A Newton–Like Method for Solving Rank Constrained Linear Matrix Inequalities ⋆

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Abstract

This paper presents a Newton–like algorithm for solving systems of rank constrained linear matrix inequalities. Though local quadratic convergence of the algorithm is not a priori guaranteed or observed in all cases, numerical experiments, including application to an output feedback stabilization problem, show the effectiveness of the algorithm.

Key words: linear matrix inequalities, rank constraints, computational methods, output feedback, stabilization, robust control.

1 Introduction

The *linear matrix inequality* (LMI) problem is a well known type of convex feasibility problem that has found many applications to controller analysis and design. The *rank constrained LMI* problem is a natural as well as important generalization of this problem. It is a nonconvex feasibility problem defined by LMI constraints together with an additional matrix rank constraint.

Interest in rank constrained LMIs arises as many important output feedback and robust control problems, that cannot always be addressed in the standard LMI framework, can be formulated as special cases of this problem [10], [37], [16], [30]. Examples include *bilinear matrix inequality* (BMI) problems, see [16] and [30], that are easily seen to be equivalent to rank one constrained LMI problems.

In addition to their importance for control, rank constrained LMI problems also appear naturally in mathematical programming and combinatorial optimization tasks: all optimization problems with polynomial objective and polynomial constraints can be reformulated as LMI optimization problems with a rank one constraint [31], [5].

In general, if the set of points that satisfy an LMI is nonempty, then a numerical solution to the LMI problem can be efficiently found using well developed interior point algorithms, see for example [40]. Lack of convexity makes the rank constrained LMI problem much harder to solve. Currently available algorithms for the rank constrained LMI problem are largely heuristic in nature and are not guaranteed to converge to a solution even if one exists. Solution methods for this problem, or certain specializations of the problem, include those based on modified interior point methods [8], [9]; linearization [11], [22], [25]; alternating projections [18], [4], [17]; trace minimization methods that try to solve the problem by solving a related convex problem [34], [29]; augmented Lagrangian methods [12], [2]; and sequential semidefinite programming [13]. Aside from [13], these methods do not have established superlinear convergence rates and the challenge remains to find numerical schemes with verifiable local quadratic convergence rates.

In this paper we present a new heuristic method for solving the rank constrained LMI problem. The method is closely related to existing alternating projection methods but is expected to have improved convergence properties due to a built-in Newton-type step. In [18] and [17] alternating projection algorithms are proposed that involve tangent-like ideas, similar to our approach. How-

⋆ This is an extended version of [33]. It contains proofs and other additional material not appearing in [33].

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ever, the implementation details are different and the connection to the Newton method is neither mentioned nor obvious. In fact, it is this established connection to Newton's method that distinguishes our approach from earlier ones.

Our method is based on the "tangent and lift" methodology [6], a generalization of Newton's method. While the classical Newton algorithm can be used to find zeroes of functions, the tangent and lift method is more general and can be used to find a point in the intersection of an affine subspace and a manifold. We show that the rank constrained LMI problem can be formulated as a problem of finding a point in the intersection of an affine subspace and another set which, though not a manifold, is a finite union of manifolds. Part of the contribution of this paper is a demonstration that tangent and lift methods can be extended to this more general setting and we present an algorithm for solving the rank constrained LMI problem based on such an extension. Numerical experiments show the effectiveness of this approach.

Since our method is based on a generalization of the Newton method, local quadratic convergence to isolated solutions is expected. However, complications arise due to the non-smoothness of the constraints as well as the possibility of continua of solutions. This makes a rigorous convergence theory difficult to develop and in fact, as some of our experiments show, local quadratic convergence cannot be expected in all cases. The challenge therefore is to single out a class of problems for which local quadratic convergence can be rigorously established.

The rest of the paper is structured as follows. Section 2 contains a statement of the rank constrained LMI problem and a reformulation of this problem into an equivalent form. Section 3 contains a discussion of the tangent and lift method and details of how we extend this methodology so that it can be applied to the rank constrained LMI problem. Section 4 discusses important geometric properties of rank constrained positive semidefinite matrices. Our algorithm for solving the rank constrained LMI problem is given in Section 5. Section 6 considers various numerical implementation issues. It includes a discussion of how the basic approach extends to enable the solution of more general problems such as those with multiple rank constraints. Section 7 reports on some numerical experiments and includes an application of the algorithm to an output feedback problem. The paper ends with some concluding remarks.

2 Problem Formulation

Let $\mathbb{R}$ denote the set of real numbers and $\mathcal{S}^n$ denote the set of real symmetric $n \times n$ matrices. For $A \in \mathcal{S}^n$, let $A \succeq 0$ denote the property that A is positive semidefinite. The rank constrained LMI problem is the following:

Problem 1 *Find $x \in \mathbb{R}^m$ such that*

$$F(x) := F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0, \tag{1}$$

$$G(x) := G_0 + \sum_{i=1}^{m} x_i G_i \succeq 0, \tag{2}$$

$$\text{rank } G(x) \leq r. \tag{3}$$

The problem data are the real symmetric matrices $F_i \in \mathcal{S}^{n_F}$ and $G_i \in \mathcal{S}^{n_G}$, and the rank bound r, which is assumed to be less than or equal to n_G.

Problem 1 consists of two LMI constraints, (1) and (2), and a rank constraint, (3). When $r = n_G$ constraint (3) is always satisfied and the problem reduces to a standard LMI feasibility problem. The more interesting case is when $r < n_G$. In this case the problem is nonconvex.

Let

$$\mathcal{S}^n_+ = \{X \in \mathcal{S}^n \mid X \succeq 0\}$$

and, for each integer s, let

$$\mathcal{S}^n_+(s) = \{X \in \mathcal{S}^n \mid X \succeq 0,\ \text{rank}(X) = s\}.$$

Define

$$\begin{aligned} \mathcal{M}_r &= \mathcal{S}^{n_F}_+ \times \bigcup_{s=0}^{r} \mathcal{S}^{n_G}_+(s) \\ &= \{(X,Y) \in \mathcal{S}^{n_F} \times \mathcal{S}^{n_G} \mid \\ &\qquad X \succeq 0,\ Y \succeq 0,\ \text{rank}(Y) \leq r\} \end{aligned} \tag{4}$$

and

$$\begin{aligned} \mathcal{L} = \{&(X,Y) \in \mathcal{S}^{n_F} \times \mathcal{S}^{n_G} \mid \\ &(X,Y) = (F(x), G(x)) \text{ for some } x \in \mathbb{R}^m\}. \end{aligned}$$

Problem 1 can be stated in the following equivalent form.

Problem 2

$$\textit{Find } (X,Y) \in \mathcal{M}_r \cap \mathcal{L}.$$

We will see that, for each s, $\mathcal{S}^n_+(s)$ is a manifold and hence that the rank constrained LMI problem is equivalent to finding a point in the intersection of an affine subspace and another set which is a finite union of manifolds. This structure will enable us to use the tangent and lift ideas that are discussed in the next section.

3 Tangent and Lift

In this section we discuss the tangent and lift methodology and present an extension that can be applied to the rank constrained LMI problem.

Before proceeding with the main discussion, a brief note on projections is required. Let x be an element in a Hilbert space H and let C be a closed (possibly nonconvex) subset of H. Any $c_0 \in C$ such that $\|x-c_0\| \leq \|x-c\|$ for all $c \in C$ will be called a *projection* of x onto C. In the cases of interest here, namely that H is a finite dimensional Hilbert space, there is always at least one such point for each x. If C is convex as well as closed then each x has exactly one such minimum distance point [27]. A function $P_C : H \to H$ will be called a *projection operator* (for C) if for each $x \in H$,

$$P_C(x) \in C \quad \text{and} \quad \|x - P_C(x)\| \leq \|x - c\| \text{ for all } c \in C.$$

The tangent and lift method is a generalization of Newton's method and can be used to find a point in the intersection of an affine subspace and a manifold. It originated in [6] and is based on a geometric interpretation of an algorithm appearing in [15].

Recall that Newton's method for finding a zero of a function $f : \mathbb{R} \to \mathbb{R}$ is iterative in nature and is given by the recursion

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Geometrically speaking, x_{n+1} is the x-axis intercept of the line which is tangent to the graph of f at $(x_n, f(x_n))$. In tangent and lift, the role of the x-axis is replaced by an affine subspace and the role of the graph of f is replaced by a manifold. More precisely, the method works as follows. Let H be a real finite dimensional Hilbert space and suppose $\mathcal{L}$ is an affine subspace of H and that $\mathcal{M}$ is a submanifold of H. Given $x_n \in \mathcal{L}$, and assuming it is possible to calculate projections onto $\mathcal{M}$, let y_n be a projection of x_n onto $\mathcal{M}$. As $\mathcal{M}$ is a manifold, it has a tangent space T at the point y_n. T has a canonical representation as a linear subspace of H and $y_n + T$ can be thought of as an affine subspace of H that is tangent to the manifold at y_n. Assuming $y_n + T$ and $\mathcal{L}$ intersect uniquely, x_{n+1} is taken to be the intersection point of $y_n + T$ and $\mathcal{L}$. As $x_{n+1} \in \mathcal{L}$, the scheme can be iterated.

A graphical representation of the algorithm is given in Figure 1(a). Here the Hilbert space H is $\mathbb{R}^2$, $\mathcal{L}$ is the x-axis, and $\mathcal{M}$ is the graph of a function $f : \mathbb{R} \to \mathbb{R}$. In this case, finding a point in $\mathcal{M} \cap \mathcal{L}$ is equivalent to finding a zero of f. Newton's method can also be employed to solve this problem and for purposes of comparison is also illustrated in Figure 1.

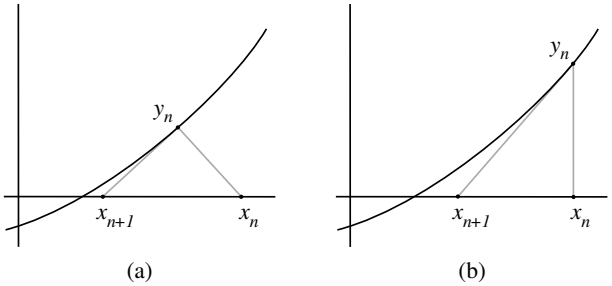

Fig. 1. Two different methods for finding a zero of a function: (a)Tangent and lift; (b)Newton's method.

Some points to note. For tangent and lift to work it must be possible to calculate projections onto $\mathcal{M}$. This step replaces the process of 'lifting' x to $(x, f(x))$ in Newton's method. In addition, at least for all points near a solution, each $y_n + T$ must intersect $\mathcal{L}$ uniquely. This essentially places a rather strong requirement on the dimensions of $\mathcal{L}$ and $\mathcal{M}$:

$$\dim \mathcal{L} + \dim \mathcal{M} = \dim H. \tag{5}$$

The reasoning is as follows. Firstly, note that if $y \in \mathcal{M}$ and T is its tangent space then $\dim \mathcal{M} = \dim T = \dim(y+T)$. Hence (5) can be interpreted in terms of the dimensions of the affine subspaces $\mathcal{L}$ and $y + T$. Now if the dimensions of two affine subspaces sum to less than the dimension of the ambient space then we would not expect them to intersect; think of two lines in $\mathbb{R}^3$. Alternatively, if the dimensions of two affine subspaces sum to more than the dimension of the ambient space then we would expect them to have multiple intersection points; think of two planes in $\mathbb{R}^3$. It is only when the dimensions of two affine subspaces sum to the dimension of the ambient space that we would expect the affine subspaces to intersect uniquely; think of a plane and a line in $\mathbb{R}^3$.

Suppose now that (5) is satisfied and that x is an intersection point of $\mathcal{M}$ and $\mathcal{L}$. If T is the tangent plane of $\mathcal{M}$ at x, and $x + T$ and $\mathcal{L}$ intersect uniquely (this will generically be the case and in the Newton scheme is equivalent to the requirement that $f'(x) \neq 0$), then, omitting the details, the tangent planes for all points in $\mathcal{M}$ near x will have this property. Hence if x_n 'close to x' implies x_{n+1} is also close to x then the algorithm will be well defined locally near x.

Though the dimension constraint (5) is not explicitly discussed in [6], it is satisfied by the problem studied in that paper and application of the tangent and lift method to that problem results in a (locally) quadratically convergent algorithm. Other applications of tangent and lift are given in [7]. As far as we are aware, tangent and lift methods have only been employed for problems that satisfy (5).

In Problem 2, $\mathcal{M}_r$ is not a manifold but rather a finite

union of pairwise disjoint manifolds, see Section 4. This means that each point in $\mathcal{M}_r$ lies in a manifold with a well defined tangent space. However, as will be shown, these manifolds are of varying dimensions. Depending on $y \in \mathcal{M}_r$, it may therefore happen that $y+T$ does not intersect $\mathcal{L}$ uniquely. The intersection may be empty or it may contain more than one point. This may happen even arbitrarily close to a solution point.

In order to apply tangent and lift ideas to Problem 2, the approach must be extended to deal with these intersection issues. Our method of doing this is as follows. We consider all points in $\mathcal{L}$ that are of minimum distance to $y_n + T$ and from these points choose x_{n+1} to be the point closest to y_n. As we will see in Section 5, x_{n+1} can be found by solving a linearly constrained least squares problem. Numerical experiments demonstrate this methodology leads to a locally convergent algorithm which, though it not always the case, often exhibits local quadratic convergence.

Regarding rigorous convergence results, though our method is based on a generalization of the Newton method, analysis of convergence is complicated due to the non-smoothness of the constraints as well as the possibility of continua of solutions. We currently have only partial results and the development of a rigorous convergence theory presents a challenge for the future. (The presence of the Newton step means that even in the absence of rank constraints it is not a priori clear whether or not the algorithm will converge. This complication seems inevitable if one looks for an algorithm with the potential of local quadratic convergence. In the absence of rank constraints, convergence would certainly hold if a standard alternating projection strategy were used but such a scheme would only converge linearly, even in the case of isolated solutions.)

4 The Geometry of Rank Constrained Positive Semidefinite Matrices

Before proceeding to describe our algorithm for solving the rank constrained LMI problem in greater detail, in this section we collect together some geometric properties of rank constrained positive semidefinite matrices. In particular, we show that $\mathcal{M}_r$ is a union of manifolds and describe the tangent spaces of these manifolds.

Theorem 3 *$\mathcal{S}_+^n(s)$ is a connected smooth manifold of dimension $\frac{1}{2}s(2n-s+1)$. The tangent space of $\mathcal{S}_+^n(s)$ at an element X is*

$$T_X\mathcal{S}_+^n(s) = \{\Omega X + X\Omega^T \,|\, \Omega \in \mathbb{R}^{n\times n}\}.$$

Proof. See for example Proposition 1.1 in Chapter 5 of [19]. ■

Corollary 4 *$\mathcal{M}_r$ is a finite union of manifolds.*

Proof. From (4) it follows that $\mathcal{M}_r$ is a finite union of terms of the form $S_+^{n_F}(s) \times S_+^{n_G}(t)$. Theorem 3 implies both $S_+^{n_F}(s)$ and $S_+^{n_G}(t)$ are manifolds and the result follows as a product of manifolds is itself a manifold. ■

As the next theorem shows, after applying an appropriate transformation, $T_X\mathcal{S}_+^n(s)$ has a rather simple form.

Theorem 5 *Given $X \in \mathcal{S}_+^n(s)$, let*

$$X = \Theta\bar{X}\Theta^T, \quad \bar{X} = \begin{bmatrix}\Lambda & 0\\ 0 & 0\end{bmatrix},$$

where $\Theta \in \mathbb{R}^{n\times n}$ is orthogonal and $\Lambda \in \mathcal{S}^s$ is a positive definite diagonal matrix. Then

$$\begin{aligned}&\Theta^T T_X\mathcal{S}_+^n(s)\Theta\\ &\quad = T_{\bar{X}}\mathcal{S}_+^n(s)\\ &\quad = \left\{\begin{bmatrix}\Omega_1 & \Omega_2^T\\ \Omega_2 & 0\end{bmatrix} \,\Big|\, \Omega_1 \in \mathcal{S}^s,\ \Omega_2 \in \mathbb{R}^{(n-s)\times s}\right\}.\end{aligned}$$

Proof. Taking the space $T_X S_+^n(s)$ and pre-multiplying by Θ^T and post-multiplying Θ gives

$$\begin{aligned}&\Theta^T T_X S_+^n(s)\Theta\\ &= \Theta^T\{\Omega X + X\Omega^T \,|\, \Omega \in \mathbb{R}^{n\times n}\}\Theta\\ &= \{(\Theta^T\Omega\Theta)(\Theta^T X\Theta) + (\Theta^T X\Theta)(\Theta^T\Omega^T\Theta) \,|\, \Omega \in \mathbb{R}^{n\times n}\}\\ &= \{\bar{\Omega}\bar{X} + \bar{X}\bar{\Omega}^T \,|\, \bar{\Omega} \in \mathbb{R}^{n\times n}\}\\ &= T_{\bar{X}} S_+^n(s).\end{aligned}$$

Here we have used the fact that Θ is orthogonal and that $\Theta^T\mathbb{R}^{n\times n}\Theta = \mathbb{R}^{n\times n}$.

In order to show the second equality of the theorem, for an arbitrary matrix $\Omega \in \mathbb{R}^{n\times n}$, consider the following partition into sub-matrices:

$$\Omega = \begin{bmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{bmatrix}.$$

Here $\Omega_{11} \in \mathbb{R}^{s\times s}$ and the remaining matrices are of the appropriate sizes. Using this partition we find

$$\Omega\bar{X} + \bar{X}\Omega^T = \begin{bmatrix}\Omega_{11}\Lambda + \Lambda\Omega_{11}^T & \Lambda\Omega_{21}^T\\ \Omega_{21}\Lambda & 0\end{bmatrix}.$$

Clearly $\Omega_{11}\Lambda + \Lambda\Omega_{11}^T \in \mathcal{S}^s$ and $\Omega_{21}\Lambda \in \mathbb{R}^{(n-s)\times s}$. Conversely, given any $\Omega_1 \in \mathcal{S}^s$ and $\Omega_2 \in \mathbb{R}^{(n-s)\times s}$, taking $\Omega_{11} = \frac{1}{2}\Omega_1\Lambda^{-1}$ and $\Omega_{21} = \Omega_2\Lambda^{-1}$ implies $\Omega_{11}\Lambda + \Lambda\Omega_{11}^T = \Omega_1$ and $\Omega_{21}\Lambda = \Omega_2$. ■

Note that Theorem 5 provides an alternate proof of the dimension of $\mathcal{S}_+^n(s)$: $\dim \mathcal{S}^s + \dim \mathbb{R}^{(n-s)\times s} = \frac{s(s+1)}{2} + (n-s)s = \frac{1}{2}s(2n-s+1)$.

The following useful fact is obvious from either Theorem 3 or Theorem 5.

Lemma 6 *$X \in T_X\mathcal{S}_+^n(s)$ for each $X \in \mathcal{S}_+^n(s)$.*

5 Algorithm

This section presents our algorithm for solving the rank constrained LMI problem. It contains a description of the algorithm at a conceptual level followed by details of the various components of the algorithm, including the required projections and initialization. The algorithm described here has been made into a freely available Matlab toolbox titled LMIRank [32]. LMIRank can be used either directly or via YALMIP [26].

In order to do projections, we need to define an appropriate Hilbert space. From now on $\mathcal{S}^n$ will be regarded as a Hilbert space with inner product

$$\langle A, B\rangle = \operatorname{tr}(AB) = \sum_{i,j} a_{ij}b_{ij}.$$

The associated norm is the Frobenius norm $\|A\| = \langle A, A\rangle^{\frac{1}{2}}$. In addition, $\mathcal{S}^{n_F} \times \mathcal{S}^{n_G}$ will be regarded as a Hilbert space with the usual inner product for a space that is a product of Hilbert spaces.

We will have need to refer to the affine tangent space of $\mathcal{S}_+^{n_F}(s)\times\mathcal{S}_+^{n_G}(t)$ at a point (X, Y) as an affine subspace of $\mathcal{S}^{n_F} \times \mathcal{S}^{n_G}$. For this purpose we introduce the following definition.

Definition 7 For $(X, Y) \in \mathcal{S}_+^{n_F}(s) \times \mathcal{S}_+^{n_G}(t)$, define

$$\mathcal{A}_{(X,Y)} = (X, Y) + T_{(X,Y)}(\mathcal{S}_+^{n_F}(s) \times \mathcal{S}_+^{n_G}(t)).$$

Lemma 6 implies that $(X, Y) \in T_{(X,Y)}(\mathcal{S}_+^{n_F}(s) \times \mathcal{S}_+^{n_G}(t))$. Hence, $\mathcal{A}_{(X,Y)} = T_{(X,Y)}(\mathcal{S}_+^{n_F}(s) \times \mathcal{S}_+^{n_G}(t))$ and $\mathcal{A}_{(X,Y)}$ is in fact a linear subspace and not just an affine subspace.

Definition 8 The distance between two non-empty subsets V and W of a vector space with norm $\|\cdot\|$ is

$$\operatorname{dist}(V, W) = \inf\{\|v - w\| \,|\, v \in V, w \in W\}.$$

Similarly, the distance between a point v and non-empty subset W is

$$\operatorname{dist}(v, W) = \inf\{\|v - w\| \,|\, w \in W\}.$$

At a conceptual level the algorithm can be stated as follows.

Algorithm:
Problem Data. $F_0, \ldots, F_m \in \mathcal{S}^{n_F}$, $G_0, \ldots, G_m \in \mathcal{S}^{n_G}$, and $0 \le r \le n_G$.
Initialization. Either choose any $(X_1, Y_1) \in \mathcal{S}^{n_F} \times \mathcal{S}^{n_G}$, or use $(X_1, Y_1) = (F(x), G(x))$ where x is the solution of the semidefinite definite program (6).
repeat
(1) Project (X_1, Y_1) onto $\mathcal{M}_r$ to give a new point (X_2, Y_2).
(2) Define $\mathcal{B} = \{(X, Y) \in \mathcal{L} \,|\, \operatorname{dist}((X, Y), \mathcal{A}_{(X_2,Y_2)}) = \operatorname{dist}(\mathcal{L}, \mathcal{A}_{(X_2,Y_2)})\}$.
(3) $(X_3, Y_3) = \arg\min_{(X,Y)\in\mathcal{B}} \|(X, Y) - (X_2, Y_2)\|$.
(4) Set $(X_1, Y_1) = (X_3, Y_3)$.
until (X_1, Y_1) converges to a solution of Problem 2. (See Section 6.1 for a precise termination criteria.)

Here are some comments regarding the above algorithm. Step 1 is readily calculated via eigenvalue-eigenvector decompositions of X_1 and Y_1. This will be shown in Section 5.2 below. In Step 2, $\mathcal{B}$ is the set of points in $\mathcal{L}$ that are of minimum distance to $\mathcal{A}_{(X_2,Y_2)}$. Step 3 is the projection of (X_2, Y_2) onto $\mathcal{B}$. Note that as $\mathcal{L}$ and $\mathcal{A}_{(X_2,Y_2)}$ are closed affine subspaces, the distance between them is zero if and only if they intersect. Whether the sets intersect or not, $\mathcal{B}$ itself will always be either a single point or an affine subspace. In the case that $\mathcal{B}$ is a single point, Step 3 is trivial. In the case that $\mathcal{B}$ is an affine subspace, Step 3 is equivalent to solving a linearly constrained least squares problem. Details of how to solve this step are given in Section 5.3 below. Finally, note that each new (X_1, Y_1) is in $\mathcal{L}$ as $(X_1, Y_1) = (X_3, Y_3) \in \mathcal{B} \subset \mathcal{L}$. Hence the termination criterion of the algorithm can be replaced by 'until $(X_1, Y_1) \in \mathcal{M}_r$'.

5.1 Initialization

There is no guarantee that the algorithm will converge from an arbitrary initial condition (X_1, Y_1). While a random choice for the initial condition does often work, an alternative choice is to use $(X_1, Y_1) = (F(x), G(x))$ where x is the solution the following semidefinite programming (SDP) problem:

$$\begin{array}{rl} \min\limits_{x\in\mathbb{R}^m} & \operatorname{tr}(G(x)) \\ \text{subject to} & F(x) \succeq 0 \\ & G(x) \succeq 0. \end{array} \tag{6}$$

This is based on the heuristic that minimizing the trace of a matrix subject to LMI constraints often leads to a low rank solution. Applied to a special case of Problem 1, the same initialization scheme is used in both [17] and [23]. This trace minimization heuristic also appears

in [34] and [29], and nice insights into why it might be effective can be found in [14].

As we will see in the results section, in some cases the solution of (6) will satisfy $\operatorname{rank} G(x) \leq r$, in which case the overall problem is solved. In general, however, the solution of the SDP gives a singular matrix $G(x)$ which does not satisfy this rank constraint.

5.2 Projecting onto $\mathcal{M}_r$

Step 1 of the algorithm is the projection of a point (X_1, Y_1) onto the set $\mathcal{M}_r$. This projection is equivalent to componentwise projection of X_1 onto $\mathcal{S}_+^{n_F} = \bigcup_{s=0}^{n_F} \mathcal{S}_+^{n_F}(s)$ and Y_1 onto $\bigcup_{s=0}^{r} \mathcal{S}_+^{n_G}(s)$.

The projection of $X \in \mathcal{S}^n$ onto $\bigcup_{s=0}^{r} \mathcal{S}_+^n(s)$ is given by Theorem 9 below. More precisely, Theorem 9 gives *a* projection of X onto $\bigcup_{s=0}^{r} \mathcal{S}_+^n(s)$ as, for r strictly less than n, the set $\bigcup_{s=0}^{r} \mathcal{S}_+^n(s)$ is nonconvex and projections onto this set are not always guaranteed to be unique. While Theorem 9 is not new (see [28] and [39], and [20] for the $r = n$ case), as far as we are aware, our proof is new.

Theorem 9 *Given $X \in \mathcal{S}^n$ and $0 \leq r \leq n$, let $X = \Theta \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\Theta^T$ with $\lambda_1 \geq \ldots \geq \lambda_n$ and Θ a real orthogonal matrix. Define $P_r : \mathcal{S}^n \to \mathcal{S}^n$ as follows,*

$$P_r(X) = \Theta \operatorname{diag}(\max\{\lambda_1, 0\}, .., \max\{\lambda_r, 0\}, 0, .., 0)\Theta^T.$$

Then $P_r(X)$ is a best approximant in $\bigcup_{s=0}^{r} \mathcal{S}_+^n(s)$ to X in the Frobenius norm.

Proof. See also [19], Chapter 5, Theorem 1.3. In order to prove the result, we will consider the function

$$f : \bigcup_{s=0}^{r} \mathcal{S}_+^n(s) \to \mathbb{R}, \quad Y \mapsto \|X - Y\|^2.$$

f is bounded below, radially unbounded, and its domain is a closed subset of the symmetric matrices. Hence it achieves a minimum value at some point $\bar{Y} \in \mathcal{S}_+^n(s)$ with $s \leq r$. If $f_s : \mathcal{S}_+^n(s) \to \mathbb{R}$ denotes f restricted to the set $\mathcal{S}_+^n(s)$, then necessarily $\bar{Y}$ will be a critical point of f_s. The derivative of f_s at $\bar{Y}$ in the tangent direction $\Omega\bar{Y} + \bar{Y}\Omega^T$ is

$$\begin{aligned} Df_s(\bar{Y})(\Omega\bar{Y} + \bar{Y}\Omega^T) &= 2 \ \operatorname{tr}((\Omega\bar{Y} + \bar{Y}\Omega^T)(\bar{Y} - X)) \\ &= 4 \ \operatorname{tr}(\bar{Y}(\bar{Y} - X)\Omega). \end{aligned}$$

The above derivative must be zero in all tangent directions. Hence, $\bar{Y}(\bar{Y} - X) = 0$ and $\bar{Y}^2 = \bar{Y}X$.

Both X and $\bar{Y}$ are symmetric and hence $X\bar{Y} = (\bar{Y}X)^T = (\bar{Y}^2)^T = \bar{Y}^2 = \bar{Y}X$. As X and $\bar{Y}$ commute and are symmetric, it follows they are simultaneously diagonalizable, that is, that there exists a real orthogonal S, and D and E diagonal, such that

$$X = SDS^T \quad \text{and} \quad \bar{Y} = SES^T \tag{7}$$

[21, Theorem 2.5.15]. Without loss of generality we can assume $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

The orthogonal invariance of the Frobenius norm together with (7) implies $\|X - \bar{Y}\| = \|D - E\|$. The quantity $\|D - E\|$ as a function of E is minimal (over the set of real diagonal positive semidefinite matrixes of rank $\leq r$) if

$$E = \operatorname{diag}(\max\{\lambda_1, 0\}, \ldots, \max\{\lambda_r, 0\}, 0, \ldots, 0).$$

The same minimum distance is achieved by $P_r(X)$ and this completes the proof. ■

We note without proof that if X has either r or fewer positive eigenvalues, or $\lambda_r > \lambda_{r+1}$, then $P_r(X)$ is the unique best approximant in $\bigcup_{s=0}^{r} \mathcal{S}_+^n(s)$ to X in the Frobenius norm. Otherwise, $P_r(X)$ is a non-unique best approximant.

5.3 Projecting onto $\mathcal{B}$

In this subsection, we will make use of the following notation. Given $X \in \mathbb{R}^{n \times n}$ and $0 \leq s \leq n$, let $X_s \in \mathbb{R}^{(n-s)\times(n-s)}$ denote the matrix consisting of the last $n - s$ rows and columns of X. That is, define X_s via

$$X = \begin{bmatrix} \star & \star \\ \star & X_s \end{bmatrix}, \quad X_s \in \mathbb{R}^{(n-s)\times(n-s)}.$$

Theorem 10 *Suppose $(X, Y) \in \mathcal{M}_r$ and define $s = \operatorname{rank}(X)$ and $t = \operatorname{rank}(Y)$. Then X and Y have eigenvalue-eigenvector decompositions*

$$X = VDV^T, \qquad D = \begin{bmatrix} \Lambda_X & 0 \\ 0 & 0 \end{bmatrix}, \tag{8}$$

$$Y = WEW^T, \qquad E = \begin{bmatrix} \Lambda_Y & 0 \\ 0 & 0 \end{bmatrix}, \tag{9}$$

where $V \in \mathbb{R}^{n_F \times n_F}$ and $W \in \mathbb{R}^{n_G \times n_G}$ are orthogonal, and $\Lambda_X \in \mathcal{S}^s$ and $\Lambda_Y \in \mathcal{S}^t$ are positive definite diagonal matrices.

Using the F_i's and G_i's of (1) and (2), and V and W from (8) and (9), define $b \in \mathbb{R}^{(n_F-s)^2+(n_G-t)^2}$ and $B \in$

$\mathbb{R}^{((n_F-s)^2+(n_G-t)^2)\times m}$ *by*

$$b = \begin{bmatrix} \mathrm{vec}((V^T F_0 V)_s) \\ \mathrm{vec}((W^T G_0 W)_t) \end{bmatrix},$$

$$B = \begin{bmatrix} \mathrm{vec}((V^T F_1 V)_s) & \dots & \mathrm{vec}((V^T F_m V)_s) \\ \mathrm{vec}((W^T G_1 W)_t) & \dots & \mathrm{vec}((W^T G_m W)_t) \end{bmatrix}.$$

If $F(\cdot)$ and $G(\cdot)$ are the functions defined in (1) and (2), and $\|\cdot\|_2$ denotes the standard vector 2-norm, then the projection of (X, Y) onto $\mathcal{B}$ equals $(F(x), G(x))$ where x is a minimizing solution of

$$\min_{x\in\mathbb{R}^m} \left\| \begin{bmatrix} \mathrm{vec}(F_1) & \dots & \mathrm{vec}(F_m) \\ \mathrm{vec}(G_1) & \dots & \mathrm{vec}(G_m) \end{bmatrix} x + \begin{bmatrix} \mathrm{vec}(F_0 - X) \\ \mathrm{vec}(G_0 - Y) \end{bmatrix} \right\|_2 \tag{10}$$

$$\text{subject to} \quad B^T B x = -B^T b. \tag{11}$$

If $(F_1, G_1), \dots, (F_m, G_m)$ are linearly independent, then x is unique.

Proof. $\mathcal{B}$ is the set of points in $\mathcal{L}$ that are of minimum distance to $\mathcal{A}_{(X,Y)}$. In order to characterize these points, the following transformation is employed. Using V and W from (8) and (9), define

$$f : \mathcal{S}^{n_F} \times \mathcal{S}^{n_G} \to \mathcal{S}^{n_F} \times \mathcal{S}^{n_G}, \qquad (Q, R) \mapsto (V^T Q V, W^T R W).$$

f is a linear isometry. As $\mathcal{B}$ is defined solely in terms of distances, it follows that its points can be characterized by using $f(\mathcal{L})$ and $f(\mathcal{A}_{(X,Y)})$ to characterize $f(\mathcal{B})$.

Theorem 5 implies

$$f(\mathcal{A}_{(X,Y)}) = \left\{ \begin{bmatrix} \Omega_1 & \Omega_2^T \\ \Omega_2 & 0 \end{bmatrix} \Big| \, \Omega_1 \in \mathcal{S}^s, \ \Omega_2 \in \mathbb{R}^{(n_F-s)\times s} \right\} \times \left\{ \begin{bmatrix} \Omega_3 & \Omega_4^T \\ \Omega_4 & 0 \end{bmatrix} \Big| \, \Omega_3 \in \mathcal{S}^t, \ \Omega_4 \in \mathbb{R}^{(n_G-t)\times t} \right\}.$$

Hence, $(Q, R) \in f(\mathcal{L})$ is a point in $f(\mathcal{B})$ if and only if it is a solution of

$$\min_{(Q,R)\in f(\mathcal{L})} \|(Q_s, R_t)\|.$$

If $\|\cdot\|_2$ denotes the standard vector 2-norm, then

$$\begin{aligned} \|(Q_s, R_t)\| &= (\|Q_s\|^2 + \|R_t\|^2)^{\frac{1}{2}} \\ &= \left\| \begin{bmatrix} \mathrm{vec}(Q_s) \\ \mathrm{vec}(R_t) \end{bmatrix} \right\|_2, \end{aligned}$$

and hence $(Q, R) \in f(\mathcal{B})$ if and only if $(Q, R) = (V^T F(x) V, W^T G(x) W)$ where x is a solution of

$$\min_{x\in\mathbb{R}^m} \quad \|Bx + b\|_2. \tag{12}$$

The minimum solutions of (12) are exactly the points that satisfy the normal equation (11). Aside from showing uniqueness, the proof is now complete.

The linear independence of $(F_1, G_1), \dots, (F_m, G_m)$ implies the cost in (10) is a strictly convex function of x. Uniqueness follows by noting that a strictly convex function is still strictly convex if it is restricted to an affine subspace. ∎

Hence projecting onto $\mathcal{B}$ is equivalent to solving the linearly constrained least squares problem (10), (11). Such problems can be solved in a number of ways, see for example [24]. A basic solution approach is as follows. First parameterize the points in the constraint set (using any particular solution and a basis for the null space of $B^T B$). Using this parametrization, transform the original constrained problem into a (lower dimensional) unconstrained least squares problem. Finally, use the solution of this new problem and the parametrization mapping to obtain a solution of the original problem.

6 Numerical Implementation Issues

In this section we mention some numerical implementation issues.

6.1 Convergence Criteria

Let $\epsilon_{\text{alg}} > 0$ be a user chosen algorithm tolerance. The convergence criteria is that the constraints (1), (2) and (3) are 'satisfied to a tolerance of ϵ_{alg}', by which we mean the following conditions are meet: $F(x) \succeq -\epsilon_{\text{alg}} I$, $G(x) \succeq -\epsilon_{\text{alg}} I$ and $G(x)$ has $n_G - r$ eigenvalues of absolute value $\leq \epsilon_{\text{alg}}$. While choosing ϵ_{alg} small guarantees that the constraints (1), (2) and (3) will be almost exactly satisfied, such a choice will lead to longer convergence times. The choice of tolerance ϵ_{alg} will be dictated by the problem being considered and it may be possible to choose a relatively large value. An example of this will be given in Section 7 when considering output feedback stabilization problems.

6.2 Additional LMI Constraints and Multiple Rank Constraints

Constraint (1) in Problem 1 can of course be used to specify multiple LMI constraints by using appropriate block diagonal matrices for the F_i's. Numerically, however, it is better to consider multiple LMI constraints individually rather than as a single LMI.

The methods described in this paper can be readily modified to deal with multiple LMI constraints. Indeed, it is even possible to have multiple rank constraints. In the rest of this subsection we briefly outline how the algorithm can be modified to incorporate both additional LMI constraints and multiple rank constraints. These extensions are incorporated into our software package LMIRank [32].

If there are q LMI constraints in total, and the ith LMI contains matrices of size $n_i \times n_i$, the optimization space is $\mathcal{S}^{n_1} \times \ldots \times \mathcal{S}^{n_q}$.

When projecting onto '$\mathcal{M}_r$', q (rather than 2) individual projections must be carried out. As before, each projection is given by Theorem 9. Note that whether or not the ith LMI is rank constrained influences the ith (and only the ith) projection. The main computational component of the ith projection is an eigenvalue-eigenvector decomposition of a $n_i \times n_i$ symmetric matrix.

Projecting onto '$\mathcal{B}$' is very similar to before. The only difference is that, in Theorem 10, rather than considering (X, Y), we now have to consider a q-tuple of symmetric matrices. As a result, each column of b and B in the theorem now consists of q (rather than 2) stacked vectors. Similarly, each column of the block matrix and block vector in (10) consists of q stacked vectors. Each of these terms are calculated in an analogous manner to the ones appearing in Theorem 10.

6.3 Linear Programming Inequality Constraints

Linear programming inequalities constraints in x of the form

$$a^T x + b \geq 0, \quad a \in \mathbb{R}^m,\ b \in \mathbb{R},$$

are just 1×1 LMIs and hence, by the prior subsection, can also be readily incorporated.

7 Numerical Experiments

This section contains some numerical experiments. Algorithm performance is investigated using both randomly generated problems and by applying the algorithm to a particular output feedback problem.

All computational results were obtained using a 3 GHz Pentium 4 machine. Our algorithm was coded using Matlab 7.0. For each problem, the initial condition was found by solving the semidefinite programming problem (6) using SeDuMi [38].

7.1 Random Problems

All results in this subsection are for randomly generated problems. Each problem is generated as follows. Let $\mathcal{N}(0,1)$ denote the normal distribution with zero mean and variance 1. Each entry of the matrices $F_1, \ldots, F_m$ and $G_1, \ldots, G_m$ is drawn from $\mathcal{N}(0,1)$. To ensure feasibility, F_0 and G_0 are set to $F_0 = V_F D_F V_F^T - \sum_{i=1}^m \xi_i F_i$ and $G_0 = V_G D_G V_G^T - \sum_{i=1}^m \xi_i G_i$, where each ξ_i is drawn from $\mathcal{N}(0,1)$; V_F and V_G are randomly generated orthogonal matrices; and D_F and D_G are randomly generated diagonal matrices: each diagonal entry in D_F is drawn from $\mathcal{N}(0,1)$ and set to zero if it is negative, while r diagonal entries in D_G are drawn from the uniform distribution on the interval $[0,1]$ and the others set to zero.

For the problems in this subsection, the algorithm tolerance was set to $\epsilon_{\text{alg}} = 10^{-12}$. (For the problems considered, typical non-zero eigenvalues have magnitudes between 10^1 and 10^{-2}.)

Table 1 contains results for $n_F = 10$, $n_G = 10$, $r = 5$ and various values of m. For each value of m in the table, the algorithm is given 1000 random problems to solve. Listed are a distribution of the number of iterations taken for the algorithm to converge, the average number of iterations, and the average CPU time. Iteration 1 is the initialization step based on the trace minimization heuristic.

Table 1
Experiments for random F and G with $n_F = 10$, $n_G = 10$ and $r = 5$. i denotes the average number of iterations and T denotes average CPU time in seconds. i and T do not include the problems that had not converged after 1000 iterations. The number of such problems for each m is given in the 'NC' or 'non-convergence after 1000 iterations' column. Tolerance $\epsilon_{\text{alg}} = 10^{-12}$.

m	iterations					i	T
	1	$2-10$	$11-20$	$21-1000$	NC		
10	965	35	0	0	0	1.1	0.16
20	333	448	84	112	23	21	0.36
30	279	559	70	71	21	21	0.48
40	756	214	19	9	2	3.2	0.46
50	967	26	6	1	0	1.5	0.54

For $m = 10$ and $m = 50$, solutions for all 1000 problems were found. In both these cases the trace minimization heuristic was very effective, finding solutions for almost all the problems. Most of the few problems that were not solved in this first iteration, were solved using a small number of additional iterations. For both $m = 20$ and $m = 30$, the trace minimization heuristic was no longer

quite as successful though it did still manage to find a solution in about 30% of cases. Overall, 95% of problems were solved in 20 iterations or less while less than 1% had not converged after 1000 iterations. Average solution times were very small.

The results indicate that on average the problems that are easiest to solve are those with either a rather small number of variables or those problems with a rather large number of variables. This situation is rather puzzling. However, as we will now explain, theoretical rank bound results do suggest why at least problems with a rather large number of variables may be easier to solve.

We will use the notation $\triangle_k$ to denote the kth triangular number, $\triangle_k := (k+1)k/2$. Consider a SDP of the form (6) where the cost is replaced by a general linear cost $c^T x$. It follows from the results in [1] that for a generic choice of c, F_i's and G_i's, a solution x of such a problem satisfies

$$\triangle_{\operatorname{rank} F(x)} + \triangle_{\operatorname{rank} G(x)} \leq \triangle_{n_F} + \triangle_{n_G} - m. \qquad (13)$$

Hence, for a given rank bound r, if m is large enough, the solution will satisfy $\operatorname{rank} G(x) \leq r$. The rank bound (13) is certainly interesting however its practical value as a means of ensuring low rank may be limited: for our largest value of m, $m = 50$, (13) only guarantees $\operatorname{rank} G(x) \leq 10$ (in the worst case scenario that $\operatorname{rank} F(x) = 0$) and hence does not even guarantee that $G(x)$ will not have full rank. Related rank bounds that apply to all SDPs and not just a generic subset can be found in [35] and [3]. (See also Section 6 of [36].)

Results for some larger problems are given in Table 2. Here again, for each value of m, the algorithm was given 1000 random problems to solve. For all values of m, performance was again very good. While average CPU times understandably increased due to the larger problem sizes, in all cases they were about 2 seconds or less. Also, as for the first set of results, the results in Table 2 seem to indicate that on average problems with a medium numbers of variables are the hardest to solve.

Table 2
Experiments for random F and G with $n_F = 20$, $n_G = 15$ and $r = 10$. Tolerance $\epsilon_{\text{alg}} = 10^{-12}$.

m	iterations					i	T
	1	$2-10$	$11-20$	$21-1000$	NC		
20	939	48	4	8	1	1.8	0.37
40	412	278	62	177	71	52	1.3
60	343	488	67	52	50	17	1.6
80	758	182	29	21	10	4.0	1.8
100	957	32	6	2	3	1.6	2.1

Lastly note that in both tables, average solution time T tends to increase with m.

7.2 Reduced Order Output Feedback

Consider a continuous time, linear time invariant (LTI) system

$$\dot{x} = Ax + Bu, \qquad y = Cx, \qquad (14)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, and $y \in \mathbb{R}^p$ is the output.

A dynamic output feedback controller of order n_c, $0 \leq n_c \leq n$, will be understood to be a controller of the form

$$\begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = K \begin{bmatrix} x_c \\ y \end{bmatrix},$$

where $K \in \mathbb{R}^{(n_c+m)\times(n_c+p)}$ is a constant matrix and $x_c \in \mathbb{R}^{n_c}$.

The problem that interests us in this subsection is the following reduced order output feedback stabilization problem.

Problem 11 *Given a system (14) and a scalar $\alpha > 0$, find a dynamic output feedback controller of order $\leq n_c$ that places the closed loop poles of the system in the set*

$$\{z \in \mathbb{C} \mid \operatorname{Re}(z) \leq -\alpha\}. \qquad (15)$$

Recall that a system with its poles in (15) is said to have stability degree (of at least) α.

Define

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0_{n_c} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} 0 & B \\ I_{n_c} & 0 \end{bmatrix}, \ \tilde{C} = \begin{bmatrix} 0 & I_{n_c} \\ C & 0 \end{bmatrix}.$$

As is well known, K is an order n_c solution of Problem 11 if and only if the augmented closed loop system matrix $\tilde{A} + \tilde{B}K\tilde{C}$ has its eigenvalues in (15). In addition, Problem 11 is solvable if and only if Problem 12, given below, is solvable (see for example [17]).

Problem 12 *Given a system (14) and a scalar $\alpha > 0$, find $X, Y \in \mathcal{S}^n$ such that*

$$-B^{\perp}(AX + XA^T + 2\alpha X)B^{\perp T} \succeq 0 \qquad (16)$$

$$-C^{T\perp}(YA + A^T Y + 2\alpha Y)C^{T\perp T} \succeq 0 \qquad (17)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \qquad (18)$$

$$\operatorname{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + n_c. \qquad (19)$$

Here $B^{\perp}$ is a matrix of maximal rank such that its rows are orthonormal and $B^{\perp}B = 0$. Similar comments apply for $C^{T\perp}$.

A solution of Problem 12 can be used to construct a solution of Problem 11 (and vice versa). As discussed in Section 6.1, our algorithm computes solutions to a user specified tolerance ϵ_{alg}. Hence, an algorithm computed solution to Problem 12 will not in general satisfy the constraints exactly. We take this fact into account in our method of controller synthesis. The first step of our method is to use the algorithm to solve the following perturbed problem.

Problem 13 *Given a system (14), a scalar $\alpha > 0$, and a tolerance $\epsilon > 0$, find $X, Y \in \mathcal{S}^n$ such that*

$$\begin{aligned} -B^{\perp}(AX + XA^T + 2\alpha X)B^{\perp T} - \epsilon I \succeq 0 \\ -C^{T\perp}(YA + A^TY + 2\alpha Y)C^{T\perp T} - \epsilon I \succeq 0 \\ \begin{bmatrix} X & I \\ I & Y \end{bmatrix} - \epsilon I \succeq 0 \\ \text{rank}\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix} - \epsilon I\right) \leq n + n_c. \end{aligned}$$

By choosing ϵ_{alg} equal to the ϵ of Problem 13, an algorithm calculated solution to Problem 13 will satisfy (16), (17) and (18). In addition, the constraint (19) will be satisfied to a tolerance of 2ϵ, that is, at least $2n-(n+n_c)$ eigenvalues of the matrix in (19) will have magnitude 2ϵ or less.

By the Schur complement result, (18) holds if and only if $Y \succ 0$ (Y is positive definite) and $X - Y^{-1} \succeq 0$. If $X - Y^{-1}$ has eigenvalue-eigenvector decomposition

$$X - Y^{-1} = V \,\text{diag}(\lambda_1, \ldots, \lambda_n)V^T, \tag{20}$$

with $\lambda_1 \geq \ldots \geq \lambda_n$, define

$$R = V(:, 1 : n_c)\,\text{diag}(\lambda_1^{\frac{1}{2}}, \ldots, \lambda_{n_c}^{\frac{1}{2}})$$

and

$$\tilde{X} = \begin{bmatrix} X & R \\ R^T & I \end{bmatrix}.$$

Notice that $\tilde{X} \succ 0$. The output feedback matrix K is reconstructed via the following SDP,

$$\begin{aligned} &\max_{\gamma \in \mathbb{R},\ K} \ \gamma \\ &\text{subject to } \ (\tilde{A} + \tilde{B}K\tilde{C})\tilde{X} + \tilde{X}(\tilde{A} + \tilde{B}K\tilde{C})^T + 2\gamma\tilde{X} \preceq 0. \end{aligned} \tag{21}$$

Remark 14 Note that the optimal γ of the SDP (21) gives a lower bound for the stability degree of the resulting closed loop system. Furthermore, a lower bound for this γ can be derived. Indeed, let $\epsilon_{\text{rof}} \geq 0$ be the smallest scalar such that $X - Y^{-1} - RR^T \preceq \epsilon_{\text{rof}} I$ (ϵ_{rof} is just λ_{n_c+1} from (20)). Considering the equality

$$\begin{bmatrix} X - Y^{-1} & 0 \\ 0 & Y \end{bmatrix} = \begin{bmatrix} I & -Y^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \begin{bmatrix} I & 0 \\ -Y^{-1} & I \end{bmatrix},$$

we would expect $X - Y^{-1}$ to have at most n_c eigenvalues that are not close to zero and hence that ϵ_{rof} will be small. By using the same type of reasoning used to prove Theorem 2.3 in [11] (we omit the details), it can be shown that

$$\gamma \geq \alpha - \epsilon_{\text{rof}} \frac{\|A\|_2 + \alpha}{\lambda_{\min}(B^{\perp}XB^{\perp T})}. \tag{22}$$

Here $\|\cdot\|_2$ is the maximum singular value norm and $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue. Hence, roughly speaking, if ϵ_{rof} is sufficiently small, the stability degree of the closed loop system will be at least nearly as large as α. □

We now consider a particular reduced order output feedback problem from [4] (see also [18], [37, Chapter 10] and [17]). The system considered is a two-mass-spring system with state space representation given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T.$$

Given $\alpha > 0$, we wish to find an order 2 dynamic controller that places the closed loop poles in (15).

The problem was solved for two of the same values of α given in [4], $\alpha = 0.2$ and $\alpha = 0.42$, and also an additional value, $\alpha = 0.46$. For each value of α, the algorithm was applied to Problem 13 with tolerances $\epsilon = 10^{-4}$ and $\epsilon = 10^{-9}$. In each case the controller matrix K was constructed via (21). The results are listed in Table 3.

Table 3
Results for the two-mass-spring system. α denotes the desired stability degree, $\hat{\alpha}$ the stability degree achieved, i the number of iterations, and T the CPU time in seconds.

	$\epsilon = 10^{-4}$			$\epsilon = 10^{-9}$		
α	$\hat{\alpha}$	i	T	$\hat{\alpha}$	i	T
0.2	0.20	59	0.55	0.21	195	1.4
0.42	0.42	644	4.2	0.42	1536	9.5
0.46	0.46	1187	8.0	0.46	2846	19

The first main point to note from these results is that it took more than twice as many iterations and more than twice as long to solve the problems using $\epsilon = 10^{-9}$ compared to $\epsilon = 10^{-4}$. Hence, taking relatively large values of ϵ such as $\epsilon = 10^{-4}$ may in general be a good solution strategy for these types of problems. The second main point to note is that, for both values of ϵ, convergence time increases with α. Hence, speed of convergence seems to be influenced by the size of the feasible set.

While $\alpha = 0.46$ was the best stability degree that we were able to achieve, it turns out this value is still not the best possible. The second order controller

$$K(s) = \frac{\frac{43}{5}s^2 - \frac{54\sqrt{15}}{125}s - \frac{27}{125}}{s^2 + \frac{6\sqrt{15}}{5}s + 7} \tag{23}$$

achieves a stability degree of $\alpha = \sqrt{15}/5 \approx 0.77$. This controller can be found by considering system and controller transfer functions and requiring that the denominator of the closed loop transfer function equal the polynomial $(s+\alpha)^6$. For comparison purposes, we also tried the cone complementarity linearization algorithm of [11] on the same problem. Our experience is that this algorithm works quite well in general though for this particular problem the greatest stability degree we were able to achieve using this algorithm was $\alpha = 0.20$.

Finally we note that for the $\alpha = 0.46$, $\epsilon = 10^{-9}$ solution, the lower bound in (22) was found to differ from α by less than 10^{-7} and hence was very tight. For the $\alpha = 0.46$, $\epsilon = 10^{-4}$ solution, the bound was found to be approximately 0.4548. This bound is still quite tight but, as is to be expected from using a larger ϵ, it is looser than the $\epsilon = 10^{-9}$ solution bound.

8 Conclusions

In this paper we have presented an algorithm for solving the rank constrained LMI problem, as well as more general LMI problems such as those with multiple rank constraints. Like all other algorithms that attempt to solve the rank constrained LMI problem, convergence from an arbitrary initial condition is not guaranteed. Though the convergence properties of the algorithm are not yet completely understood, as demonstrated by the experiments, the algorithm can be quite effective. Given that the algorithm is based on a Newton like methodology, it is not completely apparent why the algorithm is not always locally quadratically convergent and further investigations need to be made in this regard.

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