Exact Analysis of Latency of Stateless Opportunistic Forwarding

Chi-Kin Chau
University of Cambridge & University College London, UK
Chi-Kin.Chau@cl.cam.ac.uk

Prithwish Basu
BBN Technologies, Cambridge MA, USA
pbasu@bbn.com

***Abstract*—Stateless opportunistic forwarding is a simple fault-tolerant distributed approach for data delivery and information querying in wireless ad hoc networks, where packets are forwarded to the next available neighbors in a "random walk" fashion, until they reach the destinations or expire. This approach is robust against ad hoc topology changes and is amenable to computation/bandwidth/energy-constrained devices; however, it is generally difficult to predict the end-to-end latency suffered by such a random walk in a given network. In this paper, we make several contributions on this topic. First, by using spectral graph theory we derive a general formula for computing the exact hitting and commute times of weighted random walks on a finite graph with heterogeneous sojourn times at relaying nodes. Such sojourn times can model heterogeneous duty cycling rates in sensor networks, or heterogeneous delivery times in delay tolerant networks. Second, we study a common class of distance-regular networks with varying numbers of geographical neighbors, and obtain simple estimate-formulas of hitting times by numerical analysis. Third, we study the more sophisticated settings of random geographical locations and distance-dependent sojourn times through simulations. Finally, we discuss the implications of this on the optimization of latency-overhead trade-off.**

***Index Terms*—Opportunistic forwarding, Wireless sensor networks, Delay tolerant networks, Random walks on finite graphs, Spectral graph theory**

I. Introduction

Routing data to specific endpoints in ad hoc wireless networks typically requires accurate global topology information and proactive route maintenance by all participating nodes. Various flavors of ad hoc networks such as sensor networks and delay tolerant networks present challenges such as frequent disruptions of connectivity. For instance, in sensor networks relaying devices may be performing sleep-wake duty cycling, while in delay tolerant networks mobile packet carriers may be often out of reach.

Always flooding a data item may be reliable and simple but is often too bandwidth and energy intensive; hence, a simple approach is stateless opportunistic forwarding — packets are forwarded to one of the next available neighbors in a "random walk manner", until they either expire or reach the destination. For instance, ad hoc routing protocols may also use opportunistic forwarding (instead of flooding) as a single-copy routing technique in the absence of topology information. However, despite the simplicity of implementation, it is generally difficult to predict the end-to-end latency, because packets may (legitimately) travel in loops or along sub-optimal paths.

Although there have been several simulation studies and asymptotic analyses to suggest the usefulness of stateless opportunistic forwarding (e.g. [1], [2]), this paper presents exact analytical formulas to predict the latency of stateless opportunistic forwarding on finite graphs. The benefit of this style of analysis is that it can reduce the dependence of simulations[2], and is more precise than asymptotic analyses which typically hold for large networks only, whereas most practical networks are often moderate in size.

A. Motivating Scenarios

We first present two motivating scenarios where stateless forwarding may be useful:

(*i*) *Sensor Networks*: In this setting, we are given a network of battery-powered nodes that are capable of sourcing and relaying packets for delay-tolerant applications. Relaying devices perform *pseudo-random duty cycling* (sleep scheduling) to conserve energy [3]–[5], such that an awake transmitter can communicate with the receiver only when the latter is awake. For low values of duty cycling (or wake-up) probability, the network may be highly disconnected, and hence traditional routing protocols that operate upon freshly gathered topology updates may not be very suitable. So, stateless opportunistic forwarding by which the packets are forwarded to the neighbor that wakes up first, may be a viable strategy [3]. We consider the "low traffic volume" scenario, where the interference of concurrent wireless transmissions is negligible, but the estimation of end-to-end latency is a significant concern.

(*ii*): *Store-carry-forward Networks*: We follow the paradigm of so-called "pocket switched networks" [1]. The nodes of network are collection sites that can store messages, and have fixed geographical locations. The *delay-tolerant links* are the mobile message carriers (e.g. vehicles) randomly wandering between collection sites. Every message carrier has a limited reachable region and messages can only be forwarded within

[1]This research was sponsored by the U.S. Army Research Laboratory and the U.K. Ministry of Defense and was accomplished under Agreement Number W911NF-06-3-0002.l. The views and conclusions contained in this document are those of the author(s) and should not be interpreted as representing the official policies, either expressed or implied, of the U.S. Army Research Laboratory, the U.S. Government, the U.K. Ministry of Defence or the U.K. Government. The U.S. and U.K. Governments are authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright notation hereon.

[2]In our experience, one has to run several thousand simulations to get a good estimate of the mean latency even in small networks.

a certain geographical region by a single carrier. By stateless opportunistic forwarding, the packets are picked up by the first near-by carrier that approaches the respective collection site, and are off-loaded to the next collection site stopped by the carrier, independent of the paths traveled by the packets.

For the two aforementioned scenarios, stateless opportunistic forwarding can be modeled by a random walk on a finite graph. We are motivated to study several aspects of latency of random walks on graphs:

1) *Hitting time* (or access time): The expected time of a packet from the source to *hit* a certain destination.
2) *Commute time*: The expected round-trip time between source and destination.
3) *Cover time*: The expected time of a packet from the source to visit every other node in the network.

B. Our Contributions

Hitting, commute, and cover times of random walks have been studied extensively in the literature [6]–[13]. A major approach is based on spectral graph theory [6], [12]–[14]. In this paper, we study the general setting of random walks with heterogeneous sojourn times at relaying nodes. For instance, these can correspond to heterogeneous duty cycling rates in sensor networks, or heterogeneous delivery times in store-carry-forward networks. We generalize the formulas in [6], [12] to compute the *exact* hitting and commute times with heterogeneous sojourn times. Our results also enable more accurate numerical analysis of cover times via Matthew's bounds [7], [8].

Furthermore, we study random walks on a common class of distance-regular networks with varying numbers of nearest neighbors, which captures the notion of geographical proximity in sensor networks and delay tolerant networks. For instance, in an r-nearest neighbor cycle (or torus), there is an edge between every pair of neighbors of r-hops away on a cycle (or torus). The varying numbers of nearest neighbors by r's can capture the levels of overhead for maintaining local neighbors (e.g. as the levels of transmission power in sensor networks, or the areas of reachable regions of carriers in delay tolerant networks). One of our aims is to understand and optimize the latency-overhead trade-off.

Although there has been a study in [15] about the asymptotic analysis on the asymptotic order of cover times for r-nearest neighbor torus (called k-fuzz of torus) via effective resistance, in this paper we provide exact formulas via spectral graph theory to compute the hitting and commute times on r-nearest neighbor torus. This work offers the exact analyses of the latency of stateless opportunistic forwarding on finite graphs, which are more precise than asymptotic analyses on the asymptotic orders. Our formulas can be computed efficiently without relying on extensive simulations.

Finally, we study more sophisticated settings of random geographical locations and distance-dependent sojourn times through simulations, and discuss the ramifications to the optimization of latency-overhead trade-off in opportunistic forwarding algorithms.

II. Problem Formulation

In this section, we formulate a model to capture stateless opportunistic forwarding in sensor networks and delay tolerant networks, which is related to random walk on finite graphs. Also, we formulate a common class of distance-regular networks with varying numbers of nearest neighbors to model the geographical proximity in opportunistic forwarding.

A. Random Walks on Finite Graphs

To model the stateless opportunistic forwarding, we consider a finite *connected* undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes (e.g. relaying devices or collection sites) and $\mathcal{E}$ is the set of edges. Each edge means one-hop forwarding is possible between the pair of nodes (e.g. by radio transmission or packet carriers). Let $n = |\mathcal{V}|$, $m = |\mathcal{E}|$. We also let $\mathcal{N}_u \subseteq \mathcal{V}$ be the set of neighbors of u, and its degree $d_u = |\mathcal{N}_u|$.

We consider the setting of slotted time. In random walk based forwarding, a packet is stored in a node for certain time slots before there is an opportunity to be forwarded to the next hop neighbor. We assume that the forwarding operation in the current hop is stateless, being independent of the forwarding operations of previous hops.

For each neighboring pair of nodes $u, v \in \mathcal{V}$, we let $\ell_{u,v}$ be the expected sojourn time of a packet that travels from u to v. We assign a weight $w_{u,v}$ to each edge (u, v) to indicate the the availability of opportunistic forwarding operations between the nodes, such that $\rho_{u,v}$, the probability that u will forward packets to $v \in \mathcal{N}_u$, is defined as:

$$\rho_{u,v} \triangleq \frac{w_{u,v}}{w_u}, \qquad \text{where} \;\; w_u \triangleq \sum_{v' \in \mathcal{N}_u} w_{u,v'}$$

Note that in this paper we assume the symmetric case: $w_{u,v} = w_{v,u}$. Therefore, this defines a random walk on a weighted graph, or equivalently, a reversible Markov chain [7].

The hitting time $\mathbf{H}_{u,v}$ from source u to reach destination v can be computed recursively by:

$$\mathbf{H}_{u,v} = \begin{cases} \sum\limits_{w \in \mathcal{N}_u} \rho_{u,w}(\ell_{u,w} + \mathbf{H}_{w,v}) = \ell_u + \sum\limits_{w \in \mathcal{N}_u} \rho_{u,w} \mathbf{H}_{w,v} & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases} \tag{1}$$

where $\ell_u \triangleq \sum\limits_{w \in \mathcal{N}_u} \rho_{u,w} \cdot \ell_{u,w}$ is the expected sojourn time at u.

B. Examples

Example 1 (*Sensor Networks with i.i.d. Random Duty Cycling*): Let ρ_{dc} be the duty cycling rate, such that in one time slot, each node is awake with i.i.d. probability ρ_{dc}, and is dormant with probability $1 - \rho_{\mathsf{dc}}$. Hence, every neighbor has equal chance of opportunistic forwarding: $w_{u,v} = 1$. Then the waiting time for both u and v to be awake in the same time slot is a geometric random variable with parameter ρ_{dc}^2, As shown in [3], the per-hop latency is $\ell_u = \frac{1}{1-(1-\rho_{\mathsf{dc}}^2)^{d_u}}$

Example 2 (*Sensor Networks with i.i.d. Pseudo-random Duty Cycling*): As proposed in [4], if neighboring nodes first exchange the seed of pseudo-random sequence to generate duty cycling, then a node can predict the exact awake time slots of its neighbors. Let ρ_{dc} be the duty cycling rate.

Similarly, $w_{u,v} = 1$. Then the waiting time for both u and v to be awake in the same time slot is a geometric random variable with parameter ρ_{dc}, Hence, $\ell_u = \frac{1}{1-(1-\rho_{\mathsf{dc}})^{d_u}}$

Example 3 (*Delay Tolerant Networks with Geographical Dependence*): We assume that each collection site has its packet carriers commuting among its neighbors within a certain bounded reachable region. The location of packet carriers within the reachable region follows an i.i.d. stationary distribution. Hence, we assume that the per-hop latency depend linearly on the geographical distance, such that $\ell_{u,v} = c||u-v||$ and $w_{u,v} = 1$. That is, the further away the nodes are, the longer latency is incurred. Although other mobility models (e.g. random waypoint model) can also be considered, closed-form expressions for $w_{u,v}$ and $\ell_{u,v}$ will be more involved.

C. Nearest Neighbor Networks

In this paper, we consider a common class of distance-regular networks that can capture the notion of geographical proximity. For simplicity, we consider boundary-less space. Suppose there is a set of n nodes placed evenly in one dimension as a cycle, or in two dimensional space as a torus. We then define an r-nearest neighbor cycle as the graph with edges between every pair of nodes within r-hops away on the cycle. This naturally captures the notion geographical proximity, such that neighbors are within the bounded transmission range in sensor networks, or reachable by a single carrier wandering in a bounded region in delay tolerant networks. See Fig. 1 (b) for an example of 2-nearest neighbor cycle.

To generalize to two dimensional as a torus, we note that there are various ways of defining the nearest neighbors in two dimensions. One may use the L^1 norm such that there is an edge between the nodes whose shortest path is within r hops on the torus (see Fig. 2 (a)), or L^∞ norm such that the vertical and horizontal distance are both within r hops on the torus (see Fig. 2 (b)). Generally, we can allow other norms (e.g. L^2 norm to model circular transmission range), which however are less convenient to handle.

III. Hitting and Commute Times of Random Walks

In this section, we present two general techniques to solve the hitting and commute times of random walks on arbitrary graphs — effective resistance and spectral graph theory. Based on spectral graph theory, we then derive efficient general formulas for the hitting and commute times of random walk with heterogeneous sojourn times at relaying nodes

Denote π_u as the unique stationary distribution of random walk on $\mathcal{G}$. By standard Markov chain theorem [7], we obtain:

$$\pi_u = \frac{w_u}{W} \qquad \text{where} \quad W \triangleq \sum_{u' \in \mathcal{V}} w_{u'}$$

Denote the commute time between node u and node v as $\mathbf{C}_{u,v}$. Note that $\mathbf{C}_{u,v} = \mathbf{H}_{u,v} + \mathbf{H}_{v,u}$ (i.e. the sum of hitting times from u to v and vice versa). In [10], it is shown that

$$\mathbf{H}_{u,v} = \frac{1}{2}\Big(\mathbf{C}_{u,v} + \sum_{w \in \mathcal{V}} \pi_w(\mathbf{C}_{w,v} - \mathbf{C}_{w,u})\Big) \tag{2}$$

A. Effective Resistance

We first survey the approach of effective resistance [9]–[11]. Given a finite graph $\mathcal{G}$, we assign each edge $(u,v) \in \mathcal{E}$ a resistance of value $1/w_{u,v}$. For any pair of nodes $u, v \in \mathcal{V}$ (which may not be neighbors), we define the effective resistance $\mathfrak{R}_{u,v}$ as the voltage difference between u and v, when a unit current is injected at u and removed from v.

In [11], it is shown that:

$$\mathbf{C}_{u,v} = \mathfrak{R}_{u,v} \cdot \sum_{(u,v)\in\mathcal{E}} w_{u,v}(\ell_{u,v} + \ell_{v,u}) \tag{3}$$

Combining Eqns. (2)-(3), it is possible to compute hitting times via effective resistance. However, it is non-trivial to compute efficiently the exact values of effective resistances of a arbitrary network. In this paper, we rely on an alternate approach by spectral graph theory.

B. Spectral Graph Theory

Denote the adjacency matrix of $\mathcal{G}$ as $\mathbf{A}$ such that

$$\mathbf{A}_{u,v} \triangleq \begin{cases} w_{u,v} & \text{if } (u,v) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Sometimes, we denote the adjacency matrix of graph $\mathcal{G}$ as $\mathbf{A}[\mathcal{G}]$. Denote the diagonal matrix of $\mathcal{G}$ as $\mathbf{D}$ such that

$$\mathbf{D}_{u,v} \triangleq \begin{cases} w_u & \text{if } u = v \\ 0 & \text{otherwise} \end{cases}$$

We define normal matrix as $\mathbf{N} \triangleq \mathbf{D}^{-\frac{1}{2}}\mathbf{A}\mathbf{D}^{-\frac{1}{2}}$. That is,

$$\mathbf{N}_{u,v} \triangleq \begin{cases} \frac{w_{u,v}}{\sqrt{w_u w_v}} & \text{if } (u,v) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Note that $\mathbf{N}$ is symmetric, and hence there exist real eigenvalues and eigenvectors of $\mathbf{N}$. Let λ_k and $\boldsymbol{\nu}_k$ be the $(k+1)$-th[3] eigenvalue and the corresponding eigenvector of $\mathbf{N}$.

B.1. Homogeneous Sojourn Times

Consider homogeneous sojourn times: $\ell_u = 1$ for all u. In [6] (Theorem 3.1), Lovasz solved Eqn. (1) with a solution:

$$\mathbf{H}_{u,v} = \sum_{k:\lambda_k \neq 1} \frac{W}{1-\lambda_k}\Big(\frac{\boldsymbol{\nu}_{k,v}^2}{w_v} - \frac{\boldsymbol{\nu}_{k,u}\boldsymbol{\nu}_{k,v}}{\sqrt{w_u w_v}}\Big) \tag{4}$$

and hence by $\mathbf{C}_{u,v} = \mathbf{H}_{u,v} + \mathbf{H}_{v,u}$,

$$\mathbf{C}_{u,v} = \sum_{k:\lambda_k \neq 1} \frac{W}{1-\lambda_k}\Big(\frac{\boldsymbol{\nu}_{k,u}}{\sqrt{w_u}} - \frac{\boldsymbol{\nu}_{k,v}}{\sqrt{w_v}}\Big)^2 \tag{5}$$

Although Lovasz considered unweighted graphs where $w_u = d_u$ (the degree of u) and $W = 2m$ (twice of the number of edges), it can be easily generalized to weighted graphs.

In [12] (Theorem 8), Chung and Yau considered the (normalized) Laplacian of a graph defined as $\mathbf{L} \triangleq \mathbf{I} - \mathbf{N}$, and independently proved Eqn. (4) via discrete Green's function[4].

The complexity of Eqn. (4) or (5) is $O(n)$, and solving the eigen spectrum of a symmetric matrix is $O(n^3)$. Since the eigen spectrum of $\mathbf{N}$ can be reused for all source-destination pairs. Hence, the complexity of computing the hitting and

[3] The order of eigenvalues does not matter.

[4] The proof given by Lovasz in [6] is rather sketchy and some key steps have been skipped. Hence, we base our results on the ones in [12].

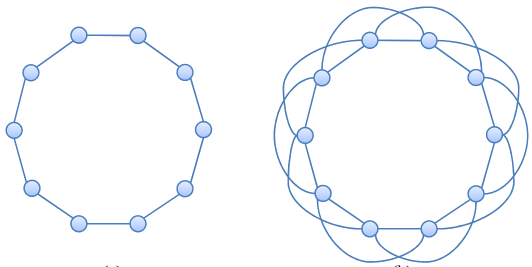
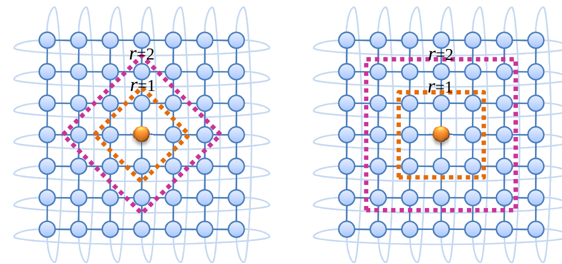

(a) (b)

Fig. 1. (a) Cycle. (b) 2-Nearest neighbor cycle.



(a) (b)

Fig. 2. (a) L^1 Nearest neighbor tori. (b) L^∞ Nearest neighbor tori. The dotted boxes indicate the different levels of neighbors of the centre node.

commute times for all n^2 source-destination pairs is $O(n^3)$.

B.2. Heterogeneous Sojourn Times

In this paper, we consider heterogeneous sojourn times ℓ_u. Note that one can set $\ell_{u,v} = 1$ for all $(u,v) \in \mathcal{E}$, equating Eqn. (3) and Eqn. (5) to obtain the effective resistance as:

$$\mathfrak{R}_{u,v} = \sum_{k:\lambda_k \neq 1} \frac{1}{1-\lambda_k}\left(\frac{\boldsymbol{\nu}_{k,u}}{\sqrt{w_u}} - \frac{\boldsymbol{\nu}_{k,v}}{\sqrt{w_v}}\right)^2 \tag{6}$$

Then one can use Eqns. (2)-(3) to compute the hitting and commute times. But the complexity of computing the hitting times for all n^2 source-destination pairs becomes $O(n^4)$.

Nonetheless, it is more efficient to use the similar formulas as Eqns. (4)-(5), which takes only $O(n^3)$. Moreover, it allows simplification for specific network topology as discussed in the next section. A major contribution of this paper is Theorem 1, which extends the formulas in [6], [12] to the setting of heterogeneous sojourn times.

First, we define latency matrix $\mathbf{S}$ as:

$$\mathbf{S}_{u,v} \triangleq \begin{cases} \ell_u & \text{if } u = v \\ 0 & \text{otherwise} \end{cases}$$

a "generalized" Laplacian as $\tilde{\mathbf{L}} \triangleq \mathbf{S}^{-\frac{1}{2}}(\mathbf{I}-\mathbf{N})\mathbf{S}^{-\frac{1}{2}}$. That is,

$$\tilde{\mathbf{L}}_{u,v} \triangleq \begin{cases} \frac{1}{\ell_u} & \text{if } u = v \\ -\frac{w_{u,v}}{\sqrt{w_u \ell_u w_v \ell_v}} & \text{if } (u,v) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Note that $\tilde{\mathbf{L}}$ is symmetric, hence there exist real eigenvalues and eigenvectors of $\tilde{\mathbf{L}}$. Let σ_k and $\boldsymbol{\mu}_k$ be the $(k+1)$-th eigenvalue and the corresponding eigenvector of $\tilde{\mathbf{L}}$.

Theorem 1: Given arbitrary sojourn time $\ell_u > 0$ for each u, the hitting time and commute time from u to v can be computed by:

$$\mathbf{H}_{u,v} = \sum_{k:\sigma_k \neq 0} \frac{\tilde{W}}{\sigma_k}\left(\frac{\boldsymbol{\mu}_{k,v}^2}{w_v \ell_v} - \frac{\boldsymbol{\mu}_{k,u}\boldsymbol{\mu}_{k,v}}{\sqrt{w_u \ell_u w_v \ell_v}}\right) \tag{7}$$

$$\mathbf{C}_{u,v} = \sum_{k:\sigma_k \neq 0} \frac{\tilde{W}}{\sigma_k}\left(\frac{\boldsymbol{\mu}_{k,u}}{\sqrt{w_u \ell_u}} - \frac{\boldsymbol{\mu}_{k,v}}{\sqrt{w_v \ell_v}}\right)^2 \tag{8}$$

where $\tilde{W} \triangleq \sum_{(u,v)\in\mathcal{E}} w_{u,v}(\ell_{u,v} + \ell_{v,u}) = \sum_{u'\in\mathcal{V}} w_{u'}\ell_{u'}$.

Proof: We generalize the proof in [12]. See Appendix. ■

When $\ell_{u,v} = 1$ for all $(u,v) \in \mathcal{E}$, it is easy to see that $\sigma_k = 1 - \lambda_k$ and $\boldsymbol{\mu}_k = \boldsymbol{\nu}_k$. Hence, Eqns. (7)-(8) reduce to Eqns. (4)-(5). Similarly, the complexity of computing the hitting and commute times with heterogeneous sojourn times by Eqn. (4) for all n^2 source-destination pairs is $O(n^3)$.

Finally, we present an instance of *Example* 3 in Fig. 3 with hitting times computed by Eqn. (7). We have verified that the hitting times in Fig. 3 satisfy Eqn. (1).

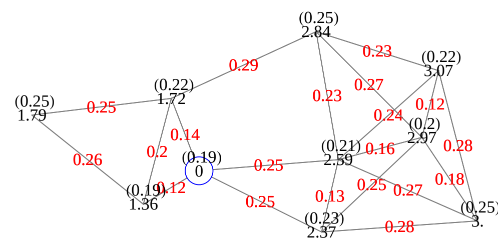


Fig. 3. A randomly generated instance of *Example* 3, where the radius of one-hop reachable region is 0.3. On each node is the hitting time to hit the blue circled node, where the red number on each edge is the Euclidean distance, and the number in brackets is the expected sojourn time at each node.

IV. Nearest Neighbor Networks

In this section, we especially study the Laplacian of nearest neighbor networks and obtain specific formulas for computing the hitting and commute times. For clarity, in this section we consider the homogeneous case: $\ell_{u,v} = 1$ for all $(u,v) \in \mathcal{E}$.

A. Nearest Neighbor Cycles

Denote a cycle of n nodes as $\mathcal{C}_n$. We construct an r-nearest neighbor cycle (denoted as $\mathcal{C}_n^r$) as the graph with edges between nodes and their r-nearest left and r-nearest right neighbors on $\mathcal{C}_n$. We label the nodes by the order around the cycle as: $0, 2, ..., n-1$. We consider the uniform symmetric case: $w_{u,u+j} = w_{u,n-j+u} = a_j$ for $1 \le j \le r$.

Lemma 1: The $(k+1)$-th eigenvalue and eigenvector of the Laplacian $\tilde{\mathbf{L}}$ of r-nearest neighbor cycle $\mathcal{C}_n^r$ are:

$$\sigma_k = 1 - \frac{\sum_{j=1}^r a_j \cos\left(\frac{2\pi jk}{n}\right)}{\sum_{j'=1}^r a_{j'}}, \ \boldsymbol{\mu}_k = \frac{1}{\sqrt{n}}\left(1, \epsilon^k, ..., \epsilon^{(n-1)k}\right)^T$$

ϵ is complex number defined as: $\epsilon \triangleq \cos\left(\frac{2\pi}{n}\right) + \mathbf{i}\sin\left(\frac{2\pi}{n}\right)$.

Proof: Because $1-\sigma_k$ and $\boldsymbol{\mu}_k$ are the eigenvalue and eigenvector of normal matrix $\mathbf{N}$ of $\mathcal{C}_n^r$. See Appendix. ■

Theorem 2: Suppose $a_j = 1$ for $1 \le j \le r$. Without loss of generality, we consider the hitting time from u to 0 on r-nearest neighbor cycle $\mathcal{C}_n^r$. Then, it is computed by:

$$\mathbf{H}_{u,0} = 2r \sum_{k=1}^{n-1} \frac{1-\cos\left(\frac{2\pi k u}{n}\right)}{(2r+1) - \frac{\sin\left(\frac{\pi k(2r+1)}{n}\right)}{\sin\left(\frac{\pi k}{n}\right)}} \tag{9}$$

Proof: Note that the degree $d_u = 2r$ is a constant for all u. Thus, $W = 2rn$. By Lemma 1 and Eqn. (4), we obtain:

$$\mathbf{H}_{u,0} = 2rn \sum_{k=1}^{n-1} \frac{1-\cos\left(\frac{2\pi ku}{n}\right) - \mathbf{i}\sin\left(\frac{2\pi ku}{n}\right)}{2rn\Big(1-\frac{1}{r}\sum_{j=1}^{r}\cos\left(\frac{2\pi jk}{n}\right)\Big)} = \sum_{k=1}^{n-1} \frac{1-\cos\left(\frac{2\pi ku}{n}\right)}{1-\frac{1}{r}\sum_{j=1}^{r}\cos\left(\frac{2\pi jk}{n}\right)}$$

This is due to the identities: $\cos\left(\frac{2\pi jk}{n}\right) = \cos\left(\frac{2\pi j(n-k)}{n}\right)$ and $\sin\left(\frac{2\pi ku}{n}\right) = -\sin\left(\frac{2\pi(n-k)u}{n}\right)$. Finally, Eqn. (9) follows from the trigonometric identity of Dirichlet kernel of Lemma 2 in Appendix. ■

Theorem 3: When $r \ll n$ and n is even, the maximum hitting time on an r-nearest neighbor cycle can be approximated by:

$$\mathbf{H}_{\frac{n}{2},0} \approx \frac{3n^2}{2(1+r)(1+2r)} \tag{10}$$

Proof: Based on careful approximation of the Taylor series expansion of Eqn. (9). See Appendix. ■

Theorem 2 generalizes our previous results in [14] on only the maximum hitting times on r-nearest neighbor cycle.

When $r = 1$, it is well known that via effective resistance [7], [8] the hitting time for a pair of farthest nodes on an n-node cycle is indeed $\frac{n^2}{4}$. Hence, Theorem 3 is accurate for 1-nearest neighbor cycles. For r-nearest neighbor cycles, Fig. 4 shows that Eqn. (10) gives a relatively accurate approximation to the exact computation based on Eqn. (9).

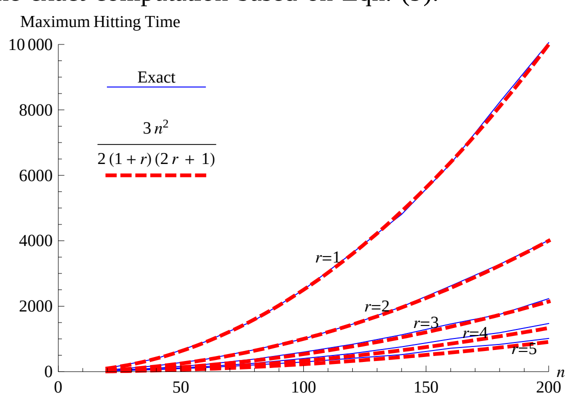


Fig. 4. $\mathbf{H}_{\frac{n}{2},0}$ computed exactly by Eqn. (9) is plotted against the approximation using Eqn. (10).

B. Nearest Neighbor Tori

Theorem 4: The hitting time $\mathbf{H}_{(u,v),(0,0)}$ from node (u, v) to node $(0, 0)$ can computed by:

1) (*Torus*):

$$\sum_{(k,l)\in[0,n-1]^2\backslash\{(0,0)\}} \frac{1-\cos\left(\frac{2\pi(ku+lv)}{n}\right)}{1-\frac{1}{2}\left(\cos\left(\frac{2\pi k}{n}\right)+\cos\left(\frac{2\pi l}{n}\right)\right)}$$

2) (L^1 *r-Nearest Neighbor Torus*):

$$\sum_{(k,l)\in[0,n-1]^2\backslash\{(0,0)\}} \frac{1-\cos\left(\frac{2\pi(ku+lv)}{n}\right)}{1-\frac{1}{2r^2+2r}\left(\sum_{i=-r}^{r}\sum_{j=-r+|i|}^{r-|i|}\cos\left(\frac{2\pi ik}{n}\right)\cos\left(\frac{2\pi jl}{n}\right)-1\right)} \tag{11}$$

3) (L$^\infty$ *r-Nearest Neighbor Torus*):

$$\sum_{(k,l)\in[0,n-1]^2\backslash\{(0,0)\}} \frac{1-\cos\left(\frac{2\pi(ku+lv)}{n}\right)}{1-\frac{1}{4r^2+4r}\left(\sum_{i=-r}^{r}\sum_{j=-r}^{r}\cos\left(\frac{2\pi ik}{n}\right)\cos\left(\frac{2\pi jl}{n}\right)-1\right)} \tag{12}$$

(*Proof idea*): The proof of Theorem 4 relies on the notion of two types of composed graphs. We specially denote the adjacency matrix of graph $\mathcal{G}$ as $\mathbf{A}[\mathcal{G}]$. We define two types of composed graphs as follows:

1) *Union Graph*: Given $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{H} = (\mathcal{V}, \mathcal{F})$, define a graph, $\mathcal{G} \cup \mathcal{H}$, such that the set of nodes is $\mathcal{V}$ and its adjacency matrix is just the sum of those of $\mathcal{G}$ and $\mathcal{H}$:
$$\mathbf{A}[\mathcal{G} \cup \mathcal{H}] = \mathbf{A}[\mathcal{G}] + \mathbf{A}[\mathcal{H}]$$
Namely, the edge weights are the sum of those of $\mathcal{G}, \mathcal{H}$.
2) *Tensor Product Graph*: Given $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, define a graph, $\mathcal{G} \times \mathcal{G}'$, such that the set of nodes is $\mathcal{V} \times \mathcal{V}'$, and its adjacency matrix is the tensor product[5] of those of $\mathcal{G}, \mathcal{G}'$:
$$\mathbf{A}[\mathcal{G} \times \mathcal{G}'] = \mathbf{A}[\mathcal{G}] \otimes \mathbf{A}[\mathcal{G}']$$
Namely, (u, u') and (v, v') are adjacent in $\mathcal{G} \times \mathcal{G}'$, if $(u, v) \in \mathcal{E}$ and $(u', v') \in \mathcal{E}'$. Particularly, if $n = |\mathcal{V}|$ and $\mathbf{I}$ is the $n \times n$ identity matrix, then $\mathbf{A}[\mathcal{G}] \otimes \mathbf{I}$ defines n disjoint copies of $\mathcal{G}$.

The combinations of union graphs and tensor product graphs over nearest neighbor cycles can generate a wide class of nearest neighbor tori with arbitrary norms. We give some examples of such constructions in Fig. 5.

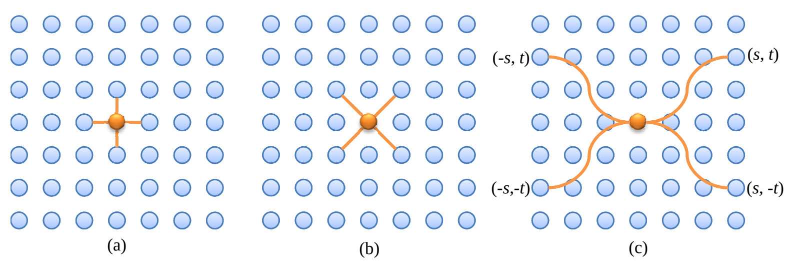


Fig. 5. The figures show different settings of neighbors of the centre node.

We consider n^2 nodes placed evenly on the two dimensional boundary-free surface of a torus. We assume that the setting of neighbors is uniform to all nodes. We label the nodes by the coordinates: (u, v), where $0 \le u, v \le n-1$ on the surface.

We explain the examples in Fig. 5 as follows:

1) In Fig. 5 (a), each node has four neighbors (two horizontal and two vertical). This indeed forms a torus, whose adjacency matrix is: $\mathbf{A}[\mathcal{C}_n] \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}[\mathcal{C}_n]$
2) In Fig. 5 (b), each node has four neighbors (all diagonal). If n is odd, then this forms a connected graph. This defines a graph, with adjacency matrix as: $\mathbf{A}[\mathcal{C}_n] \otimes \mathbf{A}[\mathcal{C}_n]$

[5]The definition of tensor product of matrices can be found in Appendix.

3) Let $\mathcal{C}_n^r(s)$ be an r-nearest neighbor cycle such that $a_s = 1$ and $a_j = 0$ for $j \neq s$. In Fig. 5 (c), each node (u, v) has only four neighbors as $(u \pm s \mod n, v \pm t \mod n)$. This defines a graph, with adjacency matrix as: $\mathbf{A}[\mathcal{C}_n^r(s)] \otimes \mathbf{A}[\mathcal{C}_n^r(t)]$

In fact, a nearest neighbor torus defined by arbitrary norm can be regarded as a union graph of a collection of graphs with suitable $\mathbf{A}[\mathcal{C}_n^r(s)] \otimes \mathbf{A}[\mathcal{C}_n^r(t)]$.

For example, for $0 \leq k, l \leq n-1$, define a vector $\boldsymbol{\mu}_{(k,l)}$:

$$\boldsymbol{\mu}_{(k,l)} \triangleq \frac{1}{n}\big(1, \epsilon^k, ..., \epsilon^{(n-1)k}\big)^T \otimes \big(1, \epsilon^l, ..., \epsilon^{(n-1)l}\big)^T$$

By Lemma 1, it follows that $\boldsymbol{\mu}_{(k,l)}$ is the $(kn + l + 1)$-th eigenvector of the Laplacian of the example in Fig. 5 (c), whose eigenvalue is $1 - \frac{1}{2}\big(\cos\big(\frac{2\pi sk}{n}\big) + \cos\big(\frac{2\pi tl}{n}\big)\big)$.

In general, it can be generalize by the following lemma.

Lemma 2: Given a set of tuples: $\{(s_1, t_1), ..., (s_r, t_r)\}$ such that $(s_j, t_j) \neq (0, 0)$ for $1 \leq j \leq r$. Consider a graph $\mathcal{G}$ with n^2 nodes, such that each node (u, v) has only four neighbors as $(u \pm s_j \mod n, v \pm t_j \mod n)$ for $1 \leq j \leq r$, whose edge weight is 1. Then the $(kn + l + 1)$-th eigenvalue of the Laplacian $\tilde{\mathbf{L}}$ of $\mathcal{G}$ is:

$$\sigma_{(k,l)} = 1 - \frac{1}{r}\sum_{j=1}^{r} \cos\big(\frac{2\pi s_j k}{n}\big) \cos\big(\frac{2\pi t_j k}{n}\big)$$

and the corresponding eigenvector is $\boldsymbol{\mu}_{(k,l)}$.

Proof: (Theorem 4) Since for a L^1 r-nearest neighbor torus the degree is $2r^2 + 2r$, and for a L^∞ r-nearest neighbor torus the degree is $4r^2 + 4r$, applying Lemma 2 we complete the proof of Theorem 4. ■

Theorem 4 can be extended to even more sophisticated geographical proximity relations, other than L^1 and L^∞ norms.

Theorem 5: When $r \ll n$, the maximum hitting time on an r-nearest neighbor torus is:

$$\mathbf{H}_{(\frac{n}{2},\frac{n}{2}),(0,0)} = \Theta\Big(\frac{n^2 \log(n)}{(1+2r)^2}\Big) \tag{13}$$

Proof: See Appendix. ■

We remark that [15] has proven similar results of Theorem 5 for the commute times on k-fuzz of torus.

Furthermore, we numerically approximate the exact formulas in Theorem 4 as follows (See Figs. 6-7):

1) (L^1 *r-Nearest Neighbor Torus*):

$$\mathbf{H}_{(\frac{n}{2},\frac{n}{2}),(0,0)} \approx \frac{3.98 n^2 \log(n)}{(1+2r)^2} + 0.96 n^2 \tag{14}$$

2) (L^∞ *r-Nearest Neighbor Torus*):

$$\mathbf{H}_{(\frac{n}{2},\frac{n}{2}),(0,0)} \approx \frac{2.34 n^2 \log(n)}{(1+2r)^2} + 0.96 n^2 \tag{15}$$

These equations will be useful for latency-overhead optimisation and prediction of TTL values in opportunistic forwarding.

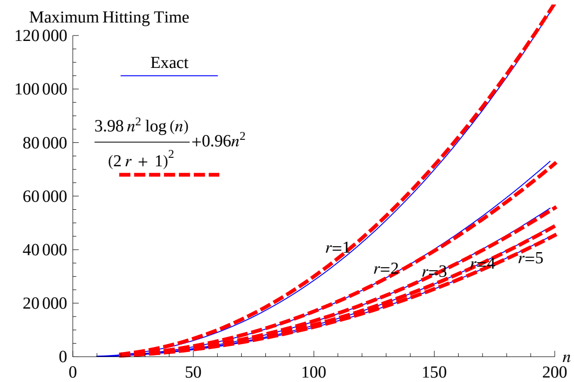


Fig. 6. For L^1 r-nearest neighbor torus, $\mathbf{H}_{(\frac{n}{2},\frac{n}{2}),(0,0)}$ computed exactly by Eqn. (14) is plotted against the approximation using Eqn. (11).

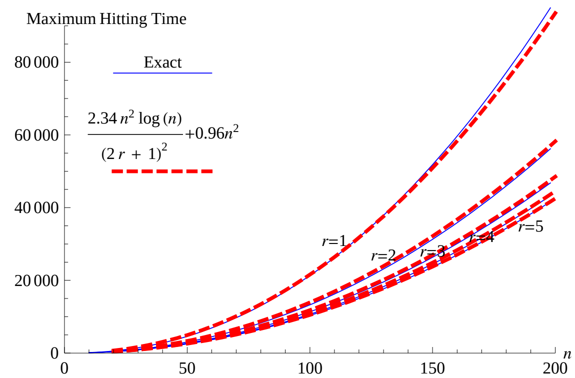


Fig. 7. For L^∞ r-nearest neighbor torus, $\mathbf{H}_{(\frac{n}{2},\frac{n}{2}),(0,0)}$ computed exactly by Eqn. (15) is plotted against the approximation using Eqn. (12).

V. Random Geographical Locations

In this section we study random geographical locations and distance-dependent weights and sojourn times based on random geometric graphs.

Random geometric graphs are widely-used for modelling diverse wireless ad hoc networks, in which nodes are randomly placed in a confined area, and communication links are established between nodes that are within a pre-defined transmission radius. In this section, we especially study the hitting time of random walks on random geometric graphs.

We denote a random geometric graph as $\mathcal{G}_{\mathsf{geo}}(N, \mathsf{R})$, which is an ensemble of N-node graph such that the position of each node is independently uniformly distributed on a 2D unit area, and there is an edge between a pair of nodes if they are within transmission radius R. First, we draw on a lemma from [16] which shows the distribution of degrees of a random geometric graph is concentrated on the mean degree.

Lemma 3: [16] (Lemma 10) Given a random geometric graph $\mathcal{G}_{\mathsf{geo}}(N, \mathsf{R})$, such that $\mathsf{R} = \Omega(\sqrt{\log N/N})$. Then the degree of every node u is:

$$d_u = N\pi\mathsf{R}^2(1 + \mathrm{o}(1)) \quad \text{w.h.p.}$$

It is easy to see that the degree distribution of a node follows the Binomial distribution, where $N\pi\mathsf{R}^2$ is the mean. Lemma 3 further suggests the degree distribution of random geometric graphs for large N and appropriate R is concentrated around the mean. Hence, it seems feasible to approximate random geometric graphs by nearest-neighbor networks to compute the hitting times.

Next, we consider *Example* 2 and *Example* 3 from Sec. II-B over a random geometric graph. Figs. 8 and 9 show the simulation results of the maximum hitting time between the farthest pair of nodes for the setting of *Example* 2 and *Example* 3 over $\mathcal{G}_{\sf geo}(N, \mathsf{R})$, where R equals the critical radius of connectivity.

The simulations are carried out as follows. We first randomly generate a geometric graph in $\mathcal{G}_{\sf geo}(N, \mathsf{R})$, such that R scales as $c\sqrt{\log N/N}$ for $c \geq 15$ to ensure high probability of a connected geometric graph. Then we apply Eqn. (7) to compute the maximum hitting time between the farthest pair of nodes, averaged over 1000 different graph instances, where $w_{u,v}$ and $\ell_{u,v}$ are set according to *Example* 2 and *Example* 3.

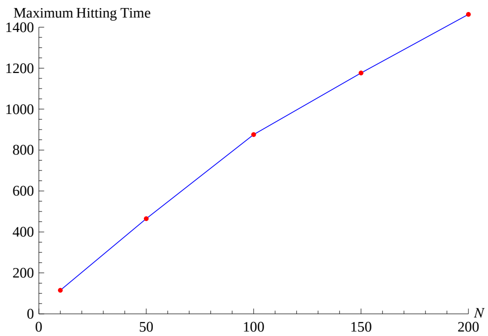


Fig. 8. *Example* 2 (sensor networks with i.i.d. pseudo-random duty cycling) over a random geometric graph $\mathcal{G}_{\sf geo}(N, \mathsf{R})$, where we set $\rho_{\sf dc} = 0.1$.

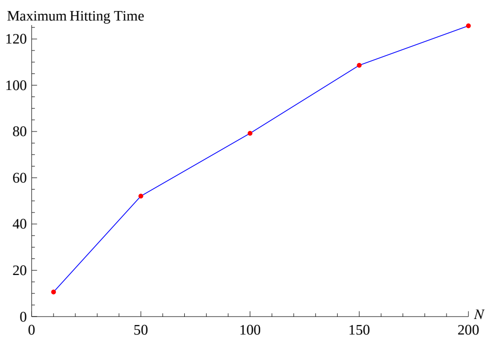


Fig. 9. *Example* 3 (store-carry-forward networks with geographical dependence) over a random geometric graph $\mathcal{G}_{\sf geo}(N, \mathsf{R})$, where we set $\ell_{u,v} = ||u - v||$ and $w_{u,v} = 1$.

We observe that both the maximum hitting times in *Example* 2 and *Example* 3 scale fairly sub-linearly in N. We can explain this observation using heuristics based on the results in the last section. From Theorem 5, we let

$$L(N, \mathsf{R}) \triangleq \frac{N \log N}{r^2}$$

where r^2 is the number of neighbors. By Lemma 3 we obtain $r^2 \sim \log N$ and $L(N, \mathsf{R}) \sim N$.

Then the maximum hitting time for *Example* 2 can be estimated by:

$$\mathbf{H}_{\max} \sim \frac{L(N, \mathsf{R})}{1 - (1 - \rho_{\sf dc})^{N\pi \mathsf{R}^2}}$$

which follows from the regularity suggested by Lemma 3. When $\rho_{\sf dc}$ is small, $(1 - \rho_{\sf dc})^{\pi c^2 \log N} \approx 1 - \rho_{\sf dc}\pi c^2 \log N$, and hence, $\frac{1}{1-(1-\rho_{\sf dc})^{\pi c^2 \log N}} \approx \frac{1}{\rho_{\sf dc}\pi c^2 \log N}$. Thus, the maximum hitting time is $\mathbf{H}_{\max} \sim \frac{N}{\log N}$.

We next consider *Example* 3. The maximum hitting time can be estimated by:

$$\mathbf{H}_{\max} \sim L(N, \mathsf{R})\bar{\ell}(N, \mathsf{R})$$

where $\bar{\ell}(N, \mathsf{R})$ is the expected per-hop latency at each node. We approximate by $\bar{\ell}(N, \mathsf{R}) \sim \mathsf{R}$. Thus, the maximum hitting time is $\mathbf{H}_{\max} \sim \sqrt{N \log N}$.

We observe that the maximum hitting time approximations match reasonably well with the trend as shown by simulations in Figs. 8-9 for moderate values of N (≤ 200).

VI. Conclusion and Discussion

In this paper, we study stateless opportunistic forwarding as random walk on finite graphs, and present several exact results for the hitting times of random walk on 1) arbitrary finite graphs with heterogeneous sojourn times at relaying nodes, 2) one dimensional r-nearest neighbor cycles, 3) two dimensional r-nearest neighbor tori. Particularly, we obtain good approximation formulas for the hitting times on r-nearest neighbor cycles and r-nearest neighbor tori. This has applications in network design as discussed below.

A. Latency-overhead Optimisation

Theorem 3 is useful to estimate the trade-off between latency and RF power consumption (i.e. different transmission radius or different number of nearest neighbors, $2r$). Therefore, $\mathbf{H}_{\frac{n}{2},0}$ is proportional to $\frac{1}{r^2}$ in the one dimensional case. It is well-known that RF power consumption is proportional to r^2. Hence, we can roughly estimate that RF power consumption is proportional to $1/$latency. We expect this can serve as a basis of estimation for the two dimensional case.

Furthermore, Eqns. (10), (14), (15) enable to set up an optimization framework, useful for the design and planning of wireless sensor networks. For instance, consider Eqn. (10) and the problem is to optimize the maximum latency subject to the constraint of the total power consumption below a certain threshold, or the dual problem to optimize the total power consumption subject to the constraint of the maximum latency below a certain threshold. This can be set up as follows:

1) $\min \frac{3n^2}{2(1+r)(1+2r)}$ subject to $n\pi r^2 \leq E_{\max}$, $r \leq r_{\max}$
2) $\min n\pi r^2$ subject to $\frac{3n^2}{2(1+r)(1+2r)} \leq L_{\max}$, $r \leq r_{\max}$

The solutions can be obtained by solving the respective Lagrangian.

References

[1] P. Hui, A. Chaintreau, R. Gass, J. Scott, J. Crowcroft, and C. Diot, "Pocket switched networking: Challenges, feasibility, and implementation issues," in *Proc. of Workshop on Autonomic Communication (WAC)*, Greece, October 2005.

[2] I. Mabrouki, X. Lagrange, and G. Froc, "Random walk based routing protocol for wireless sensor networks," in *Proc. of Conf. on Performance Evaluation Methodologies and Tools (ValueTools)*, France, 2007.

[3] P. Basu and C.-K. Chau, "Opportunistic forwarding in wireless networks with duty cycling," in *Proc. of ACM Workshop on Challenged Networks (CHANTS)*, September 2008.

[4] J. Redi, S. Kolek, K. Manning, C. Partridge, R. Rosales-Hain, R. Ramanathan, and I. Castineyra, “Javelen: An ultra-low energy ad hoc wireless network,” *Ad Hoc Networks Journal*, vol. 5, no. 8, 2008.
[5] S. Nath and P. B. Gibbons, “Communicating via fireflies: Geographic routing on duty cycled sensors,” in *Proc. of Intl. Conf. on Information Processing in Sensor Networks (IPSN)*, Cambridge, MA, April 2007.
[6] L. Lovasz, “Random walk on graphs: A survey,” *Combinatorics, Paul Erdos is Eighty*, vol. 2, pp. 1–46, 1993.
[7] D. Aldous and J. A. Fill, “Reversible Markov chains and random walks on graphs,” http://www.stat.berkeley.edu/∼aldous/RWG/book.html, 1999, lecture notes.
[8] D. A. Levin, Y. Peres, and E. L. Wilmer, “Markov chains and mixing times,” http://www.oberlin.edu/markov/, 2006, lecture notes.
[9] P. G. Doyle and J. L. Snell, *Random Walks and Electrical Networks*. Mathematical Association of America, 1984.
[10] P. Tetali, “Random walks and the effective resistance of networks,” *Journal of Theoretical Probability*, vol. 4, no. 1, pp. 1–46, 1991.
[11] A. K. Chandra, P. Raghavan, R. Smolensky, and P. Tiwari, “The electrical resistance of a graph captures its commute and cover times,” *Computational Complexity*, vol. 6, no. 4, pp. 312–340, 1996.
[12] F. Chung and S.-Y. Yau, “Discrete Green’s functions,” *Journal of Combinatorial Theory, Series A*, vol. 91, pp. 191–214, 2000.
[13] R. B. Ellis, “Discrete Green’s functions for products of regular graphs,” 2003, arXiv:math/0309080v2.
[14] P. Basu and C.-K. Chau, “Latency of opportunistic forwarding in finite regular wireless networks,” in *Proc. of ACM Intl. Workshop on Foundation of Mobile Computing (DIAL M-POMC)*, August 2008.
[15] C. Avin and G. Ercal, “On the cover time and mixing time of random geometric graphs,” *Theoretical Computer Science*, vol. 380, pp. 2–22, June 2007.
[16] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, “Randomized gossip algorithms,” *IEEE Trans. Information Theory*, vol. 52, no. 6, June 2006.
[17] D. Geller, I. Kra, S. Popescu, and S. Simanca, “On circulant matrices,” http://www.math.sunysb.edu/∼sorin/, lecture notes.
[18] “Wikipedia: Trigonometric identities,” http://en.wikipedia.org/wiki/Trigonometric_identity.

VII. APPENDIX

A. Definition of Tensor Product

Suppose $\mathbf{X}$ is a $(r+1)\times(s+1)$ matrix and $\mathbf{Y}$ is a $(p+1)\times(q+1)$ matrix. Define the tensor product matrix as $\mathbf{X}\otimes\mathbf{Y}$, which is a $(r+1)(p+1)\times(s+1)(q+1)$ matrix, such that for $0\le a\le r$, $0\le b\le p$, $0\le c\le s$, $0\le d\le q$, the entry at position $(a(p+1)+b, c(q+1)+d)$ is defined as:

$$(\mathbf{X}\otimes\mathbf{Y})_{a(p+1)+b,\ c(q+1)+d} \triangleq \mathbf{X}_{a,c}\mathbf{Y}_{b,d}$$

B. Proofs

Theorem 1: Given arbitrary sojourn time $\ell_u>0$ for each u, the hitting time and commute time from u to v can be computed by:

$$\mathbf{H}_{u,v} = \sum_{k:\sigma_k\neq 0}\frac{\tilde{W}}{\sigma_k}\Big(\frac{\boldsymbol{\mu}_{k,v}^2}{w_v\ell_v}-\frac{\boldsymbol{\mu}_{k,u}\boldsymbol{\mu}_{k,v}}{\sqrt{w_u\ell_u w_v\ell_v}}\Big)$$
$$\mathbf{C}_{u,v} = \sum_{k:\sigma_k\neq 0}\frac{\tilde{W}}{\sigma_k}\Big(\frac{\boldsymbol{\mu}_{k,u}}{\sqrt{w_u\ell_u}}-\frac{\boldsymbol{\mu}_{k,v}}{\sqrt{w_v\ell_v}}\Big)^2$$

where $\tilde{W}\triangleq\sum_{(u,v)\in\mathcal{E}} w_{u,v}(\ell_{u,v}+\ell_{v,u})=\sum_{u'\in\mathcal{V}} w_{u'}\ell_{u'}$, and σ_k and $\boldsymbol{\mu}_k$ are the k-th eigenvalue and eigenvector of $\tilde{\mathbf{L}}$.

Proof: First, we let $\boldsymbol{\omega}_u=\sqrt{w_u\ell_u}$. Then we observe that

$$\tilde{\mathbf{L}}_{u,u}\boldsymbol{\omega}_u=\sqrt{\frac{w_u}{\ell_u}} \text{ and } \sum_{v\in\mathcal{N}_u}\tilde{\mathbf{L}}_{v,u}\boldsymbol{\omega}_v=-\frac{\sum_{v\in\mathcal{N}_u} w_{u,v}}{\sqrt{w_u\ell_u}}=-\sqrt{\frac{w_u}{\ell_u}}$$

Hence, $\tilde{\mathbf{L}}\boldsymbol{\omega}=0$. That is, 0 is an eigenvalue, and $\boldsymbol{\omega}/\sqrt{\tilde{W}}$ is the corresponding eigenvector of $\tilde{\mathbf{L}}$. Note that since $\mathcal{G}$ is connected, by Frobenius-Perron Theorem, it can be shown that $\sigma_0=0$ is the unique eigenvalue.

Second, define (generalized) discrete Green’s function $\tilde{\mathbf{G}}$:

$$\tilde{\mathbf{G}}\triangleq\sum_{k:\sigma_k\neq 0}\frac{1}{\sigma_k}\boldsymbol{\mu}_k^T\boldsymbol{\mu}_k$$

This is the same one in [12], if $\ell_u=1$ for all u. Since $\boldsymbol{\omega}/\sqrt{\tilde{W}}$ is an eigenvector of $\tilde{\mathbf{L}}$, it follows that

$$\tilde{\mathbf{G}}\boldsymbol{\omega}=0 \text{ and } \tilde{\mathbf{G}}\tilde{\mathbf{L}}=\mathbf{I}-\frac{1}{\tilde{W}}\boldsymbol{\omega}^T\boldsymbol{\omega}$$

Third, denote the all-ones matrix as $\mathbf{J}$. Thus, $(\mathbf{SJ})_{u,v}=\ell_u$. We can rewrite Eqn. (1) as a matrix equation with non-diagonal entry (u,u) as:

$$\big(\mathbf{SJ}+(\mathbf{D}^{-1}\mathbf{A}-\mathbf{I})\mathbf{H}\big)_{u,v}=0,\quad\text{for}\quad u\neq v \tag{16}$$

For diagonal entries $(u=v)$, we observe that

$$\big(\pi\mathbf{SJ}+\pi(\mathbf{D}^{-1}\mathbf{A}-\mathbf{I})\mathbf{H}\big)_u^T=\big(\pi\mathbf{SJ}\big)_u^T=\textstyle\sum_{u'\in\mathcal{V}}\pi_{u'}\ell_{u'}$$

Therefore, this is a diagonal matrix. By $\pi_u=\frac{w_u}{W}$ and Eqn. (16), we can obtain the diagonal entry as:

$$\big(\mathbf{SJ}+(\mathbf{D}^{-1}\mathbf{A}-\mathbf{I})\mathbf{H}\big)_{u,u}=\frac{\sum_{u'\in\mathcal{V}}\pi_{u'}\ell_{u'}}{\pi_u}=\frac{\sum_{u'\in\mathcal{V}}w_{u'}\ell_{u'}}{w_u}=\frac{\tilde{W}}{w_u}$$

Combining these facts, we obtain:

$$\begin{aligned}
(\mathbf{I}-\mathbf{D}^{-1}\mathbf{A})\mathbf{H} &= \mathbf{SJ}-\tilde{W}\mathbf{D}^{-1}\\
\mathbf{D}^{\frac12}(\mathbf{I}-\mathbf{D}^{-1}\mathbf{A})\mathbf{D}^{-\frac12}\mathbf{D}^{\frac12}\mathbf{H}\mathbf{D}^{\frac12} &= \mathbf{D}^{\frac12}(\mathbf{SJ}-\tilde{W}\mathbf{D}^{-1})\mathbf{D}^{\frac12}\\
\mathbf{S}^{-\frac12}(\mathbf{I}-\mathbf{N})\mathbf{S}^{-\frac12}\mathbf{S}^{\frac12}\mathbf{D}^{\frac12}\mathbf{H}\mathbf{D}^{\frac12} &= \mathbf{S}^{-\frac12}\mathbf{D}^{\frac12}(\mathbf{SJ}-\tilde{W}\mathbf{D}^{-1})\mathbf{D}^{\frac12}\\
\tilde{\mathbf{L}}\mathbf{S}^{\frac12}\mathbf{D}^{\frac12}\mathbf{H}\mathbf{D}^{\frac12}\mathbf{S}^{\frac12} &= \mathbf{S}^{-\frac12}\mathbf{D}^{\frac12}(\mathbf{SJ}-\tilde{W}\mathbf{D}^{-1})\mathbf{D}^{\frac12}\mathbf{S}^{\frac12}\\
\tilde{\mathbf{L}}\mathbf{S}^{\frac12}\mathbf{D}^{\frac12}\mathbf{H}\mathbf{D}^{\frac12}\mathbf{S}^{\frac12} &= \tilde{W}(\tfrac{1}{\tilde{W}}\boldsymbol{\omega}^T\boldsymbol{\omega}-\mathbf{I})\\
\tilde{\mathbf{G}}\tilde{\mathbf{L}}\mathbf{S}^{\frac12}\mathbf{D}^{\frac12}\mathbf{H}\mathbf{D}^{\frac12}\mathbf{S}^{\frac12} &= \tilde{W}\tilde{\mathbf{G}}(\tfrac{1}{\tilde{W}}\boldsymbol{\omega}^T\boldsymbol{\omega}-\mathbf{I})\\
(\mathbf{I}-\tfrac{1}{\tilde{W}}\boldsymbol{\omega}^T\boldsymbol{\omega})\mathbf{S}^{\frac12}\mathbf{D}^{\frac12}\mathbf{H}\mathbf{D}^{\frac12}\mathbf{S}^{\frac12} &= -\tilde{W}\tilde{\mathbf{G}}
\end{aligned}$$

The diagonal entry (u,u) of the above matrix equation is:

$$w_u\ell_u\mathbf{H}_{u,u}-\frac{w_u\ell_u}{\tilde{W}}\sum_{w\in\mathcal{V}}w_w\ell_w\mathbf{H}_{w,u}=-\tilde{W}\sum_{k:\sigma_k\neq 0}\frac{1}{\sigma_k}\boldsymbol{\mu}_{k,u}^2$$

while the non-diagonal entry (u,v) is:

$$\sqrt{w_u\ell_u w_v\ell_v}\mathbf{H}_{u,v}-\frac{\sqrt{w_u\ell_u w_v\ell_v}}{\tilde{W}}\sum_{w\in\mathcal{V}}w_w\ell_w\mathbf{H}_{w,u}$$
$$=-\tilde{W}\sum_{k:\sigma_k\neq 0}\frac{1}{\sigma_k}\boldsymbol{\mu}_{k,u}\boldsymbol{\mu}_{k,v}$$

Combining the two equations and using $\mathbf{H}_{u,u}=0$, we obtain Eqn. (7). Eqn. (8) follows from $\mathbf{C}_{u,v}=\mathbf{H}_{u,v}+\mathbf{H}_{v,u}$. ■

Lemma 1: The $(k+1)$-th eigenvalue and eigenvector of the Laplacian $\tilde{\mathbf{L}}$ of r-nearest neighbor cycle $\mathcal{C}_n^r$:

$$\sigma_k=1-\frac{\sum_{j=1}^r a_j\cos\left(\frac{2\pi jk}{n}\right)}{\sum_{j'=1}^r a_{j'}},\ \boldsymbol{\mu}_k=\frac{1}{\sqrt{n}}\Big(1,\epsilon^k,...,\epsilon^{(n-1)k}\Big)^T$$

ϵ is complex number defined as: $\epsilon\triangleq\cos\left(\frac{2\pi}{n}\right)+\mathbf{i}\sin\left(\frac{2\pi}{n}\right)$.

Proof: The adjacency matrix $\mathbf{A}[\mathcal{C}_n^r]$ is:

$$\begin{pmatrix} 0 & a_1 & \dots & a_2 & a_1 \\ a_1 & 0 & \dots & a_3 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & \dots & a_1 & 0 \end{pmatrix}$$

which is a circulant matrix [17]. It is well-known that the k-th eigenvector is $\boldsymbol{\mu}_k$ and the corresponding eignevalue is:

$$a_1\epsilon^k+\dots+a_r\epsilon^{rk}+a_r\epsilon^{(r-1)k}+\dots+a_1\epsilon^{(n-1)k} = 2\sum_{j=1}^{r} a_j \cos\left(\frac{2\pi jk}{n}\right)$$

This completes the proof by $\tilde{\mathbf{L}}[\mathcal{C}_n^r] = \mathbf{I} - \frac{1}{2\sum_{j'=1}^{r} a_{j'}}\mathbf{A}[\mathcal{C}_n^r]$. ■

Lemma 2: Trigonometric identity of Dirichlet kernel [18]:

$$1 + 2\sum_{j=1}^{r}\cos(jx) = \frac{\sin\left((r+\frac{1}{2})x\right)}{\sin(\frac{x}{2})}$$

Lemma 3: When $x \longrightarrow 0$,

$$\frac{1}{1-\frac{1}{r}\sum_{j=1}^{r}\cos(jx)} \approx \frac{12}{(r+1)(2r+1)x^2} \tag{17}$$

Proof: By Lemma 2 and Taylor series expansion at $x = 0$,

$$\frac{\sin\left((r+\frac{1}{2})x\right)}{\sin(\frac{x}{2})} = (2r+1) - \frac{1}{6}r(r+1)(2r+1)x^2 + O(x^4)$$

■

By numerical study, we observe that Eqn. (17) gives a relatively good approximation even for $0 \le x \le \pi$ and small r (see Fig. 10).

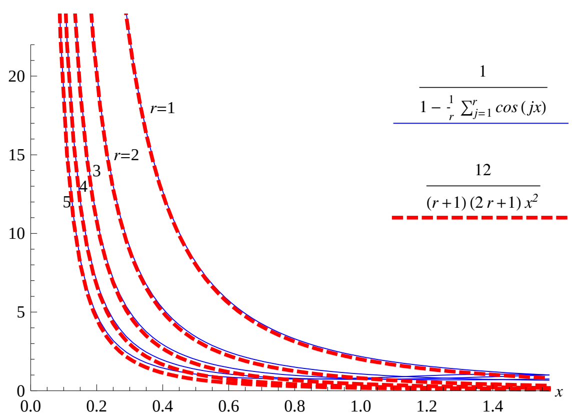


Fig. 10. The comparison of $\frac{1}{1-\frac{1}{r}\sum_{j=1}^{r}\cos(jx)}$ against $\frac{12}{(r+1)(2r+1)x^2}$ for $0 \le x \le \pi$.

Theorem 2: When $n \rightarrow \infty$, the maximum hitting time on r-nearest neighbor cycle $\mathcal{C}_n^r$ can be approximated by:

$$\mathbf{H}_{\frac{n}{2},0} \approx \frac{3n^2}{2(1+r)(1+2r)}$$

Proof: It suffices to consider the case $n = 4c$ for integer c, because we can always extrapolate in the approximation.

$$\mathbf{H}_{\frac{n}{2},0} = \sum_{k=1}^{n-1}\frac{1-\cos(\pi k)}{1-\frac{1}{r}\sum_{j=1}^{r}\cos\left(\frac{2\pi jk}{n}\right)} = \sum_{k=0}^{\frac{n}{2}}\frac{2}{1-\frac{1}{r}\sum_{j=1}^{r}\cos\left(\frac{2\pi j(2k+1)}{n}\right)}$$
$$\overset{(1)}{=} \sum_{k=0}^{\frac{n}{4}}\frac{4}{1-\frac{1}{r}\sum_{j=1}^{r}\cos\left(\frac{2\pi j(2k+1)}{n}\right)} \overset{(2)}{\approx} \sum_{k=0}^{\frac{n}{4}}\frac{12n^2}{\pi^2(r+1)(r+1)(2k+1)^2}$$

where (1) is due to: $\cos\left(\frac{2\pi j(2k+1)}{n}\right) = \cos\left(\frac{2\pi j(n-2k-1)}{n}\right)$, and (2) is due to Lemma 3. We complete the proof by the identity: $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$. ■

Theorem 3: When $r \ll n$, the maximum hitting time on an r-nearest neighbor torus is:

$$\mathbf{H}_{(\frac{n}{2},\frac{n}{2}),(0,0)} = \Theta\left(\frac{n^2\log(n)}{(1+2r)^2}\right)$$

Proof: (Sketch) Without loss of generality, we consider n is even. Since an L^∞ r-nearest neighbor torus can bound L^1 r-nearest neighbor torus, we consider L^∞ r-nearest neighbor torus. Substituting $(u,v) = (\frac{n}{2},\frac{n}{2})$ into Theorem 4,

$$\mathbf{H}_{(\frac{n}{2},\frac{n}{2}),(0,0)} = \sum_{\substack{(k,l)\neq(0,0)\\ k+l \text{ is odd}}}\frac{2}{1-\frac{1}{4r^2+4r}\left(\sum_{i=-r}^{r}\sum_{j=-r}^{r}\cos\left(\frac{2\pi ik}{n}\right)\cos\left(\frac{2\pi jl}{n}\right)-1\right)}$$

$$\text{Let } \mathfrak{a}(x,y) \triangleq \frac{1}{1-\frac{1}{4r^2+4r}\left(\sum_{i=-r}^{r}\sum_{j=-r}^{r}\cos(ix)\cos(jy)-1\right)}$$
$$= \frac{1}{1-\frac{1}{4r^2+4r}\left(\sum_{i=-r}^{r}\sum_{j=-r}^{r}\frac{\cos(ix+jy)+\cos(ix-jy)}{2}-1\right)}$$

Note that for $-\pi \le \theta \le \pi$, $1-\frac{\theta^2}{2} \le \cos(\theta) \le 1-\frac{\theta^2}{5}$. Therefore, for $\frac{-\pi}{2r} \le x,y \le \frac{\pi}{2r}$,

$$\mathfrak{a}(x,y) \ge \frac{1}{1-\frac{1}{4r^2+4r}\left(\sum_{i=-r}^{r}\sum_{j=-r}^{r}\frac{1}{2}\left(2-\frac{(ix+jy)^2}{2}-\frac{(ix-jy)^2}{2}\right)\right)} = \frac{24}{(1+2r)^2(x^2+y^2)}$$
$$\mathfrak{a}(x,y) \le \frac{1}{1-\frac{1}{4r^2+4r}\left(\sum_{i=-r}^{r}\sum_{j=-r}^{r}\frac{1}{2}\left(2-\frac{(ix+jy)^2}{5}-\frac{(ix-jy)^2}{5}\right)\right)} = \frac{62.5}{(1+2r)^2(x^2+y^2)}$$

This follows from $\sum_{i=-r}^{r}(1-\frac{(ix)^2}{k}) = \frac{(1+2r)(3k-rx^2-r^2x^2)}{3k}$, and some algebraic simplifications that:

$$\frac{1}{1-\frac{1}{4r^2+4r}\left(\sum_{i=-r}^{r}\sum_{j=-r}^{r}\frac{1}{2}\left(2-\frac{(ix+jy)^2}{k_1}-\frac{(ix-jy)^2}{k_2}\right)\right)} = \frac{24k_1k_2}{(1+2r)^2(x^2+y^2)(k_1+k_2)}$$

Hence, $\mathfrak{a}(x,y) = \Theta\left(\frac{1}{(1+2r)^2(x^2+y^2)}\right)$.

Since $r \ll n$,

$$\mathbf{H}_{(\frac{n}{2},\frac{n}{2}),(0,0)} = \sum_{\substack{(k,l)\neq(0,0)\\ k+l \text{ is odd}}} 2\mathfrak{a}\left(\frac{2\pi k}{n},\frac{2\pi l}{n}\right) \approx \Theta\left(\sum_{k=1}^{n-1}\sum_{l=1}^{n-1}\frac{n^2}{(1+2r)^2(k^2+l^2)}\right)$$

Since $\frac{1}{k^2+l^2}$ is a decreasing function in k and l, when n is large, we obtain:

$$\sum_{k=1}^{n-1}\sum_{l=1}^{n-1}\frac{1}{k^2+l^2} \approx \Theta\left(\int_1^n\int_1^n \frac{1}{k^2+l^2}\,\mathrm{d}l\,\mathrm{d}k\right)$$

Note that

$$\int_1^b \frac{1}{c^2+x^2}\,\mathrm{d}x = \frac{1}{c}\left(\arctan\left(\frac{b}{c}\right)-\arctan\left(\frac{1}{c}\right)\right)$$

Since $1 \ll n$ and $0 < \arctan(x) \le \frac{\pi}{2}$ for positive x,

$$\int_1^n\int_1^n\frac{1}{k^2+l^2}\,\mathrm{d}l\,\mathrm{d}k \approx \int_1^n \Theta\left(\frac{1}{k}\right)\mathrm{d}k = \Theta\left(\log(n)\right)$$

■