Equivalence of Categories is an Equivalence Relation

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This is an answer to the problem [BW02, §1.2.11]. Since you may have your doubts, our first definition makes it clear that there is indeed something to prove.

Definition 1. Categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* iff there is a full and faithful functor $F : \mathcal{C} \to \mathcal{D}$ such that for each object D of $\mathcal{D}$ there is some object C of $\mathcal{C}$ where $FC \cong D$.

We will call an F with these properties an *equivalence functor*. Now a couple of reminders

Definition 2. A functor is *full* if its restriction to $Hom(A, B)$ is surjective for each pair of objects A, B. It is *faithful* if each such restriction is injective.

Definition 3. An *equivalence relation* is a binary relation that is reflexive, transitive and symmetric.

So our course is set: we will prove that equivalence of categories has these three properties.

Lemma 4. *Every category is equivalent to itself.*

Proof. Each identity arrow is an isomorphism, so each object in a category is isomorphic to itself. Therefore the identitiy functor $id_{\mathcal{C}}$ satisfies $id_{\mathcal{C}}(A) \cong A$ for all objects A. This functor is also full and faithful, hence it stands as a witness to our claim. □

Lemma 5. *Equivalence of categories is transitive.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be equivalence functors, and let E be an object in $\mathcal{E}$. We have an object D of $\mathcal{D}$ such that[1] $GD \cong E$, and an object C of $\mathcal{C}$ where $FC \cong D$. Since functors preserve isomorphisms ([BW02,

[1]We leave out brackets for functor application when it looks OK. Similarly with $\circ$ for composition

§1.2.1]), we have $GFC \cong GD$. Since the composition of isomorphisms is an isomorphism, we have $GFC \cong E$. The composite of full and faithful functors is full and faithful, so the functor GF is a witness to the equivalence of $\mathcal{C}$ and $\mathcal{E}$. ☐

Now things get interesting, because given an equivalence functor $F : \mathcal{C} \to \mathcal{D}$, we must construct one $G : \mathcal{D} \to \mathcal{C}$ going in the other direction. For each object D of $\mathcal{D}$ we choose an object GD of $\mathcal{C}$ such that $FGD \cong D$, and isomorphisms $in_D : FGD \to D$ and $out_D : D \to FGD$. So, for a given arrow $f : D \to D'$ in $\mathcal{D}$, we have

$$\begin{array}{ccc} D & \xrightarrow{f} & D' \\ {\scriptstyle in_D}\uparrow & & \downarrow{\scriptstyle out_{D'}} \\ FGD & \longrightarrow & FGD' \end{array}$$

Since F is bijective on the homset $Hom(GD, GD')$, we know that there is a unique arrow $GD \to GD'$, which we shall call Gf, such that

$$FGf = out_{D'} \circ f \circ in_D$$

We now show that G is a functor.

Lemma 6. *The G just defined is a functor.*

Proof. It is enough to show that G preserves identities and composition.

For any object D of $\mathcal{D}$, Gid_D is an arrow $GD \to GD$. From our definition of G, we have

$$\begin{aligned} FGid_D &= out_D \circ id_D \circ in_D \\ &= id_{FGD} \\ &= Fid_{GD} \end{aligned}$$

Hence $Gid_D = id_{GD}$, because F is injective on $Hom(GD, GD)$.

To show that G preserves composition, first note that for arrows $f : D \to D'$ and $g : D' \to D''$ of $\mathcal{D}$, we have

$$\begin{aligned} F(Gg \circ Gf) &= FGg \circ FGf \\ &= (out_{D''} \circ g \circ in_{D'}) \circ (out_{D'} \circ f \circ in_D) \\ &= out_{D''} \circ (g \circ f) \circ in_D \\ &= FG(gf) \end{aligned}$$

Since F is bijective on $Hom(GD, GD'')$, we may conclude that $G(gf) = Gg \circ Gf$. ☐

Lemma 7. *G is full and faithful.*

Proof. Let $f, f' : D \to D'$ and assume $Gf = Gf'$. Because F is injective on this homset, we have $FGf = FGf'$. Hence we have $out_{D'} \circ f' \circ in_D = out_{D'} \circ f \circ in_D$, and because isomorphisms are epi and mono, $f' = f$. This shows that G is faithful.

Let $g : GD \to GD'$ be an arrow in $\mathcal{C}$. We require a $\mathcal{D}$ arrow $f : D \to D'$ where $Gf = g$. This is so iff $Fg = out_{D'} \circ f \circ in_D$. The required f is obtained by taking a rather indirect stroll around the diagram

$$\begin{array}{ccc} D & \xrightarrow{f} & D' \\ {\scriptstyle in_D}\uparrow & & \downarrow{\scriptstyle out_{D'}} \\ FGD & \xrightarrow[Fg]{} & FGD' \end{array}$$

We see that

$$Fg = out_{D'} \circ (out_{D'}^{-1} \circ Fg \circ in_D^{-1}) \circ in_D$$

where our required f is in the brackets, and may also be written $f = in_{D'} \circ Fg \circ out_D$. □

The salient point from the recommended [BW02, §1.2.7] is the following.

Lemma 8. *Full and faithful functors reflect isomorphisms.*

Proof. Let $H : \mathcal{C} \to \mathcal{D}$ be a full and faithful functor, $f : C \to C'$ an arrow in $\mathcal{C}$. Assume that $Hf : FC \to FC'$ is an isomorphism, with inverse $w : FC' \to FC$. As H is full, there is some $g : C' \to C$ where $Hg = w$. We have

$$\begin{aligned} H(gf) &= Hg \circ Hf \\ &= id_{HC} \\ &= Hid_C \end{aligned}$$

Because H is faithful, we may conclude that $gf = id_C$. A similar argument shows that $fg = id_{C'}$. That is, f is an isomorphism. □

Last lemma coming up

Lemma 9. *For each object C of $\mathcal{C}$, $GFC \cong C$.*

Proof. We have an isomorphism $in_D : FGD \to D$ for each object D of $\mathcal{D}$. Functors preserve isomorphisms, so $Gin_{FC} : GFGFC \to GFC$ is an isomorphism. The composite functor GF is full and faithful, because G and F are, hence GF is a bijection when restricted to $Hom(GFC, C) \to Hom(GFGFC, GFC)$. Therefore, there is some arrow $w : GFC \to C$ for which $GFw = Gin_{FC}$. Since GF reflects isomorphisms, w is the isomorphism that we require. □

We have at last established the claim that is our title

Theorem 10. *Equivalence of categories is an equivalence relation*

References

[BW02] Michael Barr and Charles Wells. *Toposes, Triples and Theories.* Springer-Verlag (first edition), Free Internet Publication (version 1.1), `http://www.cwru.edu/artsci/math/wells/pub/ttt.html`, 1983, 2002.