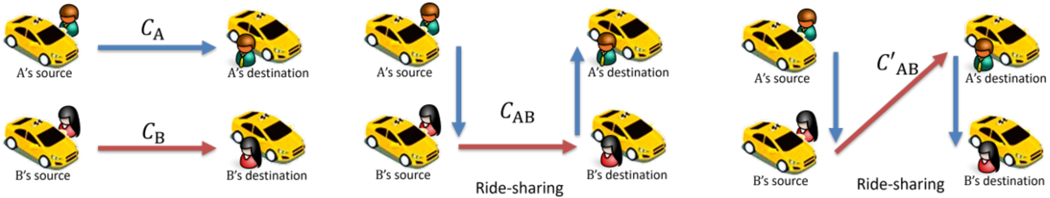
Definition (Coalition Game)

- Players can make collective change in strategies, rather than unilateral change
- A set of n self-interested players $\mathcal{N}$, who want to form a coalition $G \subseteq \mathcal{N}$
- Each coalition G has cost $C(G)$; all users in G will share $C(G)$
- **Monotonicity:** $C(G) \geq C(H)$, if $H \subseteq G$ (i.e. larger coalition incurs larger cost)
- Users split the cost according a cost-sharing scheme
  - Each player i pays $p_i(G)$ subject to $\sum_{i \in G} p_i(G) = C(G)$
- Let $C_i = C(\{i\})$ be the standalone cost, when i does not form a coalition with anyone
- Let the value of player i when joining coalition G: $u_i(G) \triangleq C_i - p_i(G)$

Coalition Game: Example

Example (Ride Sharing)

- Players form a coalition to share a ride
- C_i is the cost of a ride from i's source and i's destination
- $C(G)$ is the (minimum) cost of a shared ride among a coalition G



Possible Cost-sharing Schemes

1. **Equal-split Cost-Sharing**: The cost is split equally among all players:

$$p_i^{\mathrm{eq}}(G) \triangleq \frac{C(G)}{|G|}$$

   Namely, $u_i^{\mathrm{eq}}(G) = C_i - \frac{C(G)}{|G|}$

2. **Proportional-split Cost-Sharing**: The cost is split proportionally according to the players' standalone costs:

$$p_i^{\mathrm{pp}}(G) \triangleq \frac{C_i \cdot C(G)}{\sum_{j \in G} C_j}$$

   Namely, $u_i^{\mathrm{pp}}(G) = C_i \cdot \frac{(\sum_{j \in G} C_j) - C(G)}{\sum_{j \in G} C_j}$

Possible Cost-sharing Schemes

1. **Bargaining-based Cost-Sharing**:
   1. **Egalitarian-split Cost-Sharing** is given by:

   $$p_i^{\rm ega}(G) \triangleq C_i - \frac{(\sum_{j\in G} C_j) - C(G)}{|G|}$$

   Namely, all players $i \in G$ will receive the same utility as $u_i^{\rm ega}(G) = \frac{(\sum_{j\in G} C_j) - C(G)}{|G|}$.
   2. **Nash Bargaining Solution** is given by:

   $$\big(p_i^{\rm ns}(G)\big)_{i\in G} \in \arg\max_{(\hat{p}_i)_{i\in G}} \prod_{i\in G} u_i(\hat{p})$$

   subject to

   $$\sum_{i\in G} \hat{p}_i = C(G)$$

   Egalitarian-split cost-sharing and Nash bargaining solution are equivalent in cost-sharing

Possible Cost-sharing Schemes

	Alone	Equal-split (eq)	Proportional-split (pp)	Egalitarian-split (ega)
 A	C_A	$p_A = \frac{C_{AB}}{2}$	$p_A = \frac{C_A C_{AB}}{C_A + C_B}$	$p_A = \frac{C_{AB} + C_A - C_B}{2}$
 B	C_B	$p_B = \frac{C_{A,B}}{2}$	$p_B = \frac{C_B C_{AB}}{C_A + C_B}$	$p_B = \frac{C_{AB} + C_B - C_A}{2}$

Stable Coalition

Definition (Stable Coalition)

- Coalition structure $\mathcal{S} = (G_1, ..., G_m)$ is a collection of coalitions, such that $G_i \cap G_j = \varnothing$
- Let us fix a cost-sharing scheme $p_i(\cdot)$
- Given a coalition structure $\mathcal{P}$, a coalition of players G is called a *blocking coalition* with respect to $\mathcal{P}$, if $G \notin \mathcal{P}$, and all players in G can *strictly* decrease their costs when they form a coalition G instead of any coalition G' in $\mathcal{P}$:

$$p_i(G) < p_i(G') \text{ for all } i \in G, G' \in \mathcal{P} \text{ where } i \in G'$$

- A coalition structure $\mathcal{P}$ is called a *stable coalition structure*, if there exists no blocking coalition with respect to $\mathcal{P}$
- A stable coalition structure is also called a **strong** Nash equilibrium
  - Strong Nash equilibrium is stable despite of a collective strategy change of a group of players
  - Weak Nash equilibrium is stable despite of a unilateral strategy change of a player

Coalition Formation

Theorem

There exists a stable coalition structure in equal-split, proportional-split or egalitarian-split cost-sharing schemes

Proof:

- Define a cyclic preference as:

$$p_{i_1}(G_1) < p_{i_1}(G_2),\ p_{i_2}(G_2) < p_{i_2}(G_3),\ \ldots,\ p_{i_s}(G_s) < p_{i_s}(G_1)$$

  where $i_k \in G_k \cap G_{k+1}$ for all $k \leq s-1$, and $i_s \in G_s \cap G_1$

- If there exists no cyclic preference in a given cost-sharing scheme $p_i(\cdot)$, then there always exists a stable coalition structure

Coalition Formation

Proof (Cont.):

- We consider equal-split cost-sharing $p_i^{\text{eq}}(\cdot)$
- If there exists a cyclic preference, then

$$
\begin{aligned}
p_{i_1}^{\text{eq}}(G_1) = \frac{C(G_1)}{|G_1|} \quad &< \quad \frac{C(G_2)}{|G_2|} = p_{i_1}^{\text{eq}}(G_2), \\
p_{i_2}^{\text{eq}}(G_2) = \frac{C(G_2)}{|G_2|} \quad &< \quad \frac{C(G_3)}{|G_3|} = p_{i_2}^{\text{eq}}(G_3), \\
&\vdots \\
p_{i_s}^{\text{eq}}(G_s) = \frac{C(G_s)}{|G_s|} \quad &< \quad \frac{C(G_1)}{|G_1|} = p_{i_s}^{\text{eq}}(G_1)
\end{aligned}
$$

- Summing the above equations, one obtains a contradiction $0 < 0$
- Similarly, it can be proved for proportional-split or egalitarian-split cost-sharing schemes

Coalition Formation

- Define the *strong* price of anarchy as the ratio of the worst strong Nash equilibrium over the social optimal
- The social optimal is the coalition structure that minimizes the total cost of all players

Theorem

If the size of each coalition is at most 2 (i.e., at most a pair of players can form a coalition), then the strong price of anarchy for equal-split, proportional-split or egalitarian-split cost-sharing schemes is $\leq \frac{3}{2}$

Proof:

- We consider equal-split cost-sharing $p_i^{\rm eq}(\cdot)$
- Let $\hat{\mathcal{S}}^{\rm eq}$ be a stable coalition structure and Opt be the social optimal coalition structure
- Suppose $\{i, j\} \in \hat{\mathcal{S}}^{\rm eq} \backslash$Opt. Then there must exist (k, l), such that $\{i, k\}, \{j, l\} \in$ Opt$\backslash \hat{\mathcal{S}}^{\rm eq}$
- Assume $i \neq k$ and $j \neq l$ (otherwise, $i = k$ or $j = l$ can be proven easily)

Proof (Cont.):

- Since $\{i,k\},\{j,l\} \in$ Opt, we obtain

$$p_i^{\rm eq}(\mathsf{Opt}) + p_j^{\rm eq}(\mathsf{Opt}) + p_k^{\rm eq}(\mathsf{Opt}) + p_l^{\rm eq}(\mathsf{Opt}) = C(\{i,k\}) + C(\{j,l\})$$

- Since $\hat{\mathcal{S}}^{\rm eq}$ is a stable coalition structure, we obtain

$$p_i^{\rm eq}(\hat{\mathcal{S}}^{\rm eq}) + p_j^{\rm eq}(\hat{\mathcal{S}}^{\rm eq}) = C(\{i,j\}),\ p_k^{\rm eq}(\hat{\mathcal{S}}^{\rm eq}) \le C_k,\ p_l^{\rm eq}(\hat{\mathcal{S}}^{\rm eq}) \le C_l$$

- Because $\hat{\mathcal{S}}^{\rm eq}$ is a stable coalition structure and $i \ne k$ and $j \ne l$, we obtain

$$\tfrac{C(\{i,j\})}{2} = p_i^{\rm eq}(\hat{\mathcal{S}}^{\rm eq}) \le p_i^{\rm eq}(\mathsf{Opt}) = \tfrac{C(\{i,k\})}{2},\ \ \tfrac{C(\{i,j\})}{2} = p_j^{\rm eq}(\hat{\mathcal{S}}^{\rm eq}) \le p_j^{\rm eq}(\mathsf{Opt}) = \tfrac{C(\{j,l\})}{2}$$

- Note that $C(\{i,k\}) \ge \max\{C_i, C_k\}$ and $C(\{j,l\}) \ge \max\{C_j, C_l\}$ by monotonicity

$$3\big(p_i^{\rm eq}(\mathsf{Opt}) + p_j^{\rm eq}(\mathsf{Opt}) + p_k^{\rm eq}(\mathsf{Opt}) + p_l^{\rm eq}(\mathsf{Opt})\big) = 3\big(C(\{i,k\}) + C(\{j,l\})\big)$$
$$\ge C(\{i,k\}) + C(\{j,l\}) + C_k + C_l + C_k + C_l \ge 2\big(C(\{i,j\}) + C_k + C_l\big)$$
$$\ge 2\big(p_i^{\rm eq}(\hat{\mathcal{S}}^{\rm eq}) + p_j^{\rm eq}(\hat{\mathcal{S}}^{\rm eq}) + p_k^{\rm eq}(\hat{\mathcal{S}}^{\rm eq}) + p_l^{\rm eq}(\hat{\mathcal{S}}^{\rm eq})\big)$$

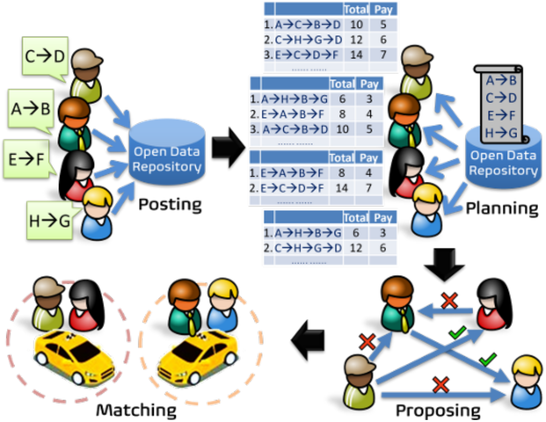
Coalition Formation

Theorem

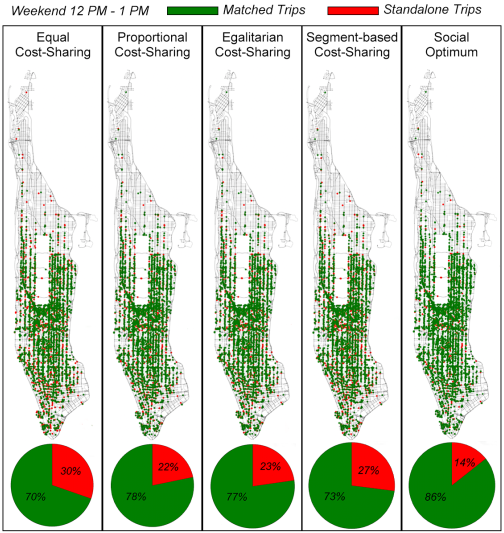
If the size of each coalition is at most K (i.e., at most K players can form a coalition), then the strong price of anarchy for equal-split, proportional-split or egalitarian-split cost-sharing schemes is $\mathsf{O}(\log K)$

- Implications: Coalition formation can be performed in a decentralized manner, without a centralized manager
- Each player proposes coalitions to each other, and accept the coalitions that minimize the costs by individual players
- Players will always reach a stable coalition structure
- The strong price of anarchy for stable coalition formation is only $\mathsf{O}(\log K)$, which is small compared to the social optimal (that may not be stable with respect to collective changes in strategy)

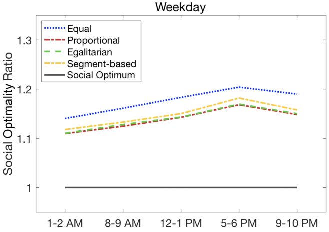
Case Study: Ride-sharing

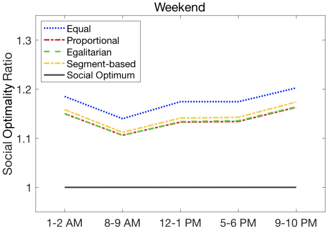


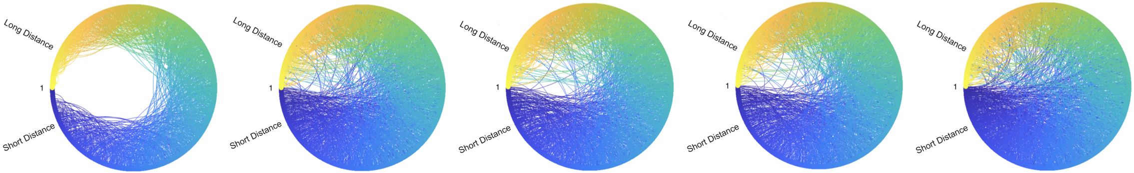
Empirical Studies



Empirical Studies







(a) Equal (b) Proportional (c) Egalitarian (d) Segment-based (e) Social optimal

Figure: Stable matching structures based on different cost-sharing schemes

References

Reference Materials

- Algorithmic Game Theory (Edited by Nisan, Roughgarden, Tardos, Vazirani)
  - Chapters 17-18

Recommended Materials

- "*Decentralized Ride-Sharing and Vehicle-Pooling Based on Fair Cost-Sharing Mechanisms*", (Chau, Shen, Zhou), IEEE Trans. on Intelligent Transportation Systems, 2020. `http://arxiv.org/abs/2007.08064`
- "*Quantifying Inefficiency of Fair Cost-Sharing Mechanisms for Sharing Economy*", (Chau, Elbassioni), IEEE Trans. on Control of Network Systems, 2018. `https://arxiv.org/abs/1511.05270`