Combinatorial Optimization of Electric Vehicle Charging in AC Power Distribution Networks

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***Abstract*—This paper studies the scheduling optimization problem of electric vehicle (EV) charging considering two salient characteristics: (1) discrete charging rates with minimum power requirements in common EV charging standards, and (2) nodal voltage and line capacity constraints of alternating current (AC) power flows in electricity distribution networks. We present approximation algorithms to solve scheduling optimization problem of EV charging, which have a provably small parameterized approximation ratio. Simulations show our algorithms can produce close-to-optimal solutions in practice.**

I. Introduction

To satisfy the increasing number of electric vehicles, electricity distribution networks need to address the growing power loads. However, recent studies show that most distribution networks have legacy transformers and transmissions lines that are unable to handle the loads of electric vehicle charging. Therefore, the notion of "smart" charging has been introduced to optimize the charging schedules and dispatching operations of EVs considering the dynamic generation, transmission and distribution capacities of electricity grid. Although there has been extant research work about the scheduling optimization of EV charging, there are several practical limitations in the extent solutions. Thus, this paper provides a more comprehensive solution by considering more holistic aspects.

The intermediary between a power source and the EV's charging port is called Electric Vehicle Service Equipment (EVSE). Currently, there are three main categories of EVSE: *Level 1* charging with cord-set singe-phase connections to a regular household outlet of 115V AC and 15A in America (also, 230V AC and 6A in Europe), which requires a power demand around 1.5kW. *Level 2* wall-mount three-phase connections with 230V AC and 30A two pole, which requires a power demand around 7kW. *Level 3* DC fast charger with 400-600V DC and up to 300A, which bypasses the on-board EV charger and converting the power directly into the battery. Level 3 chargers require up to 150kW.

It is worth noting that none of these current popular charging standards allows continuously controllable charging power at an arbitrary rate. To ensure reliable charging, there requires a delicate control system for the supplied charging power. As a consequence, the injected voltage and current to charge EV battery packs ought to be compliant with certain standard charging rates. Hence, the charging power normally varies within a limited discrete set of nearly constant values [1]. The extant studies assuming arbitrarily time-varying charging rate are inapplicable to the current EV charging systems.

Therefore, this paper studies a more realistic setting of EV charging systems, requiring a *fixed* charging rate in a discrete manner. An EV consumes a fixed power when it is being charged. Essentially, an EV only selects the starting time of charging, after which it has to be charged at a fixed rate until its scheduled ending time. However, we can also relax the restriction of a constant charging rate by a variable rate with a minimum power requirement. Note that there are software-controlled EV chargers (e.g., [2]) that allow possible programmable switching among discrete charging modes if different charging ports are plugged in.

A major factor affecting EV charging is the presence of grid-wide characteristics. There are various power system attributes that are important metrics for the reliability of the electricity grid. Deviations from the prescribed ranges of these attributes will be an indicator for an imminent serious issue in electricity grid, if no immediate corrective action is taken. It has been a primary concern for power system engineers to ensure the electricity grid within its stable operating limits, in particular, when facing large loads due to EV charging. The optimization of electricity grid operations subject to a variety of operating limits, such as power capacity, current thermal, and voltage constraints, is formulated as the alternating current (AC) optimal power flow (OPF) problem. OPF is hard to solve, because of non-convex operating constraints of power systems [3], not to mention the presence of discrete control variables (e.g., discrete charging rates).

This paper studies the scheduling optimization of EV charging considering: (1) discrete charging rates with minimum power requirements in common EV charging standards, and (2) nodal voltage and line capacity constraints of AC power flows in distribution networks. We present *approximation algorithms* for scheduling optimization of EV charging, with a provably small parameterized *approximation ratio*.

II. Problem Definition and Notations

A. EV Charging

We consider a constant number of discrete timeslots, denoted by the set $\mathcal{T}$, with a fixed interval ΔT. There are a set of EVs, denoted by $\mathcal{A}$. Each $a \in \mathcal{A}$ has its charging requirement characterized by $(\mathcal{T}^a, \mathsf{loc}^a, \mathsf{usg}^a)$, where $\mathcal{T}^a \subseteq \mathcal{T}$ is a subset of timeslots that a is allowed to be recharged at charging station loc^a, requiring a total amount of energy usg^a to charge the EV battery for later usage.

There are a set of charging options available to each charging station loc^a, denoted by $\mathcal{C}^a$. For each $c \in \mathcal{C}^a$, the charging option c is characterized by a tuple $\big(\mathsf{rate}_c, \mathsf{cost}_c(t)\big)$, where rate_c is the charging rate and $\mathsf{cost}_c(t)$ is the time-varying charging cost per unit time. Note that rate_c draws electricity power for Alternating Current (AC) electric power grid, and hence, is represented by a complex number in the

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standard power systems literature. $\mathsf{cost}_c(t)$ may depend on the availability of renewable energy generation.

We consider software-controlled EV chargers (e.g., [2]) such that it can be programmed to switch among charging modes. Define binary decision variable $x^a_c(t) \in \{0,1\}$, such that $x^a_c(t) = 1$, if a recharges at time $t \in \mathcal{T}^a$ using charging option $c \in \mathcal{C}^a$, otherwise $x^a_c(t) = 0$. Denote $x^a \triangleq \big(x^a_c(t)\big)_{c\in\mathcal{C},t\in\mathcal{T}^a}$. Define binary variable $y^a \in \{0,1\}$ to indicate whether a's charging requirement is satisfied:

$$y^a \triangleq \begin{cases} 1, \text{ if } \mathsf{usg}^a \le \sum_{t\in\mathcal{T}^a}\sum_{c\in\mathcal{C}^a} \Delta T \cdot \mathsf{rate}_c \cdot x^a_c(t) \\ 0, \text{ otherwise.} \end{cases} \quad (1)$$

Define the *gain* for a with respect to charging decision x^a by

$$\mathsf{G}^a(x^a, y^a) \triangleq \mathsf{u}^a y^a - \sum_{t\in\mathcal{T}^a}\sum_{c\in\mathcal{C}^a} \mathsf{cost}_c(t) x^a_c(t), \quad (2)$$

where u^a is the utility if a's charging requirement is satisfied. It is natural to assume that u^a is sufficiently large so that $G^a(x^a, y^a)$ is non-negative, for all $x^a_c(t), y^a \in [0,1]$.

B. AC Power Flow

As in the previous work [4], [5], this paper considers a radial (tree) electric distribution network, represented by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The set of nodes $\mathcal{V} = \{0, ..., m\}$ denotes the electric buses, and the set of edges $\mathcal{E}$ denotes the distribution lines. For edge $(i,j) \in \mathcal{E}$, denote its impedance by $z_{i,j}$. Let $\mathcal{V}^+ \triangleq \mathcal{V}\setminus\{0\}$. A substation feeder is attached to the root of the tree, denoted by node 0. Assume root 0 is only connected to node 1 via a single edge (0,1). Since $\mathcal{G}$ is a tree, $|\mathcal{V}^+| = |\mathcal{E}| = m$. Let $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$ be the subtree rooted at node i. We denote the unique path from node j to the root 0 by $\mathcal{P}_j$. Suppose the EV charging stations are connected to the set of nodes $\mathcal{V}$, namely, $\mathsf{loc}^a \in \mathcal{V}$. Let $\mathcal{U}_j \triangleq \{a \in \mathcal{A} : \ \mathsf{loc}^a = j\}$ and $\mathcal{A}_j \triangleq \bigcup_{i\in\mathcal{V}_j} \mathcal{U}_i$. For $a \in \mathcal{A}$, let $\mathcal{P}_a$ denote th path from the root to node $j \in \mathcal{V}^+$ such that $a \in \mathcal{U}_j$.

For a given time t, let $v_j(t)$ and $\ell_{i,j}(t)$ be the voltage and current magnitude square at node j and edge (i,j), respectively. Let $S_{i,j}(t)$ be the power flowing from node i towards node j. Note that $S_{i,j}(t)$ is not symmetric, namely, $S_{i,j}(t) \ne S_{j,i}(t)$. At each node j, there is a net power demand, denoted by $s_j(t)$. The power supply at root 0 is denoted by s_0. For a complex number $\nu \in \mathbb{C}$, we denote the *magnitude* of ν by $|\nu|$, the *phase angle* (or argument) that ν makes with the real axis by $\angle\nu$, and the complex *conjugate* of ν by ν^*.

The AC power flows in the electricity grid can be described by the branch flow model (BFM) [1] representation:

$$\ell_{i,j}(t) = \frac{|S_{i,j}(t)|^2}{v_i(t)}, \forall (i,j) \in \mathcal{E}, \quad (3)$$

$$S_{i,j}(t) = s_j(t) + \sum_{l:(j,l)\in\mathcal{E}} S_{j,l}(t) + z_{i,j}\ell_{i,j}(t), \forall (i,j) \in \mathcal{E}, \quad (4)$$

$$S_{0,1}(t) = -s_0(t), \quad (5)$$

$$v_j(t) = v_i(t) + |z_{i,j}|^2 \ell_{i,j}(t) \text{-} 2\mathrm{Re}(z^*_{i,j} S_{i,j}(t)), \forall (i,j) \in \mathcal{E}, \quad (6)$$

[1]To be precise, this model is called branch flow model with angle relaxation [3], as it omits the phase angles of voltages and currents. But it is always possible to recover the phase angles in a radial network. This paper adopts the convention to denote a power supply by a complex number with negative real part and a power demand by a complex number with positive real part (i.e., $\mathrm{Re}(s_0(t)) \le 0$ and $\mathrm{Re}(s_j(t)) \ge 0$ for all $j \in \mathcal{V}$).

For each $j \in \mathcal{V}$, let $d_j(t) \in \mathbb{C}$ be the background power demand from other residential and commercial loads. We assume $\mathrm{Re}(d_j(t)) \ge 0$ (but $\mathrm{Im}(d_j(t))$ may be negative) for all $j \in \mathcal{V}$.

Given the charging decisions $(x^a)_{a\in\mathcal{A}}$, the total power demand at each node j is given by

$$s_j(t) = d_j(t) + \sum_{a\in\mathcal{U}_j : t\in\mathcal{T}^a}\sum_{c\in\mathcal{C}^a} \mathsf{rate}_c x^a_c(t)$$

The operating constraints of power systems are represented by

$$(\textit{Voltage Constraints}): \underline{v}_j \le v_j(t) \le \overline{v}_j, \forall j \in \mathcal{V}^+, \quad (7)$$

$$(\textit{Capacity Constraints}): |S_{i,j}(t)| \le \overline{S}_{i,j}, \forall (i,j) \in \mathcal{E}, \quad (8)$$

$$(\textit{Current Thermal Constraints}): \ell_{i,j}(t) \le \overline{\ell}_{i,j}, \forall (i,j) \in \mathcal{E}, \quad (9)$$

$\underline{v}_j, \overline{v}_j \in \mathbb{R}^+$ denote the minimum and maximum allowable voltage magnitude square at j, and $\overline{S}_{i,j}, \overline{\ell}_{i,j} \in \mathbb{R}^+$ denote the maximum allowable apparent power and current on (i,j). In the following, a subscript is omitted from a variable to denote a vector.

C. Scheduling Optimization Problem

The goal of scheduling of EV charging is to find an assignment for the charging decision vectors x, y and supply s_0 that maximizes a non-negative concave utility function f:

$$f(s_0, x, y) \triangleq \sum_{t=1}^{T} f_0\big(\mathrm{Re}(-s_0(t))\big) + \sum_{a\in\mathcal{A}} \mathsf{G}^a(x^a, y^a), \quad (10)$$

where f_0 is non-negative and non-increasing utility function of the total active power supply $\mathrm{Re}(-s_0(t))$[2].

Integrating the aforementioned constraints and objectives, the EV Charging Scheduling Problem is given by the mixed integer programming problem (EVSP).

$$(\text{EVSP}) \quad \max_{s_0, s, S, v, \ell, x, y} f(s_0, x, y)$$

$$\text{subject to} \quad \forall t \in \mathcal{T}, (3) - (9)$$

$$s_j(t) = d_j(t) + \sum_{a\in\mathcal{U}_j : t\in\mathcal{T}^a}\sum_{c\in\mathcal{C}^a} \mathsf{rate}_c \cdot x^a_c(t), \forall j \in \mathcal{V}^+, \quad (11)$$

$$y^a \mathsf{usg}^a \le \sum_{t\in\mathcal{T}^a}\sum_{c\in\mathcal{C}^a} \Delta T \cdot \mathsf{rate}_c \cdot x^a_c(t), \forall a \in \mathcal{A}, \quad (12)$$

$$\sum_{c\in\mathcal{C}^a} x^a_c(t) \le 1, \forall t \in \mathcal{T}^a, \forall a \in \mathcal{A}, \quad (13)$$

$$v_j(t) \in \mathbb{R}^+, j \in \mathcal{V}^+, \ell_{i,j}(t) \in \mathbb{R}^+, \ S_{i,j}(t) \in \mathbb{C}, \forall (i,j), \quad (14)$$

$$x^a_c(t) \in \{0,1\}, \forall c \in \mathcal{C}^a, t \in \mathcal{T}^a, y^a \in \{0,1\}, \forall a \in \mathcal{A}. \quad (15)$$

D. Approximation Solutions

This paper provides an efficient approximation algorithm to solve EVSP. We define some standard terminology for approximation algorithms. Consider a maximization problem $\mathcal{A}$ with non-negative objective function $f(\cdot)$, let F be a feasible solution to $\mathcal{A}$ and $F^\star$ be an optimal solution to $\mathcal{A}$. $f(F)$ denotes the objective value of F. Let $\textsc{Opt} = f(F^\star)$ be the optimal objective value of $F^\star$. A common definition of approximation solution is α-approximation, where α characterizes the approximation ratio between the approximation solution and an optimal solution.

[2]Note that the negative sign is to indicate s_0 is supply rather than demand. Thus, the less generated power, the higher the utility, which follows the convention of power system engineering.

Definition 1. *For $\alpha \in (0,1)$, an α-approximation to maximization problem $\mathcal{A}$ is a feasible solution F such that $f(F) \geq \alpha \cdot$ OPT.*

In particular, *a polynomial-time approximation scheme* (PTAS) is a $(1-\epsilon)$-approximation algorithm to a maximization problem, for any $\epsilon > 0$. The running time of a PTAS is polynomial in the input size for every fixed ϵ, but the exponent of the polynomial might depend on $1/\epsilon$. Namely, a PTAS allows a parametrized approximation ratio as the running time.

E. Alternate Reformulation

To derive PTAS, we need to show that the convex relaxation has an (near) optimal solution with the property that the number of fractional components is small. Unfortunately, that is not the case for the relaxation of (EVSP). To resolve this issue, we consider an alternative formulation below.

For each EV $a \in \mathcal{A}$, let Γ^a be a set of feasible choices for a. Each choice $\gamma^a \in \Gamma^a$ is described by an assignment of all the variables $x_c^a(t)$, satisfying (13) and (15):

$$\Gamma^a \triangleq \Big\{ \big(x_c^a(t) \in \{0,1\}\big)_{c\in\mathcal{C}^a, t\in T^a} : \sum_{c\in\mathcal{C}^a} x_c^a(t) \leq 1,\ \forall t \in \mathcal{T}^a \Big\}$$

The number of choices for each a is $|\Gamma^a| \leq (|C^a|+1)^{|T^a|}$, which is polynomial for constant $|T^a|$. For each choice $\gamma^a \in \Gamma^a$, we can define a gain function $\mathsf{G}^a(\gamma^a)$ by (1) and (2) and a demand function $d^a(\gamma^a, t) \triangleq \sum_{c\in C^a} \mathsf{rate}_c x_c^a(t)$, for $t \in T^a$. Define also a binary variable $X^a(\gamma^a)$ that takes value 1 if and only if choice γ^a is selected for EV a. Let

$$f(s_0, X) \triangleq \sum_{t=1}^{T} f_0\big(\mathrm{Re}(-s_0(t))\big) + \sum_{a\in\mathcal{A}} \sum_{\gamma^a\in\Gamma^a} \mathsf{G}^a(\gamma^a) X^a(\gamma^a),$$

Then we can rewrite (EVSP) as follows.

$$\text{(A-EVSP)} \quad \max_{s_0,s,S,v,\ell,X} f(s_0, X)$$

$$\text{subject to} \quad \forall t \in \mathcal{T}, (3)-(9),\ (14),$$

$$s_j(t) = d_j(t) + \sum_{a\in\mathcal{U}_j} \sum_{\gamma^a\in\Gamma^a} d^a(\gamma^a, t) X^a(\gamma^a), \forall j \in \mathcal{V}^+, \quad (16)$$

$$\sum_{\gamma^a\in\Gamma^a} X^a(\gamma^a) \leq 1, \forall a \in \mathcal{A}, \quad (17)$$

$$X^a(\gamma^a) \in \{0,1\},\ \forall \gamma^a \in \Gamma^a, \forall a \in \mathcal{A}. \quad (18)$$

Remark 1. *In the above formulation, it is (implicitly) assumed that the assignment $X^a(\gamma^a) = 0$, for all $\gamma^a \in \Gamma^a$, corresponds to setting $x_c^a(t) = 0$ for all $c \in \mathcal{C}^a$ and all $t \in \mathcal{T}^a$.*

F. Assumptions

We make some assumptions to facilitate our algorithms:

- A0: $f_0(-s_0^{\mathrm{R}})$ is non-increasing in $-s_0^{\mathrm{R}} \in \mathbb{R}^+$.
- A1: $z_e \geq 0, \forall e \in \mathcal{E}$, which naturally hold in distribution networks.
- A2: $v_0(t) < \overline{v}_j, \forall j \in \mathcal{V}^+,\ t \in \mathcal{T}$, which is also assumed in [4]. Typically in a distribution network, $v_0(t) = 1$ (per unit), $\underline{v}_j = (.95)^2$ and $\overline{v}_j = (1.05)^2$; in other words, 5% deviation from the nominal voltage is allowed.
- A3: $\mathrm{Re}(z_e^* d_j(t)) \geq 0, \forall j \in \mathcal{V}^+,\ t \in \mathcal{T}, e \in \mathcal{E}$. Intuitively, A3 requires that the phase angle difference between any z_e and $d_j(t)$ is at most $\frac{\pi}{2}$. This assumption holds, if the background demands have small negative reactive power.
- A4: $\big|\angle d_j(t) - \angle d_{j'}(t)\big| \leq \frac{\pi}{2}$ for any $j, j' \in \mathcal{V}^+,\ t \in \mathcal{T}$. Intuitively, A4 requires that the background demands have "similar" power factors. A4 can also be stated as $\mathrm{Re}(d_j^*(t) d_{j'}(t')) \geq 0$.

Assumptions A3 and A4 are motivated, from a theoretical point of view, by the inapproximability results in [6] (if either assumption does not hold, then the problem cannot be approximated within any polynomial factor unless P=NP; see [6] for details). Assumption A3 also holds in reasonable practical settings [4]. In the next subsection, by performing an axis rotation, we may assume by A4 that $d_j(t) \geq 0$. Clearly, under this and assumption A1, the reverse power constraint in (8) is implied by the forward power constraint ($|S_e(t)| \leq \overline{S}_e$). It will also be seen that under assumptions A1, A2 and A3, the voltage upper bounds in (7) can be dropped.

III. Preliminaries of OPF

In this section, we present key preliminary results needed for our algorithm presented in Sec. IV. For a more detailed treatment, we refer the reader to [7].

A. Rotational Invariance of EVSP

We note that if we rotate all complex quantities in the EVSP problem (namely, $z_e, d_k(t)$) by a fixed angle ϕ, then the problem structure remains the same. Therefore, we chose ϕ such that, after the rotation, all complex quantities have positive real and imaginary components (i.e., lie in the first quadrant). This property is necessary in the analysis of our algorithm in Sec. IV. The rotation allows us to replace assumptions A0 and A4 by the following assumptions:

- A0′: $f_0(-s_0^{\mathrm{R}} \cos\phi - s_0^{\mathrm{I}} \sin\phi)$ is non-decreasing in $-s_0^{\mathrm{R}}, -s_0^{\mathrm{I}}$.
- A4′: $d_j(t) \geq 0$ for all $j \in \mathcal{V}^+,\ t \in \mathcal{T}$. This is because all demand sets satisfying A4 are now in the first quadrant after the rotation by ϕ.

Note that assumption A1 continues to hold, assuming the original A-EVSP problem satisfies A3: $z_e e^{\mathbf{i}\phi} \geq 0, \forall e \in \mathcal{E}$. This is because of A3, namely, $\mathrm{Re}(z_e^* d_j(t)) \geq 0, \forall j \in \mathcal{V}^+,\ t \in \mathcal{T},\ e \in \mathcal{E}$, such that the phase angle difference between z_e and $d_j(t)$ is at most $\frac{\pi}{2}$. Note also that A1 and A4′ already imply A3.

B. Convex Relaxation

A *Second Order Cone Programming* (SOCP) relaxation of A-EVSP is obtained by replacing Cons. (3) by $\ell_{i,j}(t) \geq \frac{|S_{i,j}(t)|^2}{v_i(t)}$, and replacing the discrete constraints in (18) by $X^a(\gamma^a) \in [0,1]$ for all $\gamma^a \in \Gamma^a,\ a \in \mathcal{A}$:

$$\text{(CA-EVSP)} \quad \max_{s_0,s,S,v,\ell,X} f(s_0, X)$$

$$\text{subject to}, \forall t \in \mathcal{T}, (4)-(9),\ (14), (16),\ (17)$$

$$\ell_{i,j}(t) \geq \frac{|S_{i,j}(t)|^2}{v_i(t)}, \forall (i,j) \in \mathcal{E}, \quad (19)$$

$$X^a(\gamma^a) \in [0,1],\ \forall \gamma^a \in \Gamma^a, \forall a \in \mathcal{A}. \quad (20)$$

We will denote the set of all binary assignments satisfying (17) and (20) by $\mathcal{X}$. For a given $\widehat{X} \in \mathcal{X}$, we denote by A-EVSP$[\widehat{X}]$ (resp., CA-EVSP$[\widehat{X}]$) the restriction of A-EVSP (resp., CA-EVSP) where we set $X = \widehat{X}$.

For convenience, we rewrite Cons. (6) based the recursive "unfolding" of Eqns. (4)-(6) (at each $t \in \mathcal{T}$), $v_j(t) = v_0 - 2\sum_{a\in\mathcal{A}} D_j(\gamma^a, t) - 2\sum_{h\in T_j} D_h(t) - \Big(2\sum_{(h,u)\in\mathcal{P}_j} \mathrm{Re}\big(z^*_{h,u}\sum_{e\in\mathcal{E}_u} z_e\ell_e\big) + \sum_{(h,u)\in\mathcal{P}_j} |z_{h,u}|^2\ell_{h,u}\Big)$, where $D_j(t) \triangleq \mathrm{Re}\Big(\sum_{k\in\mathcal{V}}\sum_{(h,u)\in\mathcal{P}_k\cap\mathcal{P}_j} z^*_{h,u}d_k(t)\Big)$ and $D_j(\gamma^a, t) \triangleq \mathrm{Re}\Big(\sum_{(h,u)\in\mathcal{P}_a\cap\mathcal{P}_j} z^*_{h,u}d^a(\gamma^a, t)\Big)$.

Corollary 1 ([8]). *Let $F' \triangleq \big(s'_0, s', S', v', \ell', X'\big)$ be a feasible solution to* CA-EVSP$[X']$ *and $\widehat{X} \in \mathcal{X}$ be a given vector such that, for some $\epsilon > 0$,*

$$\sum_{a\in\mathcal{A}}\sum_{\gamma^a\in\Gamma^a}\mathsf{G}^a(\gamma^a)\widehat{X}^a(\gamma^a) \geq \sum_{a\in\mathcal{A}}\sum_{\gamma^a\in\Gamma^a}\mathsf{G}^a(\gamma^a)X'^a(\gamma^a) - \epsilon f(s'_0, X'),$$

$$\sum_{a\in\mathcal{A}}\sum_{\gamma^a\in\Gamma^a}D_j(\gamma^a, t)\widehat{X}^a(\gamma^a) \leq \sum_{a\in\mathcal{A}}\sum_{\gamma^a\in\Gamma^a}D_j(\gamma^a, t)X'^a(\gamma^a), \forall j,$$

$$\sum_{a\in\mathcal{A}_j}\sum_{\gamma^a\in\Gamma^a}d^a(\gamma^a, t)\widehat{X}^a(\gamma^a) \leq \sum_{a\in\mathcal{A}_j}\sum_{\gamma^a\in\Gamma^a}d^a(\gamma^a, t)X'^a(\gamma^a), \forall j.$$

Then under assumptions A0$'$, A1, A2, A3 and A4$'$ *, we can find in polynomial time a feasible solution $\hat{F} = \big(\hat{s}_0, \hat{s}, \hat{S}, \hat{v}, \hat{\ell}, \widehat{X}\big)$ to* A-EVSP$[\widehat{X}]$ *such that $f(\hat{F}) \geq (1-\epsilon)f(F')$.*

IV. PTAS

This section presents a $(1-\epsilon)$-approximation algorithm (PTAS) for A-EVSP. Note that we consider the number of links in the distribution network (i.e., $|\mathcal{V}^+| = |\mathcal{E}| = m$) and the number of time slots $\mathcal{T}$ to be constants. We will need here that assumptions A0, A1, A2, A3 and A4 hold. We may assume after rotation with an appropriate angel ϕ that A0$'$, A1, A2, A3 and A4$'$ hold instead. As mentioned earlier, we will denote for convenience the rotated problem by A-EVSP.

After convex relaxation and rotation, we enumerate possible partial guesses for configuring the control variables of a small subset of EVs. For each guess, we solve the remaining subproblem by relaxing the other discrete control variables to be continuous control variables, and then rounding the continuous control variables to obtain a feasible solution. This algorithm can attain a parameterized approximation ratio by carefully adjusting the number of partial guesses and rounding.

A formal description of the PTAS algorithm (called PTAS-A-EVSP) is presented as follows.

1) First, define a *partial guess* by $\mathcal{A}_1 \subseteq \mathcal{A}$ and vector $\hat{\gamma} = \big(\hat{\gamma}^a\big)_{a\in\mathcal{A}_1}$. For each guess, we set $X^a(\hat{\gamma}^a) = 1$, and $X^a(\gamma^a) = 0$ for $\gamma^a \neq \hat{\gamma}^a$, for all $a \in \mathcal{A}_1$.
2) Define a variant of CA-EVSP with partially pre-configured and partially relaxed discrete control variables, denoted by P1$[\mathcal{A}_1, \hat{\gamma}]$, as follows.

$$(\text{P1}[\mathcal{A}_1, \hat{\gamma}]) \max_{s_0, s, S, v, \ell, X} f(s_0, X)$$

subject to, $\forall t \in \mathcal{T}$,

$$(4) - (9),\ (14), (16),\ (17) \tag{21}$$

$$X(\hat{\gamma}^a) = 1, X^a(\gamma^a) = 0 \text{ for } \gamma^a \neq \hat{\gamma}^a,\ \forall a \in \mathcal{A}_1, \tag{22}$$

$$X^a(\gamma^a) = 0,\ \forall \gamma^a \in \Gamma^a\backslash\widetilde{\Gamma}^a, \forall a \in \mathcal{A}', \tag{23}$$

$$X^a(\gamma^a) \in [0, 1],\ \forall \gamma^a \in \widetilde{\Gamma}^a, \forall a \in \mathcal{A}', \tag{24}$$

where

$$\widetilde{\Gamma}^a \triangleq \Gamma^a\backslash\big\{\gamma^a \in \Gamma^a :\ \mathsf{G}^a(\gamma^a) \geq \min_{a'\in\mathcal{A}_1}\{\mathsf{G}^{a'}(\hat{\gamma}^{a'})\}\big\}, \tag{25}$$

for $a \in \mathcal{A}' \triangleq \mathcal{A}\backslash\mathcal{A}_1$. Note that P1$[\mathcal{A}_1, \hat{\gamma}]$ is an SOCP (and hence is solvable in polynomial time). We then solve this relaxation to obtain an optimal solution $F' = \big(s'_0, s', S', v', \ell', X'\big)$. Note that F' may not satisfy the discrete demand constraints (18) in A-EVSP. Next, F' will be rounded to obtain a feasible solution to A-EVSP.
3) Define P2$[F', \mathcal{A}']$ as follows.

$$(\text{P2}[F', \mathcal{A}']) \max_{X_a(\gamma^a)\in[0,1]} \sum_{a\in\mathcal{A}'}\sum_{\gamma^a\in\widetilde{\Gamma}^a}\mathsf{G}^a(\gamma^a)X^a(\gamma^a)$$

subject to

$$\sum_{a\in\mathcal{A}}\sum_{\gamma^a\in\widetilde{\Gamma}^a} D_j(\gamma^a, t)X^a(\gamma^a) \leq \sum_{a\in\mathcal{A}}\sum_{\gamma^a\in\widetilde{\Gamma}^a} D_j(\gamma^a, t)X'^a(\gamma^a),\ \forall j \in \mathcal{V}^+, t \in \mathcal{T} \tag{26}$$

$$\sum_{a\in\mathcal{A}_j}\sum_{\gamma^a\in\widetilde{\Gamma}^a} d^a(\gamma^a, t)X^a(\gamma^a) \leq \sum_{a\in\mathcal{A}_j}\sum_{\gamma^a\in\widetilde{\Gamma}^a} d^a(\gamma^a, t)X'^a(\gamma^a),\ \forall j \in \mathcal{V}^+, t \in \mathcal{T} \tag{27}$$

$$\sum_{\gamma^a\in\widetilde{\Gamma}^a} X^a(\gamma^a) \leq 1, \forall a \in \mathcal{A}'. \tag{28}$$

Note that P2$[F', \mathcal{A}']$ is an LP.
4) Suppose $X'' = \big(X''^a(\gamma^a)\big)_{a\in\mathcal{A}', \gamma^a\in\widetilde{\Gamma}^a}$ is an optimal basic feasible solution (BFS) of P2$[F', \mathcal{A}']$. We define an integral solution $\widehat{X}$, as follows

$$\widehat{X}^a(\gamma^a) = \lfloor X''^a(\gamma^a)\rfloor,\ \forall a \in \mathcal{A},\ \gamma^a \in \widetilde{\Gamma}^a. \tag{29}$$

5) Then, obtain the corresponding $\hat{s}_0, \hat{s}, \hat{S}, \hat{\ell}, \hat{v}$ by invoking Corollary 1 with $\widehat{X}$ defined as in (22), (23) and (29).
6) The output solution will be the one having the maximal objective value among all guesses.

The pseudo-codes of PTAS-A-EVSP are given in Alg. 1.

Algorithm 1 PTAS-A-EVSP

Input: $\epsilon, v_0, (\underline{v}_j, \overline{v}_j)_{j\in\mathcal{V}^+}, (\overline{S}_e, \overline{\ell}_e, z_e)_{e\in\mathcal{E}}, (d_j(t))_{j\in\mathcal{V}^+,\ t\in\mathcal{T}}$
Output: Solution $\hat{F} = (\hat{s}_0, \hat{s}, \hat{S}, \hat{v}, \hat{\ell}, \widehat{X})$ to A-EVSP

```
f_max ← −∞
for each set A_1 ⊆ A and γ̂ = (γ̂^a)_{a∈A_1} such that |A_1| ≤ 6m|T|/ϵ do
    Γ̃^a ≜ Γ^a \ {γ^a ∈ Γ^a : G^a(γ^a) ≥ min_{a'∈A_1}{G^{a'}(γ̂^{a'})}}
    A' ← A \ A_1
    if P1[A_1, γ̂] is feasible then
        F' ← Optimal solution of P1[A_1, γ̂]
        (X''^a(γ^a))_{a∈A', γ^a∈Γ̃^a} ← Optimal BFS of P2[F', A']
        (X̂^a(γ^a))_{a∈A', γ^a∈Γ̃^a} ← rounded solution according to (29).
        X̂(γ̂^a) = 1, X^a(γ^a) = 0 for γ^a ≠ γ̂^a, ∀a ∈ A_1
        X̂^a(γ^a) = 0, ∀γ^a ∈ Γ^a \ Γ̃^a, ∀a ∈ A'

        (ŝ_0, ŝ, Ŝ, v̂, ℓ̂, X̂) ← solution returned by Corollary 1
        if f_max < f(ŝ_0, X̂) then
            F̂ ← (ŝ_0, ŝ, Ŝ, v̂, ℓ̂, X̂)
            f_max ← f(ŝ_0, X̂)
        end if
    end if
end for
return F̂
```

A. Analysis of Approximation Ratio

In this section, the approximation ratio of PTAS-A-EVSP will be shown to be $(1-\epsilon)$, if one sets the size of partial guesses of satisfiable discrete demands as $|\mathcal{A}_1| \le \frac{6m|\mathcal{T}|}{\epsilon}$, where m is the number of edges in distribution network. Therefore, one can adjust the approximation ratio by limiting the size of $\mathcal{A}_1$ in the partial guessing.

Remark 2. *To (practically) speed up* PTAS-A-EVSP*, one can apply* brunch and bound *approach to prune the search space of* $\mathcal{A}_1$.

We will use the following lemma in our derivation of the approximation ratio of PTAS-A-EVSP.

Lemma 2 ([9], [10], [11])**.** *Let X be a BFS of P2$[F', \mathcal{A}']$. Then X has at most $6m|\mathcal{T}|$ fractional components:* $|\{(\gamma^a \in \Gamma^a,\ a \in \mathcal{A}') \mid X^a(\gamma^a) \in (0,1)\}| \le 6m|\mathcal{T}|$.

Theorem 3. *With assumptions* A0, A1', A2, A3, A4', A5*, for any fixed $\epsilon > 0$,* PTAS-A-EVSP *provides a $(1-\epsilon)$-approximate solution for* A-EVSP*, in time polynomial in* $n = |\mathcal{V}| + |\mathcal{A}| + \sum_{a\in\mathcal{A}} |\mathcal{C}^a|$*, assuming* $|\mathcal{T}| = O(1)$.

Proof. It is easy to see that the running time of PTAS-A-EVSP is polynomial in n, for any fixed $\epsilon > 0$ and $|\mathcal{T}| = O(1)$. Next, we show that the output solution $\hat{F}$ is $(1-\epsilon)$-approximation for A-EVSP. Let $F^\star = (s_0^\star, s^\star, S^\star, v^\star, \ell^\star, X^\star)$ be an optimal solution of A-EVSP. Define

$$\mathcal{A}_1^\star \triangleq \{a \in \mathcal{A} \mid \exists \gamma^{\star a} \in \Gamma^a :\ X^{\star a}(\gamma^{\star a}) = 1\}. \quad (30)$$

There are two cases:

1) If $|\mathcal{A}_1^\star| \le \frac{6m|\mathcal{T}|}{\epsilon}$, then there exists a partial guess $(\mathcal{A}_1, \hat{\gamma})$, such that $\mathcal{A}_1 = \mathcal{A}_1^\star$ and $\hat{\gamma}^a = \gamma^{\star a}$ for $a \in \mathcal{A}_1^\star$. Thus, PTAS-A-EVSP can find an optimal solution $F^\star$ of A-EVSP by enumerating all possible $\mathcal{A}_1$ and $\hat{\gamma}$ such that $|\mathcal{A}_1| \le \frac{6m|\mathcal{T}|}{\epsilon}$.
2) Otherwise, $|\mathcal{A}_1^\star| > \frac{6m|\mathcal{T}|}{\epsilon}$; then PTAS-A-EVSP can still find some $\mathcal{A}_1$ (and a corresponding assignment $\hat{\gamma} = (\gamma^{\star a})_{a\in\mathcal{A}_1}$), which is a subset of EVs in $\mathcal{A}_1^\star$ with a number of $\lfloor \frac{6m|\mathcal{T}|}{\epsilon} \rfloor$ highest $\mathsf{G}^a(\gamma^{\star a})$:

$$\mathcal{A}_1 \subseteq \mathcal{A}_1^\star, \quad |\mathcal{A}_1| = \lfloor \tfrac{6m|\mathcal{T}|}{\epsilon} \rfloor, \quad (31)$$

$$\min_{a'\in\mathcal{A}_1} \{\mathsf{G}^{a'}(\hat{\gamma}^{a'})\} > \max_{a'\in\mathcal{A}^\star\setminus\mathcal{A}_1} \{\mathsf{G}^{a'}(\gamma^{\star a'})\}. \quad (32)$$

Next, we assume $\mathcal{A}_1$ and $\hat{\gamma}$ satisfying (31) and (32).

Then, we focus on case 2. Let us consider an optimal solution $F' = (s_0', s', S', v', \ell', X')$ of P1$[\mathcal{A}_1, \hat{\gamma}]$, where $\mathcal{A}_1$ satisfies (31) and $\hat{\gamma}^a = \gamma^{\star a}$ for $a \in \mathcal{A}_1$. Since $F^\star$ is feasible for P1$[\mathcal{A}_1, \hat{\gamma}]$, it follows that

$$f(s_0', X') \ge f(s_0^\star, X^\star). \quad (33)$$

Next, let us consider an optimal BFS $(X''^a(\gamma^a))_{a\in\mathcal{A}',\gamma^a\in\widetilde{\Gamma}^a}$ of P2$[F', \mathcal{A}']$. Note that X' is a feasible solution to P2$[F', \mathcal{A}']$ (where Cons. (26) and (27) are tight) . It follows that

$$\sum_{a\in\mathcal{A}'} \sum_{\gamma^a\in\widetilde{\Gamma}^a} \mathsf{G}^a(\gamma^a) X''^a(\gamma^a) \ge \sum_{a\in\mathcal{A}'} \sum_{\gamma^a\in\widetilde{\Gamma}^a} \mathsf{G}^a(\gamma^a) X'^a(\gamma^a). \quad (34)$$

For each $a \in \mathcal{A}'$ and $\gamma^a \in \widetilde{\Gamma}^a$, one has by (25)

$$\mathsf{G}^a(\gamma^a) \le \min_{a'\in\mathcal{A}_1} \mathsf{G}^{a'}(\hat{\gamma}^{a'}) \le \frac{1}{|\mathcal{A}_1|} \sum_{a'\in\mathcal{A}_1} \mathsf{G}^{a'}(\hat{\gamma}^{a'}). \quad (35)$$

By Lemma 2, at most $6m|\mathcal{T}|$ components in $(X''^a(\gamma^a))_{a\in\mathcal{A}',\gamma^a\in\widetilde{\Gamma}^a}$ are fractional. Therefore, by our rounding Step (29), we set $\sum_{\gamma^a\in\widetilde{\Gamma}^a} X''^a(\gamma^a)$ to 0 for at most $6m|\mathcal{T}|$ EVs $a \in \mathcal{A}'$. Thus, by (34), (35) and the non-negativity of $f_0(\cdot)$, we have

$$\begin{aligned}
&\sum_{a\in\mathcal{A}'} \sum_{\gamma^a\in\widetilde{\Gamma}^a} \mathsf{G}^a(\gamma^a) \widehat{X}^a(\gamma^a) \\
&\ge \sum_{a\in\mathcal{A}'} \sum_{\gamma^a\in\widetilde{\Gamma}^a} \mathsf{G}^a(\gamma^a) X''^a(\gamma^a) - \frac{6m|\mathcal{T}|}{|\mathcal{A}_1|} \sum_{a'\in\mathcal{A}_1} \mathsf{G}^{a'}(\hat{\gamma}^{a'}) \\
&> \sum_{a\in\mathcal{A}'} \sum_{\gamma^a\in\widetilde{\Gamma}^a} \mathsf{G}^a(\gamma^a) X''^a(\gamma^a) + \epsilon \sum_{a'\in\mathcal{A}_1} \mathsf{G}^{a'}(\hat{\gamma}^{a'}) \\
&\ge \sum_{a\in\mathcal{A}'} \sum_{\gamma^a\in\widetilde{\Gamma}^a} \mathsf{G}^a(\gamma^a) X''^a(\gamma^a) + \epsilon f(s_0', X'). \quad (36)
\end{aligned}$$

Finally, by (36) and Corollary 1, one obtains that the solution returned in Step 12 satisfies

$$f(\hat{s}_0, \widehat{X}) \ge (1-\epsilon) f(s_0', X') \ge (1-\epsilon) f(s_0^\star, X^\star),$$

which completes the proof. □

V. Evaluation Studies

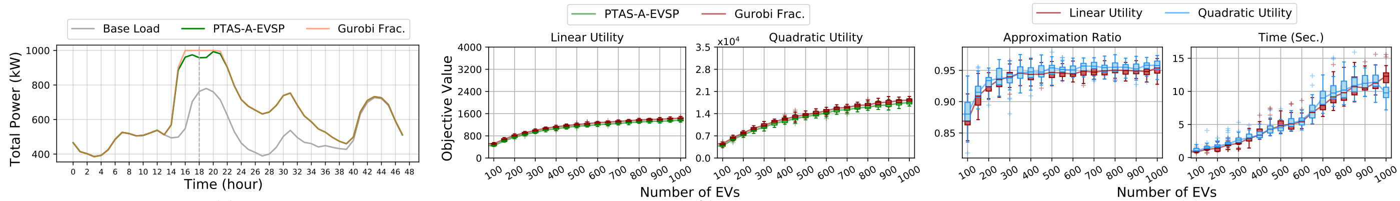
In this section, the performance of PTAS-A-EVSP is evaluated by simulations in terms of optimality and running time. For simplicity, we assume EV charging is not interrupted. This could be a desired property in a solution since intermittency in charging shortens the lifespan of an EV battery [1]. We assume a single charging option during charging interval $\mathcal{T}^a$.

A. Simulation Settings

1) Distribution network setting: We consider a 38-node system adopted from [12] (the settings of line impedance and maximum capacity are provided in [12]). The capacity of the substation is 1MVA. We choose the average household load profile in the service area of South California Edison from 00:00, January 3, 2011, to 23:59, January 4, 2011 [13], shown in Fig. 1a (gray line). We consider different penetration levels of EVs in 650 households (distributed at random locations in $\mathcal{V}^+$). For simplicity, we assume all loads have unity power factor. Each EV is assigned with at most three charging options with power rates of 1.5kW, 7kW, and 50kW, respectively.

2) Scheduling horizon: We consider a 48-hour scheduling horizon, divided into 192 slots of 15 minutes. Typically, most EV users start charging when returned home at 18:00, and more than 90% of EVs charging start at between 13:00 and 23:00. Therefore, and according to [14], the start time can be modeled as a normal distribution with a mean μ of 18:00 and a standard deviation σ of 5 hours. For each charging option c, we set its start time at random (following a normal distribution) and its length to the minimum time needed to satisfy the energy requirement usg^a using charging option c.

3) EV Battery size and initial SOC: The initial *state-of-charge* (SOC) of an EV battery is modeled as a truncated normal distribution that takes values between 20% to 80% with $\mu = 50\%$ and $\sigma = 30\%$ [15]. The battery size B^a is also modeled as a truncated normal distribution with values within 24kW and 100kW, $\mu = 30$kW, and $\sigma = 10$kW. We set the energy requirement for each EV a by $\mathsf{usg}^a = (1 - \frac{\mathsf{SOC}}{100}) \cdot B^a$.



(a) (b) (c)

Fig. 1: (a) The total power delivered over time; the gray line shows the base load profile of 650 households, and the remaining lines indicate the total power after applying the respective algorithms. (b) The median of objective values for PTAS-A-EVSP, Gurobi numerical solver fractional solution (as the upper bounds to the true optimal values) using linear and quadratic utility settings, against the number of EVs. (c) The empirical approximation ratio and running time of PTAS-A-EVSP for linear and quadratic utility, against the number of EVs.

4) Objective function: The cost of energy is based on South California Edison TOU rate plans [13]. We consider two settings for utility values: i) *Linear utility*: $u^a = 0.36 \cdot \mathsf{usg}^a$, ii) *Quadratic utility*: $u^a = (0.36 \cdot \mathsf{usg}^a)^2$, where 0.36 is the peak energy price. The utility value in this context can be interpreted as the maximum amount a customer is willing to pay to fulfill the charging requirement, and the gain $G^a(x^a, y^a)$ as the amount the customer saves.

In order to evaluate the performance of our algorithms, Gurobi numerical solver is used as a benchmark to obtain numerical solutions. Note that there is no guarantee that Gurobi will terminate in a reasonable time; therefore, we use solutions obtained from solving the relaxed problem CA-EVSP, which is a SOCP (and always terminates using Gurobi optimizer). Notice that the objective value of optimal solutions for CA-EVSP upper bounds that of A-EVSP. Therefore, we use it instead of that of A-EVSP to benchmark the solution quality of our algorithms.

The simulations were evaluated using Intel i7-3770 CPU 3.40GHz processor with 32GB of RAM. The algorithms were implemented using Python 2.7 programming language with Scipy library for scientific computation.

B. Evaluation Results

We first observed that our algorithm successfully shifts EV demands to time periods with lower cost [13] (see Fig. 1a).

1) Optimality: Fig 1b presents the objective values obtained by PTAS-A-EVSP, and the upper bounds to the true optimal values by fractional solutions with relaxed discrete constraints (and the convex relaxation CA-EVSP) for up to 1000 EVs. Each run was evaluated with over 40 random instances. PTAS-A-EVSP will terminate when its objective value is close to the lower bound. The objective values of PTAS-A-EVSP are often close to the optimal values of the relaxed problem (which upper bounds the true optimal values). This is because the number of fractional components in the relaxed problem CA-EVSP is often small. The empirical approximation ratios are plotted in Fig. 1c against the number of EVs. We observe that the empirical approximation ratio is close to 1 for linear and quadratic utility settings.

2) Running Time: The computation time of PTAS-A-EVSP is plotted in Fig. 1c under the two utility settings. Computation time is significantly important when implemented in a controller in practice, and this will have implications to the overall resilience of power grid. Although the current implementations are not fully optimized, the running time is quite reasonable. However, the running time of Gurobi with discrete variables is much higher, and in many cases, it does not provide any guarantee on the termination of execution.

VI. Conclusion

This paper presents a polynomial-time approximation algorithm (PTAS) to solve the scheduling optimization problem of EV charging in realistic settings, in the presence of (1) discrete charging options with minimum power requirements in various charging modes, and (2) practical operating constraints of alternating current (AC) power flows. Earlier fundamental hardness results for OPF show that our PTAS is among the best achievable in theory. Further simulations show our algorithms can produce close-to-optimal solutions in practice.

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