Preorder Topology

Greg O'Keefe

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The aim of this development is to show that the upward closed sets (upsets) of a preorder are a topology.

theory *PreorderTopology* = *closure*:

Let $\sqsubseteq$ be a preorder (reflexive and transitive relation) on a set X. We call a subset $A \subseteq X$ a $\sqsubseteq$ *upset* if anything bigger than something in A is also in A. Symbolically, A is an upset iff $\forall x, y.\ x \in Ax \sqsubseteq y \rightarrow y \in A$. Then the set $T = \{A.\ A \text{ is a } \sqsubseteq \text{ upset}\}$ is a topology over X.

locale *preorder* = *var* X + *var* R +
  assumes *on-carrier*:
    $\forall\, x\ y.\ R\ x\ y \longrightarrow x \in X \wedge y \in X$
  and *reflexive*:
    $\forall\, x{\in}X.\ R\ x\ x$
  and *transitive*:
    $\forall\, x{\in}X.\ \forall\, y{\in}X.\ \forall\, z{\in}X.\ R\ x\ y \wedge R\ y\ z \longrightarrow R\ x\ z$
  fixes *upset*
  defines *upset* $A \equiv \forall\, x\ y.\ x \in A \wedge R\ x\ y \longrightarrow y \in A$

Why is a finite intersection of upsets an upset? Let F be a finite family of upsets, and x be in the intersection, and y be bigger than X. Then x is in all the upsets in F, so y is too, so y is in the intersection.

lemma (**in** *preorder*) *finite-intersection*:
  assumes *1*: $F \subseteq \{A.\ A{\subseteq}X\ \&\ \mathit{upset}\ A\}$
  and *3*: $F \neq \{\}$
  shows $\bigcap F \in \{A.\ A{\subseteq}X\ \&\ \mathit{upset}\ A\}$
proof (*simp only*: *mem-Collect-eq*, *intro conjI*)
  from *1* **and** *3* **show** $\bigcap F \subseteq X$ **by** *auto*
  show *upset* $(\bigcap F)$
  proof (*unfold upset-def*, *intro allI impI*, *elim conjE*)
    fix x **and** y
    assume *4*: $x \in \bigcap F$
      and *5*: $R\ x\ y$
    from *4* **have** $\forall\, A{\in}F.\ x \in A$
      by *auto*
    have $\forall\, A{\in}F.\ y \in A$
      proof

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      fix A
      assume 6: A ∈ F
      with 1 have 7: upset A by auto
      from 4 and 6 have x ∈ A by auto
      with 5 and 7 show y ∈ A
        by (unfold upset-def) blast
    qed
  thus y ∈ ⋂F
    by auto
 qed
qed
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Why is a union of upsets an upset? Anything x in the union is in one of the members A of the family, so anything y bigger than x is in A and hence in the union.

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lemma (in preorder) arbitrary-union:
  assumes 1: F ⊆ {A. A⊆X & upset A}
  shows ⋃F ∈ {A. A⊆X & upset A}
proof (simp only: mem-Collect-eq, intro conjI)
  from 1 show ⋃F ⊆ X by auto
  show upset (⋃F)
  proof (unfold upset-def, intro allI impI, elim conjE)
    fix x and y
    assume 2:x ∈ ⋃F
      and 3: R x y
    from 1 and 2 obtain A
      where 4: x ∈ A
      and 5: A ∈ F
      by auto
    from 1 and 5 have upset A
      by auto
    with 3 and 4 have y ∈ A
      by (unfold upset-def) blast
    with 5 show y ∈ ⋃F
      by auto
  qed
qed
```

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theorem (in preorder) upset-topology:
  assumes X ≠ {}
  shows topological-space X {A. A⊆X & upset A}
proof (intro topological-space.intro)
  let ?T = {A. A⊆X & upset A}
  show X ≠ {} .
  show ∀A∈?T. A ⊆ X
    by auto
  from on-carrier
  show X ∈ ?T
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by (*simp only*: *upset-def mem-Collect-eq*) *auto*
show $\{\} \in$ *?T*
by (*unfold upset-def*) *auto*
from *finite-intersection*
show $\forall F.\ F \subseteq$ *?T* $\wedge$ *finite* $F \wedge F \neq \{\} \longrightarrow \bigcap F \in$ *?T*
by *auto*
from *arbitrary-union*
show $\forall F.\ F \subseteq$ *?T* $\longrightarrow \bigcup F \in$ *?T*
by *auto*
qed

end