Continuous Wireless Communications

Leif Hanlen
National ICT Australia,
Canberra, Australia
leif.hanlen@nicta.com.au

February 2005

Schauder Basis

DEFINITION 1 (SCHAUDER BASIS).
If, for every $\mathbf{x} \in \mathcal{X}$, there is a unique representation of scalars x_i such that

$$\lim_{n\to\infty} \left\| \mathbf{x} - \sum_{i=0}^{n} x_i \mathbf{e}_i \right\| = 0$$

holds in a normed space $\mathcal{X}$, then the set $\{\mathbf{e}_1, \ldots\} = \{\mathbf{e}_i\}_{i=0}^{\infty}$ is called a ***Schauder Basis*** *or basis for $\mathcal{X}$, and $\lim_{n\to\infty} \|\cdot\|$ is called an expansion of $\mathbf{x}$.*

- has same interpretation as linear algebra basis
- No inner product required.
- So why do the Fourier coefficients use one?

Compact

Recall the question about convergence versus Cauchy sequences.

David Hilbert and Augustine Cauchy are playing dodgeball in a Banach Space $\mathcal{K}$. David is running around in $\mathcal{K}$, and being a mathematician he need not move continuously, he appears over here, then there, then somewhere else.... And Augustine's too old to be running around

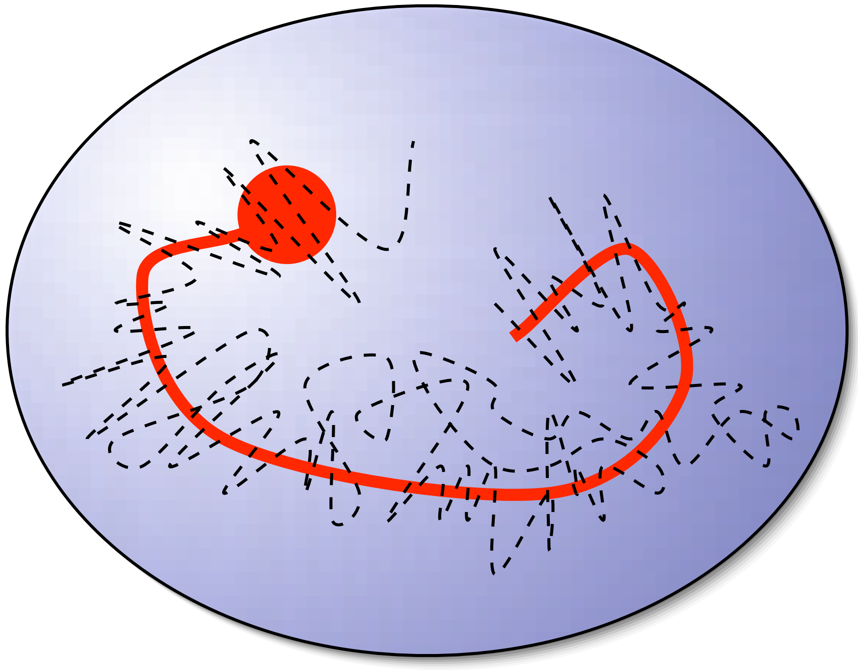
However, Augustine has (sneakily) attached a $\mathcal{K}$-compatible global positioning system to Hilbert and is reading off the locations as David moves. We have an infinite sequence. Cauchy wins if he's guaranteed to hit Hilbert with the ball, ie. if the sequence converges. (why?)

So the rules are:

- David picks all his positions out ahead of time
- Augustine sits back and waits with the ball

Is there a space where Cauchy always wins?

Compact



Now Cauchy's lazy (or faster aiming!), and just waits for Hilbert to appear *some of the time* at the appropriate spot.

Are there space(s) now where Cauchy is guaranteed to win?

Compact

What do we need to force Cauchy to *always* win, regardless of what David does?
Hint: think of a sub-sequence of Hilbert's trajectory
Hint: think of noise

Essential Dimension

DEFINITION 2 (COMPACT SPACE: SHOCK AND AWE REVIEWERS). *A compact space* $\mathcal{K}$ *is one for which* every *sequence* $\mathbf{x} = \{x_i\}_{i=1}^{\infty}$ *has a sub-sequence* $\mathbf{x}' = \{x_k\}_{k\in\mathbb{Z}}$, $\mathbf{x}' \subseteq \mathbf{x} \in \mathcal{K}$, *such that* x' *converges.*

- Consider the space of $n \times n$, $n < \infty$ dimensional matrices with complex entries: $\mathbb{C}^{n\times n}$.
- Can we use a member $\mathbf{x} \in \mathbb{C}^{n\times n}$ to represent an element $\mathbf{y} \in \mathbb{C}^{n+1\times n+1}$
- Why/why not? If so how? Think: finite precision.

Essential Dimension

DEFINITION 3 (ESSENTIAL DIMENSION).
A compact space $\mathcal{K}$ with dimension $\aleph_0$ has essential dimension N if, for every vector $\mathbf{x} \in \mathcal{K}$ we may write:

$$\left\| \mathbf{x} - \sum_{i=0}^{N} x_i \mathbf{e}_i \right\| < \epsilon$$

for a fixed $\epsilon \geq 0$

What role does ϵ play?

T is for Operator

- Recall Functions.

-

$$T : \mathcal{X} \mapsto \mathcal{Y}$$

$$T\mathbf{x} = \mathbf{y} \quad \mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}$$

- We will look at linear bounded operators acting between Hilbert Spaces over the complex field

- act on vectors $\mathbf{x} \in \mathcal{X}$ to produce (other) vectors $\mathbf{y} \in \mathcal{Y}$.

- What properties would we expect from linear operators? Superposition?

Examples

- Identity: $I_{\mathcal{X}} : \mathcal{X} \mapsto \mathcal{X}$, $\quad I_{\mathcal{X}} x = x$
- Zero: $0 : \mathcal{X} \mapsto \mathcal{Y}$
- Integration
- Differentiation
- Spot an operator: $\nabla \mathbf{x} + k\mathbf{x} = 0$

Bounded Linear Operators

DEFINITION 4 (BOUNDED LINEAR OPERATOR).
For normed spaces $\mathcal{X}$ and $\mathcal{Y}$, with operator $T : \mathfrak{D}(T) \mapsto \mathcal{Y}$ and $\mathfrak{D}(T) \subseteq \mathcal{X}$ then T is bounded *by a real scalar constant c if,*

$$\|y\|_{\mathcal{Y}} = \|Tx\|_{\mathcal{Y}} \leq c\|x\|_{\mathcal{X}} \quad \forall x \in \mathfrak{D}(T), y = Tx$$

where we have noted that the norms are over different spaces.

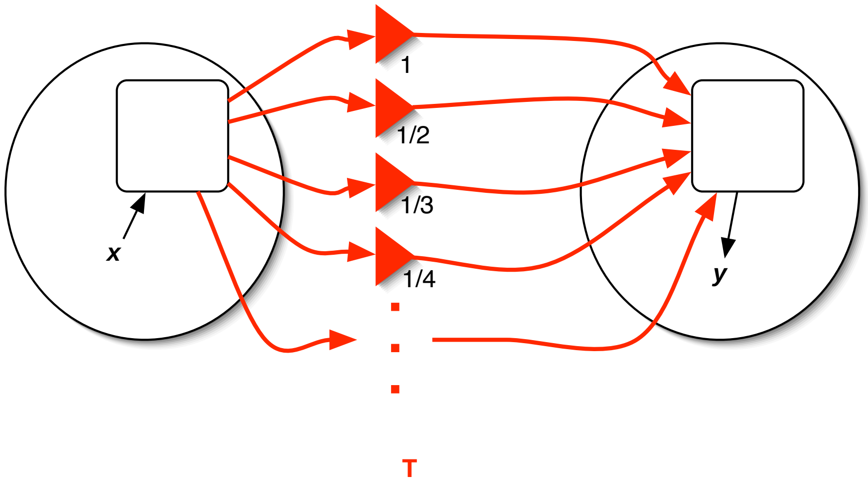
- May be written as (infinite dimension) matrices.
- Outline: write $T\mathbf{x} = \mathbf{y}$ and expand $\mathbf{x}$ and $\mathbf{y}$

Operator Norm

DEFINITION 5 (NORM OF OPERATOR T).
The norm of an operator is given by the supremum over all outputs for inputs of a fixed size

$$\|T\| \triangleq \sup_{\substack{x \in \mathfrak{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in \mathfrak{D}(T) \\ \|x\| = 1}} \|Tx\|$$

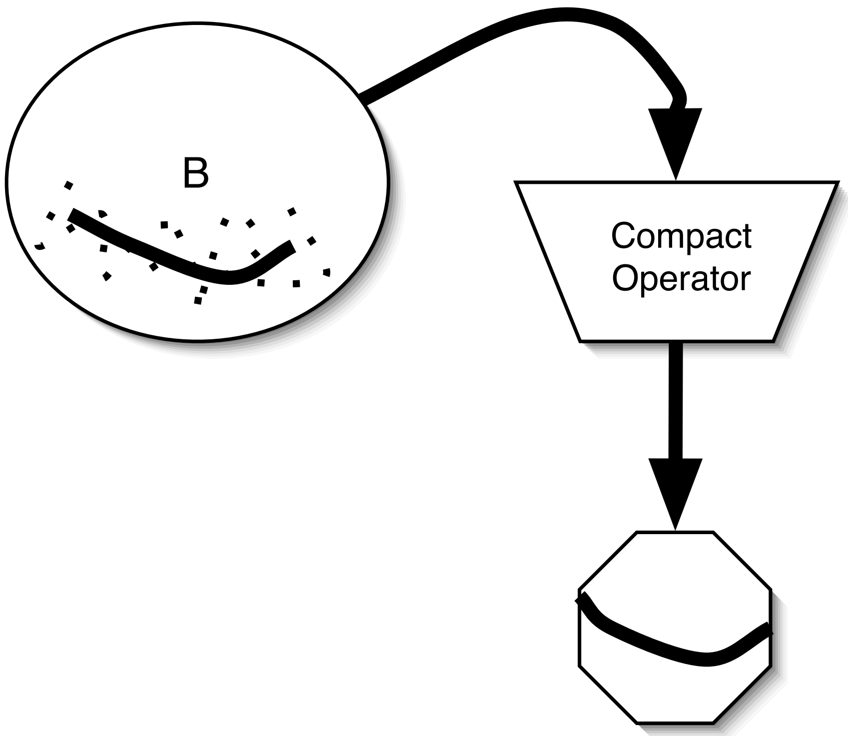
Operator Action



- Operator acts on all vectors $\mathbf{x} \in \mathcal{X}$
- Can use basis expansion in $\mathcal{X}$.
- Think of operator in terms of effect on the unit ball $\mathbb{B} = \{\mathbf{x} : \|\mathbf{x}\| < c\}$ in $\mathcal{X}$.

Compact Operators

- A special type of linear bounded operator
- Maps Bounded Balls to Compact Balls.
- Smoothing/Filtering Effect.



Compact Operators

Example 1. *Consider the following "sequence" which is the Fourier expansion of a function* $f(t)$*, under a particular complete orthonormal sequence* $\{\varphi_i(t)\}_{i=0}^{\infty}$*:*

$$f(t) = \sum_{i=0}^{\infty} f_i \varphi(t)$$

The operator

$$K : Kf(t) = \sum_{i=0}^{\infty} f_i \exp\{-i^2\}$$

is compact.

Example 2. *The identity operator in infinite dimensions is not compact.*

Compact Operators: Hilbert-Schmidt

$$Tf(t) = \int_a^b k(t,\tau) f(\tau)\, d\tau$$

T is compact for $|k(t,\tau)| < \infty$

- Hilbert-Schmidt operator has "trace-rule"

$$\sum_{ij} |T_{ij}|^2 = K$$

- Look for H-S in Miller

[Mil00] David A. Miller. Communicating with waves between volumes: evaluating orthogonal spatial channels and limits on coupling strengths. *Applied Optics*, 39(11):1681–1699, April 2000.

Information Theory Crash Course

- Entropy
- Mutual Information
- Capacity
- Next week: Fidelity & Random MIMO

Before the crash: Probability notation

- Random objects $\mathbf{x}$ live in spaces $\Omega_{\mathbf{x}}$
- Probability functions define measures on the space.
- Most of today's work will be from [Gal68] chap.2 and chap.4

[Gal68] Robert G. Gallager. *Information Theory and Reliable Communication*. John Wiley & Sons, New York, USA, 1968.

Mutual Information?

DEFINITION 6 (MUTUAL INFORMATION).
Information about the event $\mathbf{x} = \alpha$ *by another event* $\mathbf{y} = \beta$

$$\mathcal{I}_{\Omega_{\mathbf{x}};\Omega_{\mathbf{y}}}(\mathbf{x} = \alpha; \mathbf{y} = \beta) = \log \frac{P_{\Omega_{\mathbf{x}}|\Omega_{\mathbf{y}}}(\mathbf{x} = \alpha|\mathbf{y} = \beta)}{P_{\Omega_{\mathbf{x}}}(\mathbf{x} = \alpha)}$$

$\mathbf{x} \in \Omega_{\mathbf{x}}$ *and* $\mathbf{y} \in \Omega_{\mathbf{y}}$ *where* $\Omega_{\mathbf{x}}$ *and* $\Omega_{\mathbf{y}}$ *are (separate) probability spaces*

Mutual Information?

DEFINITION 7 (SELF INFORMATION).
Information required to specify the event $\mathbf{x} = \alpha$.

$$\mathcal{I}_{\Omega_{\mathbf{x}}}(\mathbf{x} = \alpha) = \log \frac{1}{P_{\Omega_{\mathbf{x}}}(\mathbf{x} = \alpha)} = -\log P_{\Omega_{\mathbf{x}}}(\mathbf{x} = \alpha)$$

Event $\mathbf{y} = \beta$ *uniquely specifies* $\mathbf{x} = \alpha$ *so* $P_{\Omega_{\mathbf{x}}|\Omega_{\mathbf{y}}}(\mathbf{x} = \alpha | \mathbf{y} = \beta) = 1$.

DEFINITION 8 (CONDITIONAL SELF INFORMATION).
Information required to specify the event $\mathbf{x} = \alpha$,
given another event $\mathbf{y} = \beta$ *occurred*

$$\mathcal{I}_{\Omega_{\mathbf{x}}|\Omega_{\mathbf{y}}}(\mathbf{x} = \alpha; \mathbf{y} = \beta) = \log \frac{1}{P_{\Omega_{\mathbf{x}}|\Omega_{\mathbf{y}}}(\mathbf{x} = \alpha | \mathbf{y} = \beta)}$$
$$= -\log P_{\Omega_{\mathbf{x}}|\Omega_{\mathbf{y}}}(\mathbf{x} = \alpha | \mathbf{y} = \beta)$$

Mutual Information

- $\mathcal{I}_{\Omega_{\mathbf{x}};\Omega_{\mathbf{y}}}(\mathbf{x}=\alpha;\mathbf{y}=\beta)$ and $\mathcal{I}_{\Omega_{\mathbf{x}}}(\mathbf{x}=\alpha)$ are random variables.

$$\mathcal{I}_{\Omega_{\mathbf{x}};\Omega_{\mathbf{y}}}(\mathbf{x}=\alpha;\mathbf{y}=\beta)=\mathcal{I}_{\Omega_{\mathbf{x}}}(\mathbf{x}=\alpha)-\mathcal{I}_{\Omega_{\mathbf{x}}|\Omega_{\mathbf{y}}}(\mathbf{x}=\alpha|\mathbf{y}=\beta)$$

- Can average these over the ensembles $\Omega_{\mathbf{x}}$ and $\Omega_{\mathbf{y}}$
- We can have expected values and variance

Entropy

aka. Average Self Information, only for discrete r.v.

DEFINITION 9 (ENTROPY).
The "amount" of information required to describe a probability space.

$$\mathbf{H}(\Omega_{\mathbf{x}}) = \mathbf{E}_{\Omega_{\mathbf{x}}}\{-\log\left(P_{\Omega_{\mathbf{x}}}(\mathbf{x})\right)\}$$
$$-\int_{\Omega_{\mathbf{x}}} P_{\Omega_{\mathbf{x}}}(t)\log\left(P_{\Omega_{\mathbf{x}}}(t)\right)\,dt$$

DEFINITION 10 (CONDITIONAL ENTROPY).

$$\mathbf{H}(\Omega_{\mathbf{x}}|\Omega_{\mathbf{y}}) = \mathbf{E}_{\Omega_{\mathbf{x}},\Omega_{\mathbf{y}}}\left\{-\log\left(P_{\Omega_{\mathbf{x}}|\Omega_{\mathbf{y}}}(\mathbf{x}|\mathbf{y})\right)\right\}$$
$$-\int_{t\in\Omega_{\mathbf{x}}}\int_{\tau\in\Omega_{\mathbf{y}}} P_{\Omega_{\mathbf{x}}|\Omega_{\mathbf{y}}}(t,\tau)\log\left(P_{\Omega_{\mathbf{x}}|\Omega_{\mathbf{y}}}(t|\tau)\right)\,dt\,d\tau$$

Average Mutual Info

$$\mathbf{I}(\Omega_{\mathbf{x}};\Omega_{\mathbf{y}}) = \mathbf{E}_{\Omega_{\mathbf{x}},\Omega_{\mathbf{y}}}\left\{\mathcal{I}_{\Omega_{\mathbf{x}};\Omega_{\mathbf{y}}}(\mathbf{x}=\alpha;\mathbf{y}=\beta)\right\}$$

- What is this as an integral?
- Build $\mathbf{I}(\Omega_{\mathbf{x}};\Omega_{\mathbf{y}})$ from Entropy.

Average Mutual Info Example

$$\mathbf{y} = \mathbf{x} + \mathbf{z}$$

where $\mathbf{z} \sim \mathcal{N}_{,}\left(0, \sigma^2\right)$ and $\mathbf{x} \sim \mathcal{N}_{,}\left(0, p^2\right)$

Find $\mathbf{I}(\Omega_{\mathbf{x}}; \Omega_{\mathbf{y}})$

Average Mutual Info Properties

$$\mathbf{I}(\Omega_{\mathbf{x}};\Omega_{\mathbf{y}}) \geq 0$$

$$\mathbf{I}(\Omega_{\mathbf{x}};\Omega_{\mathbf{y}}|\Omega_{\mathbf{z}}) \geq 0$$

Capacity

$$C = \max_{P_{\Omega_{\mathbf{x}}}(\mathbf{x})} \mathbf{I}(\Omega_{\mathbf{x}}; \Omega_{\mathbf{y}})$$

$$= \max_{P_{\Omega_{\mathbf{x}}}(\mathbf{x})} \mathbf{E}_{\Omega_{\mathbf{x}},\Omega_{\mathbf{y}}} \{ \mathcal{I}_{\Omega_{\mathbf{x}};\Omega_{\mathbf{y}}}(\mathbf{x}; \mathbf{y}) \}$$

- Maximum is over all probability assignments

Capacity for vector inputs

$$\mathbf{I}\left(\Omega_{\mathbf{x}}{}^{N};\Omega_{\mathbf{y}}{}^{N}\right) \leq \sum_{k} \mathbf{I}\left(\Omega_{\mathbf{x}k};\Omega_{\mathbf{y}k}\right)$$

$$\mathbf{I}\left(\Omega_{\mathbf{x}}{}^{N};\Omega_{\mathbf{y}}{}^{N}\right) \leq NC$$

Equality in first, iff inputs independent.

Equality in first, iff inputs independent.

References

[1] N. Chiurtu, B. Rimoldi, and E. Telatar. Dense multiple antenna systems. In *IEEE Information Theory Workshop*, pages 108–109, Cairns, Australia, September 2–7 2001.

[2] B. R. Frieden. Evaluation, design and extrapolation methods for optical signals, based on use of the prolate functions. *Progress in Optics IX*, pages 311–407, 1971.

Excellent reference for use of Prolate Spheroidal functions.

[3] Robert G. Gallager. *Information Theory and Reliable Communication*. John Wiley & Sons, New York, USA, 1968.

[4] D. Gesbert, H. Bölcskei, D. Gore, and A. Paulraj. MIMO wireless channels: Capacity and performance prediction. In *Proceedings IEEE Global Communications Conference, Globecom*, volume 2, pages 1083–1088, 2000.

[5] D. Gesbert, T. Ekman, and N. Christophersen. Capacity limits of dense palm-sized MIMO arrays. In *Proceedings IEEE Global Communications Conference, Globecom*, volume 2, pages 1187 – 1191, November 17–21 2002.

[6] Leif W. Hanlen. *Channel Modelling and Capacity Analysis for Multi-input, Multi-output, Wireless Communication Systems*. PhD thesis, The University of Newcastle, June 2003.

[7] J.Y. Hui, Chunyu B., and Hongxia S. Spatial communication capacity based on electromagnetic wave equations. In *Proceedings IEEE International Symposium on Information Theory, ISIT*, page 342, Washington USA, June 24–29 2001.

Although the authors made the claim that "EM Wave equations can be used to evaluate the capacity of the channel" and "full proofs to appear", they do not provide results. In 2004, Joseph Hui said there was no full paper, as the results were "difficult."

[8] H. Jones. *On Multipath Spatial Diversity in Wireless Multiuser Communications*. PhD thesis, The Australian National University, 2001.

[9] Haley M. Jones, Rodney A. Kennedy, and Thushara D. Abhayapala. On dimensionality of multipath fields: Spatial extent and richness. In *Proceedings IEEE International Conference on Acoustics Speech and Signal Processing, ICASSP*, volume 3, pages 2837–2840, May 2002.

[10] Rodney A. Kennedy and Thushara D. Abhayapala. Spatial concentration of wave-fields: Towards spatial information content in arbitrary multipath scattering. In *4th Australian Communications Theory Workshop, AusCTW*, pages 38–45, Melbourne, Australia, February 4–5 2003.

[11] Rodney A. Kennedy, Thushara D. Abhayapala, and Haley M. Jones. Bounds on the spatial richness of multipath. In *3rd Australian Communications Theory Workshop, AusCTW*, pages 76–80, Canberra, Australia, February 4–5 2002.

[12] Erwin Kreysig. *Introductory functional analysis with applications*. John Wiley & Sons, New York, USA, 1978.

Classic text. Covers spaces, convergence, norms and orthonormal sequences before moving on to operators, spectra and a brief dip into uncertainty principles.

[13] H. Krim and M. Viberg. Two decades of array signal processing. *IEEE Signal Processing Magazine*, pages 67–94, July 1996.

Very detailed review of beamforming theory prior to “space-time” coding. Extensive bibiography.

[14] P. Kyritsi and D. Cox. Effect of element polarization on the capacity of a MIMO system. In *Proceedings IEEE Wireless Communinications and Networking Conference, WCNC*, pages 892–896, 2002.

Polarization can be used to provide additional diversity, line of sight component still causes headaches.

[15] David A. Miller. Communicating with waves between volumes: evaluating orthogonal spatial channels and limits on coupling strengths. *Applied Optics*, 39(11):1681–1699, April 2000.

A good introduction to the use of continuous systems to model communications in an electromagnetic setting. Considers single-frequency scenario. Optical viewpoint. See also [16] and [21].

[16] David A. Miller and R. Piestun. Electromagnetic degrees of freedom of an optical system. *Journal of the Optical Society of America (A)*, 17(6):892–902, May 2000.

[17] J. R. Minkel. The hollow universe. *New Scientist Magazine*, 27 April 2002.

Discussion of information theoretic aspects of 2 dimensional universe

[18] A. L. Moustakas, H. U. Baranger, L. Balents, A. M. Sengupta, and S. H. Simon. Communication through a diffusive medium: coherence and capacity. *Science*, 287:287–290, January 1 2000.

[19] H. Nyquist. Certain topics in telegraph transmission theory. *Transactions of the AIEE*, pages 617–644, February 1928.

[20] M. Petrich. On the number of orthogonal signals which can be placed in a WT-product. *Journal of the Society for Industrial and Applied Mathematics*, 11(4):936–940, December 1963.

[21] R. Piestun and D. A. Miller. Degrees of freedom of an electromagnetic wave. In *18th Congress of the International Commission for Optics: Optics for the Next Millennium*, pages 110–111, San Francisco, CA, August 1999.

[22] Tony S. Pollock. *On Limits of Multi-Antenna Wireless Communications in Spatially Selective Channels*. PhD thesis, The Australian National University, July 2003.

[23] Claude E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27:379–423, 623–656, July 1948.

[24] T. Svantesson and J. W Wallace. On signal strength and multipath richness in multi-input multi-output systems. In *Proceedings IEEE International Conference on Communications, ICC*, pages 2683–2687, Anchorage, Alaska, May 11–15 2003.

[25] I. E. Telatar. Capacity of multi-antenna Gaussian channels. Technical Report # BL0112170-950615-07TM, AT & T Bell Laboratories, 1995.

The result $\log\det(I + pHH^*)$ *is famously derived in this paper, although subsequent authors have often mis-applied it.*

[26] I. Emre Telatar. Capacity of multi-antenna Gaussian channels. *European Transactions on Telecommunications*, 10(6):585–595, November 1999.

Although this is the "official" reference of this paper, most authors use the technical report [25]. Together these papers form an exceptional tutorial in how to derive fundamental MIMO results.

[27] I. Emre Telatar and David N. C. Tse. Capacity and mutual information of wideband multipath fading channels. *IEEE Transactions on Information Theory*, 46(4):1384–1400, July 2000.

[28] J. W. Wallace and M. A. Jensen. The capacity of MIMO wireless systems with mutual coupling. In *Proceedings IEEE Vehicular Technology Conference, VTC*, pages 696–700, Vancouver, Canada, Fall 2002.