Combinatorial Optimization of AC Electric Power Systems

Bridging Power Engineering & Computer Science

[Tutorial]

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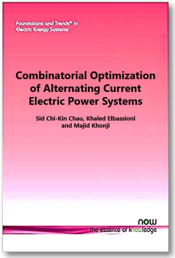

https://users.cecs.anu.edu.au/~sid.chau/FnT.html

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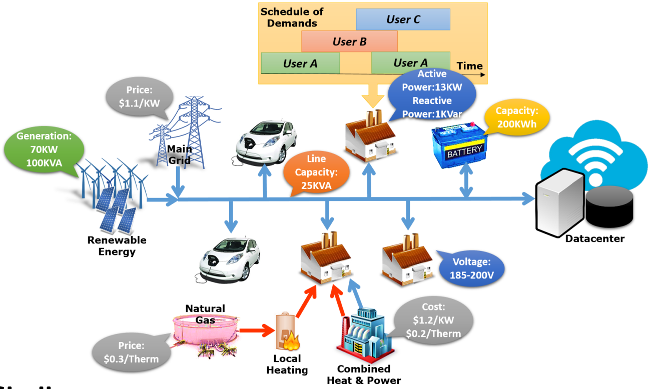
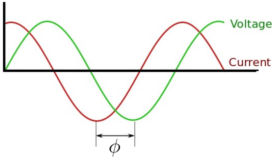
Big Picture

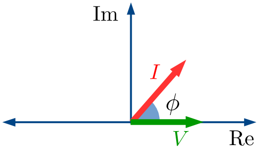


- **Challenges**
  - *Complexity*: Non-linear constraints of AC power systems, energy storage, (combined heat & power) generators, etc.
  - *Uncertainty*: Intermittent renewable energy, dynamic market prices, fluctuating demands (e.g. EVs, datacenters)
- **Goal:** How to optimize the management of energy demands and resources efficiently and intelligently?

AC Electrical Systems 101

- Circular motion of dynamo generator $\Rightarrow$
  Periodic current and voltage





- **Expressed by complex numbers**:
  $V = |V|e^{\mathbf{i}\omega t}$, $I = |I|e^{\mathbf{i}(\omega t+\phi)}$

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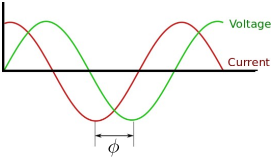

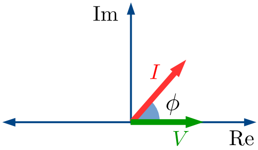

- **Expressed by complex numbers**:
  $V = |V|e^{\mathbf{i}\omega t}$, $I = |I|e^{\mathbf{i}(\omega t+\phi)}$
- Power: $S = V \times I^*$ (also a complex number)
  - *Active* power: $\mathrm{Re}(S)$
  - *Reactive* power: $\mathrm{Im}(S)$
  - *Apparent* power: $|S| = \sqrt{\mathrm{Re}(S)^2 + \mathrm{Im}(S)^2}$

AC Electrical Systems 101

- **Active power** $(\mathrm{Re}(S))$
  - Deliver useful work at loads (unit: Watt)
  - Demands: *positive* active power
  - Supplies: *negative* active power
- **Reactive power** $(\mathrm{Im}(S))$
  - Contribute to electricity flows (unit: VAR):
  - Inductors: *positive* reactive power
  - Capacitors: *negative* reactive power

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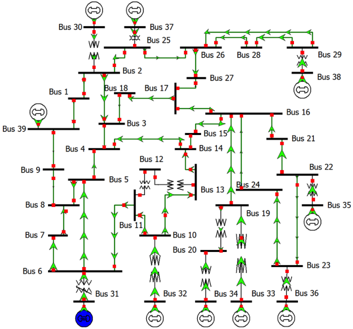
  - Contribute to electricity flows (unit: VAR):
  - Inductors: *positive* reactive power
  - Capacitors: *negative* reactive power
- **Power factor** $(\mathrm{PF} = \frac{\mathrm{Re}(S)}{|S|})$
  - Normalized measure of reactive power
  - Power electronic standards usually require limited reactive power in most domestic appliances $(\mathrm{PF} \geq 0.8)$

	Compressors/Pumps	Motors	Machining
Power factor	0.75-0.8	0.5-0.8	0.4-0.65

Power Networks 101



- Electric networks: set of nodes ($\mathcal{V}$), set of edges ($\mathcal{E}$)
  - Voltage at node i: V_i
  - Impedance between i & j: $z_{i,j}$
  - Current from i to j: $I_{i,j}$
  - Transmitted power from i to j: $S_{i,j}$

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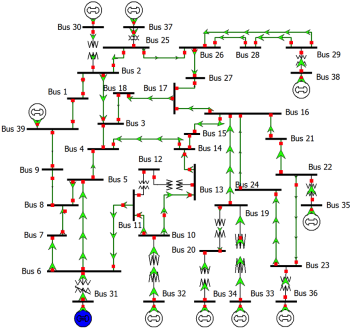

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- **Direct Current (DC) electric systems**:
  - $(V_i, I_{i,j}, z_{i,j}, S_{i,j})$ are *real* numbers ($\mathbb{R}$)
- **Alternating Current (AC) electric systems**:
  - $(V_i, I_{i,j}, z_{i,j}, S_{i,j})$ are *complex* numbers ($\mathbb{C}$)

Power Flow Model

- Basic electricity laws:
  - *Ohm's Law*: For each edge $(i, j) \in \mathcal{E}$,

$$V_i - V_j = z_{i,j} I_{i,j}.$$

  - *Kirchhoff's Current Law*: For each node $i \in \mathcal{V}$,

$$\sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} I_{i,j} = 0.$$

  - *Electric Power Formula*: For each edge $(i, j) \in \mathcal{E}$,

$$S_{i,j} = V_i I^*_{i,j}.$$

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  $$S_{i,j} = V_i I_{i,j}^*.$$

Definition (Branch Flow Model)

$$S_{i,j} = z_{i,j}|I_{i,j}|^2 + \sum_{l:(j,l) \in \mathcal{E}} S_{j,l}, \qquad \text{for all } j \in \mathcal{V},$$
$$V_i - V_j = z_{i,j} I_{i,j}, \qquad \text{for all } (i,j) \in \mathcal{E},$$
$$S_{i,j} = V_i I_{i,j}^*, \qquad \text{for all } (i,j) \in \mathcal{E}.$$

Power Flow Model

Definition (Branch Flow Model with Angle Relaxation)

Let $v_i = |V_i|^2$ and $\ell_{i,j} = |I_{i,j}|^2$. Omit the phase angles:

$$S_{i,j} = z_{i,j}\ell_{i,j} + \sum_{l:(j,l)\in\mathcal{E}} S_{j,l}, \qquad \text{for all } j \in \mathcal{V},$$

$$v_i - v_j = 2\mathrm{Re}(z_{i,j}^* S_{i,j}) - |z_{i,j}|^2 \ell_{i,j}, \qquad \text{for all } (i,j) \in \mathcal{E},$$

$$|S_{i,j}|^2 = v_i \ell_{i,j}, \qquad \text{for all } (i,j) \in \mathcal{E}.$$

- Angle relaxation reduces complex-valued variables, and hence is more tractable
- Always possible to recover $(V_i, I_{i,j})$ from $(v_i, \ell_{i,j})$, when it is a **tree** network [Low et al.]
- Assume a tree (radial) distribution network

Control Variables & Operating Constraints

- Each user $k \in \mathcal{N}$ can control individual demand s_k
- Some have **discrete** (inelastic) power demands ($\mathcal{I} \subseteq \mathcal{N}$)
  - A discrete demand is either completely satisfied or dropped
  - E.g., equipment is either switched on with a fixed power consumption rate or completely off
  - $s_k = \bar{s}_k x_k$, where $x_k \in \{0, 1\}$
- Others have **continuous** (elastic) demands ($\mathcal{F} \subseteq \mathcal{N}$)
  - $\underline{s}_k \leq s_k \leq \bar{s}_k$

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- Operating Constraints of Power Systems:
  - *Power Capacity Constraints*: $|S_{i,j}| \leq \overline{S}_{i,j}$
  - *Current Thermal Constraints*: $\ell_{i,j} \leq \overline{\ell}_{i,j}$
  - *Voltage Constraints*: $\underline{v}_j \leq v_j \leq \overline{v}_j$

Optimal Power Flow Problem (OPF)

Definition (Optimal Power Flow Problem)

$$(\text{OPF}) \quad \max_{s_0, s, x, S, v, \ell} f(s_0, s)$$

subject to

$$\ell_{i,j} = \frac{|S_{i,j}|^2}{v_i}, \qquad \forall (i,j) \in \mathcal{E},$$

$$S_{i,j} = \sum_{k \in \mathcal{U}_j} s_k + \sum_{l:(j,l) \in \mathcal{E}} S_{j,l} + z_{i,j}\ell_{i,j}, \qquad \forall (i,j) \in \mathcal{E},$$

$$v_j = v_i + |z_{i,j}|^2 \ell_{i,j} - 2\text{Re}(z_{i,j}^* S_{i,j}), \qquad \forall (i,j) \in \mathcal{E},$$

$$\underline{v}_j \le v_j \le \overline{v}_j, \qquad \forall j \in \mathcal{V}^+,$$

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$$s_k = \bar{s}_k x_k, \quad x_k \in \{0, 1\}, \qquad \forall k \in \mathcal{I},$$

$$v_j \in \mathbb{R}^+, \forall j \in \mathcal{V}^+, \ \ell_{i,j} \in \mathbb{R}^+, S_{i,j} \in \mathbb{C}, \quad \forall (i,j) \in \mathcal{E}.$$

Hardness of OPF

- **Non-Convex Constraints** involving complex-valued variables and parameters of AC electric power systems
  - E.g. $\ell_{i,j} = \frac{|S_{i,j}|^2}{v_i}$
- **Combinatoric Constraints** involving binary control decision variables for the operations of power systems
  - E.g. $s_k = \bar{s}_k x_k, \quad x_k \in \{0, 1\}$
- Even without combinatoric constraints, checking feasibility of OPF (with voltage & capacity constraints) is NP-hard
  - Ref. [Lehmann et al.] [Verma]
- Need to relax some constraints
  - Namely, considering a less restrictive optimization problem

Convex Relaxation of OPF

Definition (Convex Relaxed OPF)

$$\text{(cOPF)} \quad \max_{s_0,s,x,S,v,\ell} f(s_0, s)$$

subject to

$$\boxed{\ell_{i,j} \geq \frac{|S_{i,j}|^2}{v_i}}, \quad \forall (i,j) \in \mathcal{E},$$

$$S_{i,j} = \sum_{k \in \mathcal{U}_j} s_k + \sum_{l:(j,l) \in \mathcal{E}} S_{j,l} + z_{i,j}\ell_{i,j}, \quad \forall (i,j) \in \mathcal{E},$$

$$v_j = v_i + |z_{i,j}|^2 \ell_{i,j} - 2\mathrm{Re}(z_{i,j}^* S_{i,j}), \quad \forall (i,j) \in \mathcal{E},$$

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$$\ell_{i,j} \leq \overline{\ell}_{i,j}, \quad \forall (i,j) \in \mathcal{E},$$

$$\underline{s}_k \leq s_k \leq \bar{s}_k, \quad \forall k \in \mathcal{F},$$

$$s_k = \bar{s}_k x_k, \quad x_k \in \{0,1\}, \quad \forall k \in \mathcal{I},$$

$$v_j \in \mathbb{R}^+, \forall j \in \mathcal{V}^+, \ \ell_{i,j} \in \mathbb{R}^+, S_{i,j} \in \mathbb{C}, \quad \forall (i,j) \in \mathcal{E}.$$

Convex Relaxation of OPF

- *Second Order Cone Problem*: Easier convex problem with polynomial-time algorithms
- *Exactness*: Solution of $\mathrm{cOPF} \Rightarrow$ Solution of OPF
  - Under certain sufficient conditions [Low et al.] [Huang et al.]
  - Conversion is polynomial-time

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Assumptions

A1: $z_e \geq 0, \forall e \in \mathcal{E}$, naturally holds in distribution networks

A2: $\underline{v}_j < v_0 < \overline{v}_j, \forall j \in \mathcal{V}^+$

C2: Given a solution s, it satisfies

$$\sum_{k \in \mathcal{N}_j} \text{Re}(z^*_{h,l} s_k) \geq 0 \quad \forall j \in \mathcal{V}^+, (h,l) \in \mathcal{E}_j \cup \{(i,j) \in \mathcal{E}\}$$

where $\mathcal{N}_j$ is the set of attached users within subtree from node j, and $\mathcal{E}_j$ is the set of edges of subtree node j

Convex Relaxation of OPF

Theorem

Assuming A1,A2,C2, *an optimal solution to* cOPF, $F^{\star}= (s_0^{\star}, s^{\star}, x^{\star}, S^{\star}, v^{\star}, \ell^{\star})$ *can be converted to an optimal solution to* OPF, $\hat{F}^{\star}= (\hat{s}_0^{\star}, s^{\star}, x^{\star}, \hat{S}^{\star}, \hat{v}^{\star}, \hat{\ell}^{\star})$, *in polynomial-time, by solving the following convex problem:*

$$(\widehat{\mathrm{cOPF}}[F^{\star}]) \min_{s_0, S, v, \ell} \sum_{e \in \mathcal{E}} \ell_e$$

$$\text{subject to} \quad \text{constraints of } \mathrm{cOPF}$$

$$s = s^{\star}$$

$$f(s_0, s^{\star}) \geq f(s_0^{\star}, s^{\star})$$

Convex Relaxation of OPF

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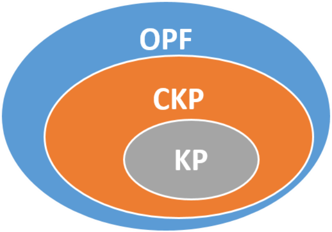
Question

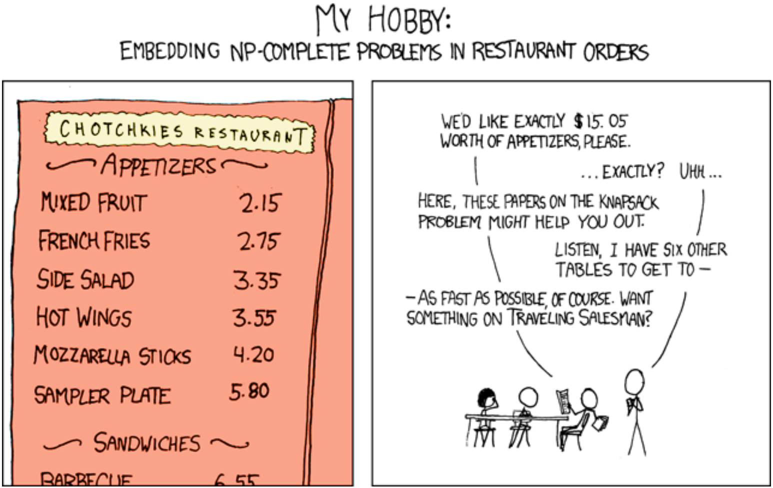
How to solve cOPF with discrete demands?

Main Ideas



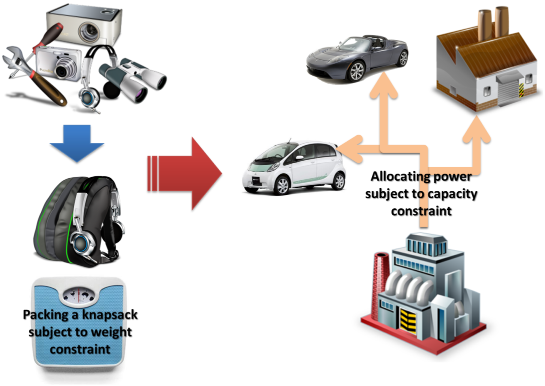
- Solve OPF by studying simplified problems
- **Complex-demand Knapsack Problem** (CKP)
  - Single-capacitated AC system without impedance
  - Demands are complex numbers
- **Knapsack Problem** (KP)
  - Classical computer science problem
  - Packing discrete items subject to capacity constraint
  - Demands are non-negative real numbers

What is Knapsack Problem?



Complex-demand Knapsack Problem (CKP)

Knapsack Problem ⇒ Complex-demand Knapsack Problem



(1D) Knapsack Problem (KP)

- $\mathcal{N} = \{1, \ldots, n\}$: a set of users (or items)
- s_k^{R}: positive real-valued demand of k-th user (e.g. weight)
- u_k: utility of k-th user when s_k^{R} is satisfied (e.g. value)
- C: real-valued capacity
- x_k: decision variable of allocation
  - $x_k = 1$, if k-th user's demand is satisfied
  - $x_k = 0$, otherwise

Definition (1DKP)

$$\max_{x_k \in \{0,1\}} \sum_{k \in \mathcal{N}} x_k u_k$$

subject to

$$\sum_{k \in \mathcal{N}} s_k^{\mathrm{R}} x_k \leq C$$

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- 1DKP is NP-Hard, but can be approximately solved efficiently

Complex-demand Knapsack Problem (CKP)

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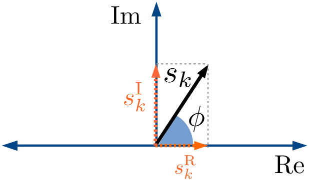

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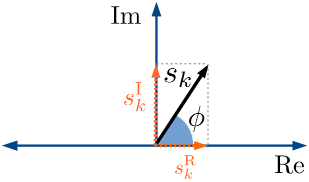

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- Let $u(x) \triangleq \sum_k u_k x_k$ be the objective value of x

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Definition

Given $\alpha \in (0,1]$ and $\beta \geq 1$, a bi-criteria (α, β)-approximation to CKP is $(\hat{x}_k)_{k \in \mathcal{N}}$ satisfying:

$$u(\hat{x}) \geq \alpha \cdot u(x^*)$$
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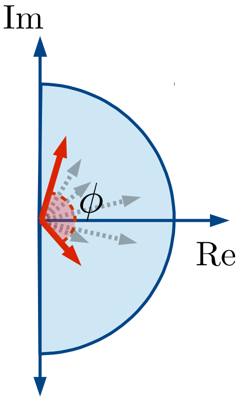
- **FPTAS**: PTAS with running time also polynomial in $1/\epsilon$

Results for CKP

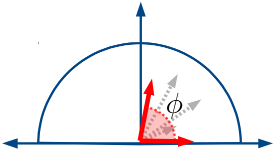


- Let ϕ be the maximum angle between any two demands
- The (in)approximability is dependent on ϕ
- CKP is *rotational invariant*
  - When all demands are rotated by the same angle

Results for CKP

$\text{CKP}[0, \frac{\pi}{2}]$:

- **PTAS** $(1-\epsilon, 1)$-approx
- **No FPTAS** unless P=NP



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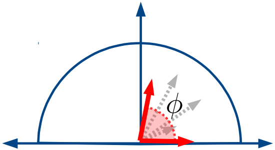
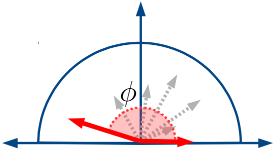


$\mathrm{CKP}[\frac{\pi}{2}, \pi - \varepsilon]$:

- $\varepsilon = 1/\mathrm{poly}(n)$
- **Bi-criteria FPTAS** $(1, 1{+}\epsilon)$-approx
- **No** $(\alpha, 1)$**-approximation** unless $\mathrm{P{=}NP}$



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$\mathrm{CKP}[\frac{\pi}{2}, \pi - \varepsilon]$:

- $\varepsilon = 1/\text{poly}(n)$
- **Bi-criteria FPTAS** $(1, 1{+}\epsilon)$-approx
- **No** $(\alpha, 1)$**-approximation** unless $\mathrm{P{=}NP}$

$\mathrm{CKP}[\frac{\pi}{2}, \pi - \varepsilon]$:

- $\varepsilon = 1/\text{super-pol}(n)$
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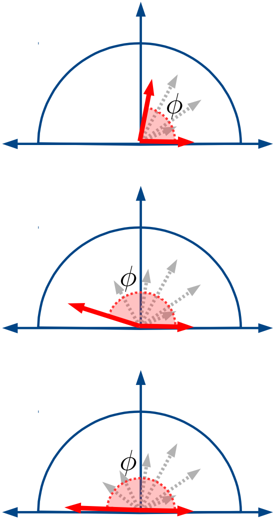
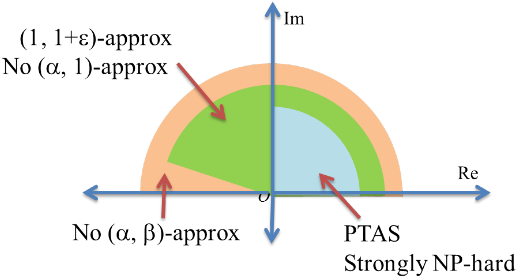
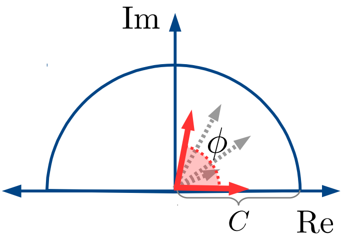


Results for CKP



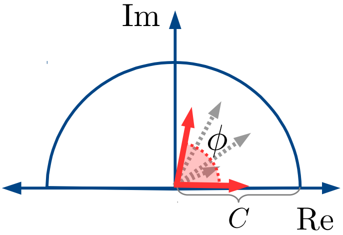
- Other results: Greedy Algorithm
  - Running time $O(n \log n)$
  - Constant factor approximation algorithm ($\alpha = \frac{1}{2}\cos\frac{\phi}{2}$)

PTAS for $\mathrm{CKP}[0, \frac{\pi}{2}]$



- Assume $\phi \in [0, \frac{\pi}{2}]$ (i.e. bounded power factor ≥ 0.75)

PTAS for $\mathrm{CKP}[0, \frac{\pi}{2}]$



- Assume $\phi \in [0, \frac{\pi}{2}]$ (i.e. bounded power factor ≥ 0.75)
- Basic Ideas:
  1. Fix partial solution by guessing
  2. Solve convex relaxation with fractional solution
  3. Round the obtained fractional solution to integral solution
  4. Enumerate all possible partial guessing and find the best solution

PTAS for $\mathrm{CKP}[0, \frac{\pi}{2}]$

- Recall integer quadratic formulation:

$$\max_{x_k} \sum_{k \in \mathcal{N}} u_k x_k$$

$$\text{subject to } \Big(\sum_{k \in \mathcal{N}} s_k^{\mathrm{R}} \cdot x_k\Big)^2 + \Big(\sum_{k \in \mathcal{N}} s_k^{\mathrm{I}} \cdot x_k\Big)^2 \le C^2,$$

$$x_k \in \{0, 1\}, \ \forall k \in \mathcal{N}$$

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Definition (Quadratic Relaxation)

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$$\text{subject to } \Big(\sum_{k \in \mathcal{N}} s_k^{\mathrm{R}} \cdot x_k \Big)^2 + \Big(\sum_{k \in \mathcal{N}} s_k^{\mathrm{I}} \cdot x_k \Big)^2 \le C^2,$$

$$x_k \in [0, 1], \ \forall k \in \mathcal{N}$$

- The relaxation is *convex* because $s_k^{\mathrm{R}}, s_k^{\mathrm{I}} \ge 0$ for all k

PTAS for $\mathrm{CKP}[0, \frac{\pi}{2}]$

- Fix some subsets $I_1, I_0 \subseteq \mathcal{N}$
- Quadratic Relaxation with *partial guessing* of variables $\{x_k\}$:

Definition ($\mathrm{rCKP}[I_1, I_0]$)

$$\max_{x_k} \sum_{k \in \mathcal{N}} u_k x_k$$

$$\text{subject to } \Big(\sum_{k \in \mathcal{N}} s_k^{\mathrm{R}} \cdot x_k\Big)^2 + \Big(\sum_{k \in \mathcal{N}} s_k^{\mathrm{I}} \cdot x_k\Big)^2 \leq C^2,$$

$$x_k \in [0,1], \ \forall k \in \mathcal{N} \backslash (I_0 \cup I_1)$$

$$x_k = 0, \ \forall k \in I_0$$

$$x_k = 1, \ \forall k \in I_1$$

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- Carefully selection of I_1, I_0:
  - $|I_1| \le \frac{4}{\epsilon}$ (depending on ϵ)
  - $I_0 = \{k \in \mathcal{I} \backslash I_1 \mid u_k > \min_{k' \in I_1} u_{k'}\}$
  - *Intuition*: I_0, I_1 are some users with large utilities

PTAS for $\mathrm{CKP}[0, \frac{\pi}{2}]$

- Let x' be an optimal solution to $\mathrm{rCKP}[I_1, I_0]$

Definition ($\mathrm{LP}[x', I_1 \cup I_0]$)

$$\max_{x_k} \sum_{k \in \mathcal{N}} u_k x_k$$

$$\text{subject to } \sum_{k \in \mathcal{N}} s_k^{\mathrm{R}} \cdot x_k \le \sum_{k \in \mathcal{N}} s_k^{\mathrm{R}} \cdot x'_k,$$

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$$x_k = x'_k, \ \forall k \in I_1 \cup I_0$$

- Because of basic solution of LP, at most 2 coordinates in optimal basic solution to $\mathrm{LP}[x', I_1 \cup I_0]$ are fractional
  - Basic solutions are *vertices* in the polytope of feasible solution set
  - Rounding fractional components down to integral components has limited deviation from optimal

PTAS for $\mathrm{CKP}[0, \frac{\pi}{2}]$

Algorithm $\mathrm{PTAS\text{-}CKP}$

- Pick a guess of partial solution by choosing I_1, I_0
  - Each guess sets variables x_k in I_1 be 1 and I_0 be 0
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$$\hat{x}_k = \lfloor x''_k \rfloor, \quad \forall k \in \mathcal{N}$$

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Theorem

For any $\epsilon > 0$, $\mathrm{PTAS\text{-}CKP}$ obtains $(1-\epsilon, 1)$-approximation in polynomial time.

Convex Relaxation of OPF

Definition (Convex Relaxed OPF)

$$\text{(cOPF)} \quad \max_{s_0,s,x,S,v,\ell} f(s_0, s)$$

subject to

$$\boxed{\ell_{i,j} \geq \frac{|S_{i,j}|^2}{v_i}}, \qquad \forall (i,j) \in \mathcal{E},$$

$$S_{i,j} = \sum_{k \in \mathcal{U}_j} s_k + \sum_{l:(j,l)\in\mathcal{E}} S_{j,l} + z_{i,j}\ell_{i,j}, \qquad \forall (i,j) \in \mathcal{E},$$

$$v_j = v_i + |z_{i,j}|^2 \ell_{i,j} - 2\mathrm{Re}(z_{i,j}^* S_{i,j}), \qquad \forall (i,j) \in \mathcal{E},$$

$$\underline{v}_j \leq v_j \leq \overline{v}_j, \qquad \forall j \in \mathcal{V}^+,$$

$$|S_{i,j}| \leq \overline{S}_{i,j}, \ |S_{j,i}| \leq \overline{S}_{i,j}, \qquad \forall (i,j) \in \mathcal{E},$$

$$\ell_{i,j} \leq \overline{\ell}_{i,j}, \qquad \forall (i,j) \in \mathcal{E},$$

$$\underline{s}_k \leq s_k \leq \bar{s}_k, \qquad \forall k \in \mathcal{F},$$

$$s_k = \bar{s}_k x_k, \quad x_k \in \{0,1\}, \qquad \forall k \in \mathcal{I},$$

$$v_j \in \mathbb{R}^+, \forall j \in \mathcal{V}^+, \ \ell_{i,j} \in \mathbb{R}^+, S_{i,j} \in \mathbb{C}, \quad \forall (i,j) \in \mathcal{E}.$$

Hardness Results for OPF

- OPF_V: OPF with voltage constraints only ($\underline{v}_j \le v_j \le \overline{v}_j$)

Theorem

Unless P=NP, there is no (α, β)-approximation for OPF_V (even when $|\mathcal{E}| = 1$), for any α and β that have polynomial number of bits in n.

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Remark

To obtain approximation algorithms, one has to relax some constraints

Hardness Results for OPF

- OPF_C: OPF with capacity constraints only $(|S_{i,j}| \le \overline{S}_{i,j})$

Theorem

Unless $\mathrm{P{=}NP}$*, there exists no* (α, β)*-approximation for* OPF_C *in general networks, even in purely resistive electric networks (i.e.* $\mathrm{Im}(z_{i,j}) = 0$ *for all* $(i,j) \in \mathcal{E}$ *and* $\mathrm{Im}(s_k) = 0$ *for all* $k \in \mathcal{N}$*).*

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Remark

To obtain approximation algorithms, one has to consider acyclic networks (i.e. trees)

Assumptions on OPF

- OPF with discrete demands is hard to solve
- Some assumptions are required to facilitate the solutions

Assumptions

A1: $z_e \geq 0, \forall e \in \mathcal{E}$, naturally holds in distribution networks

A2: $\underline{v}_j < v_0 < \overline{v}_j, \forall j \in \mathcal{V}^+$

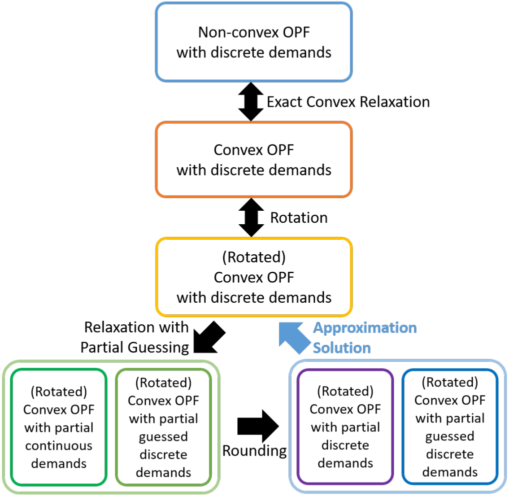
A3: $\mathrm{Re}(z_e^* \overline{s}_k) \geq 0, \forall k \in \mathcal{I}, e \in \mathcal{E}$

- Namely, the phase angle difference between any z_e and s_k for $k \in \mathcal{I}$ is at most $\frac{\pi}{2}$
- This assumption holds, if discrete demands do not have large negative reactive power

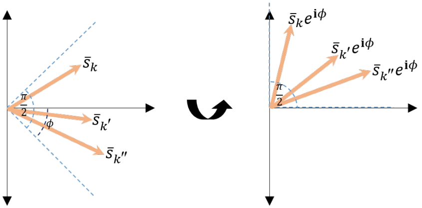
A4: $\left| \angle \overline{s}_k - \angle \overline{s}_{k'} \right| \leq \frac{\pi}{2}$ for any $k, k' \in \mathcal{I}$

- Namely, discrete demands have similar power factors
- It can also be restated as $\mathrm{Re}(\overline{s}_k^* \overline{s}_{k'}) \geq 0$

Basic Ideas of PTAS for OPF



Rotational Invariance



- If complex-valued parameters z_e and s_k are rotated by the same angle (say ϕ)
- And objective func $f(s_0, s)$ is counter-rotated by ϕ in s_0

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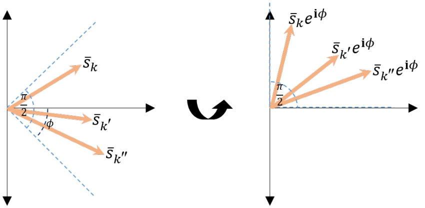

- If complex-valued parameters z_e and s_k are rotated by the same angle (say ϕ)
- And objective func $f(s_0, s)$ is counter-rotated by ϕ in s_0
- Then there is a bijection between the rotated OPF and original OPF

$$\left(z_e, s_k, f(s_0, s)\right) \Leftrightarrow \left(z_e e^{\mathbf{i}\phi}, s_k e^{\mathbf{i}\phi}, f(s_0 e^{-\mathbf{i}\phi}, s)\right)$$

- Therefore, assume all s_k, z_e are in the first quadrant

PTAS for OPF

- Assume constant-sized network ($|\mathcal{V}^+| = |\mathcal{E}| = m$)
- But number of users ($|\mathcal{N}| = n$) is a scalable parameter
- Define a variant of cOPF with partially guessing:

Definition (P1[I_0, I_1])

$$\max_{s_0,s,x,S,v,\ell} \quad f(s_0, s)$$

subject to Constraints of cOPF

$$s_k = \overline{s}_k x_k, \ \forall k \in \mathcal{I},$$

$$x_k = 1, \ \forall k \in I_1, \quad x_k = 0, \ \forall k \in I_0,$$

$$x_k \in [0,1], \ \forall k \in \mathcal{I} \backslash (I_0 \cup I_1)$$

- Let optimal solution of $\mathrm{P1}$ be $F' = \big(s'_0, s', x', S', v', \ell'\big)$

PTAS for OPF

- Define $\overline{f}_k \triangleq f_k(1)$ for $k \in \mathcal{I}$, and define LP as:

Definition (P2$[F', I_0 \cup I_1]$)

$$\max_{x_k \in [0,1], k \in \mathcal{I}'} \sum_{k \in \mathcal{I}} \overline{f}_k x_k$$

subject to $0 \leq \sum_{k \in \mathcal{N}} \mathrm{Re}\Big(\sum_{(h,l) \in \mathcal{P}_k \cap \mathcal{P}_j} z_{h,l}^* s_k \Big)$

$$\leq \sum_{k \in \mathcal{N}} \mathrm{Re}\Big(\sum_{(h,l) \in \mathcal{P}_k \cap \mathcal{P}_j} z_{h,l}^* s'_k \Big), \forall j \in \mathcal{V}^+$$

$$\sum_{k \in \mathcal{N}_j} \mathrm{Re}(s_k) \leq \sum_{k \in \mathcal{N}_j} \mathrm{Re}(s'_k), \forall j \in \mathcal{V}^+$$

$$\sum_{k \in \mathcal{N}_j} \mathrm{Im}(s_k) \leq \sum_{k \in \mathcal{N}_j} \mathrm{Im}(s'_k), \forall j \in \mathcal{V}^+$$

$$s_k = \overline{s}_k x_k, \ \forall k \in \mathcal{N} \setminus (I_0 \cup I_1)$$

$$s_k = s'_k, \ \forall I_0 \cup I_1$$

PTAS for OPF

Algorithm PTAS-cOPF

- Guess partial solution by I_1, I_0, where $|I_1| \leq \frac{4m}{\epsilon}$
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Theorem

Assuming A1,A2,A3,A4, *for any* $\epsilon > 0$, PTAS-cOPF *obtains* $(1-\epsilon, 1)$*-approximation in polynomial time for constant-sized tree electric networks.*

Simulation Settings

- Compare approx algorithm against the optimal solutions
- Use numerical solver (Gurobi) to obtain optimal solution

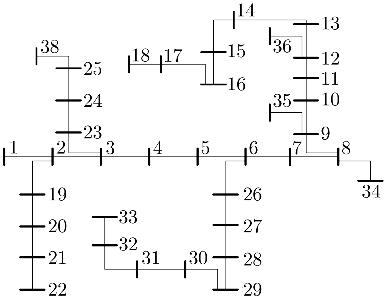


Figure: Test electric network from [Singh, Baran]

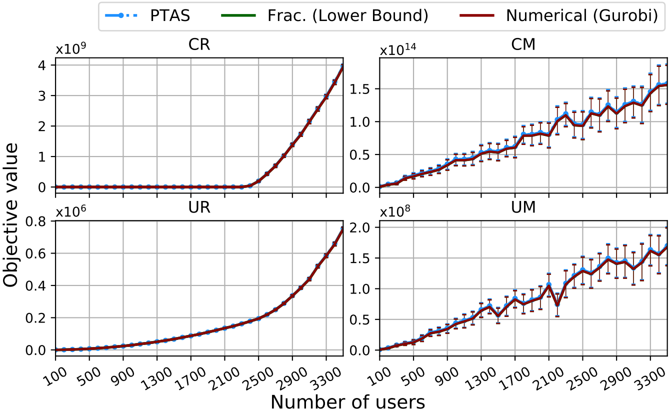
Simulation Settings

- **User Types**:
  1. *Residential* (R): Users have small power demands ranging from 500VA to 5KVA
  2. *Industrial* (I): Users have big demands ranging from 300KVA to 1MVA with non-negative reactive power
  3. *Mixed* (M): Users consist of a mix of industrial and residential users, with less than 20% industrial users
- **Cost-Demand Correlation**:
  1. *Correlated Setting* (C): The objective of each user is a function of demand:

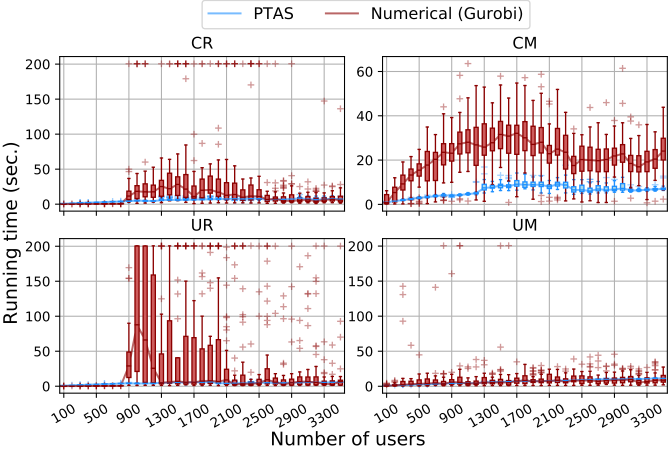
$$f_k(|s_k|) = a \cdot |s_k|^2 + b \cdot |s_k| + c$$

  2. *Uncorrelated Setting* (U): Coefficients in f_k of each user are generated randomly according to $|s_{\max}(k)|$
     - For industrial user, $|s_{\max}(k)| = 1$MVA, otherwise $|s_{\max}(k)| = 5$KVA

Simulation Results



Running Time



Summary

- Complex-demand Knapsack Problem (CKP):
  - Hardness results
  - PTAS algorithm
- Optimal Power Flow with discrete demands (OPF):
  - Hardness results
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- Optimal Power Flow with discrete demands (OPF):
  - Hardness results
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- *Other results*:
  - Bi-criteria FPTAS
  - Scheduling problem
  - Scalable-sized networks
  - Truthful mechanisms

Publications

- Related Papers
  - Chau, Elbassioni, Khonji. *"Truthful Mechanisms for Combinatorial Allocation of Electric Power in Alternating Current Electric Systems for Smart Grid"*, **ACM Trans. on Economics and Computation (TEAC)**, 2016
  - Karapetyan, Khonji, Chau, Elbassioni, Zeineldin. *"Efficient Algorithm for Scalable Event-based Demand Response Management in Microgrids"*, **IEEE Trans. on Smart Grid (TSG)**, 2018
  - Khonji, Chau, Elbassioni. *"Optimal Power Flow with Inelastic Demands for Demand Response in Radial Distribution Networks"*, **IEEE Trans. on Control of Network Systems (TCNS)**, 2018
  - Khonji, Chau, Elbassioni. *"Combinatorial Optimization of AC Optimal Power Flow with Discrete Demands in Radial Networks"*, **IEEE Trans. on Control of Network Systems (TCNS)**, 2020
  - Karapetyan, Khonji, Chau, Elbassioni, et al. *"A Competitive Scheduling Algorithm for Online Demand Response in Islanded Microgrids"*, **IEEE Trans. on Power Systems (TPS)**, 2021
  - Khonji, Karapetyan, Chau, Elbassioni. *"Complex-demand Scheduling Problem with Application in Smart Grid"*, **Theoretical Computer Science**, 2018
  - Karapety, Elbassioni, Khonji, Chau. *"Approximations for Generalized Unsplittable Flow on Paths with Application to Power Systems Optimization"*, **Annals of Operations Research**, 2022
  - Chau, Elbassioni, Khonji. *"Combinatorial Optimization of Alternating Current Electric Power Systems"*, (Book), **Foundations and Trends in Electric Energy Systems, Now Publishers Inc.**, 2018, ISBN: 978-1-68083-514-4. https://users.cecs.anu.edu.au/~sid.chau/FnT.html