Continuous Wireless Communications

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NATIONAL ICT AUSTRALIA

THE AUSTRALIAN NATIONAL UNIVERSITY

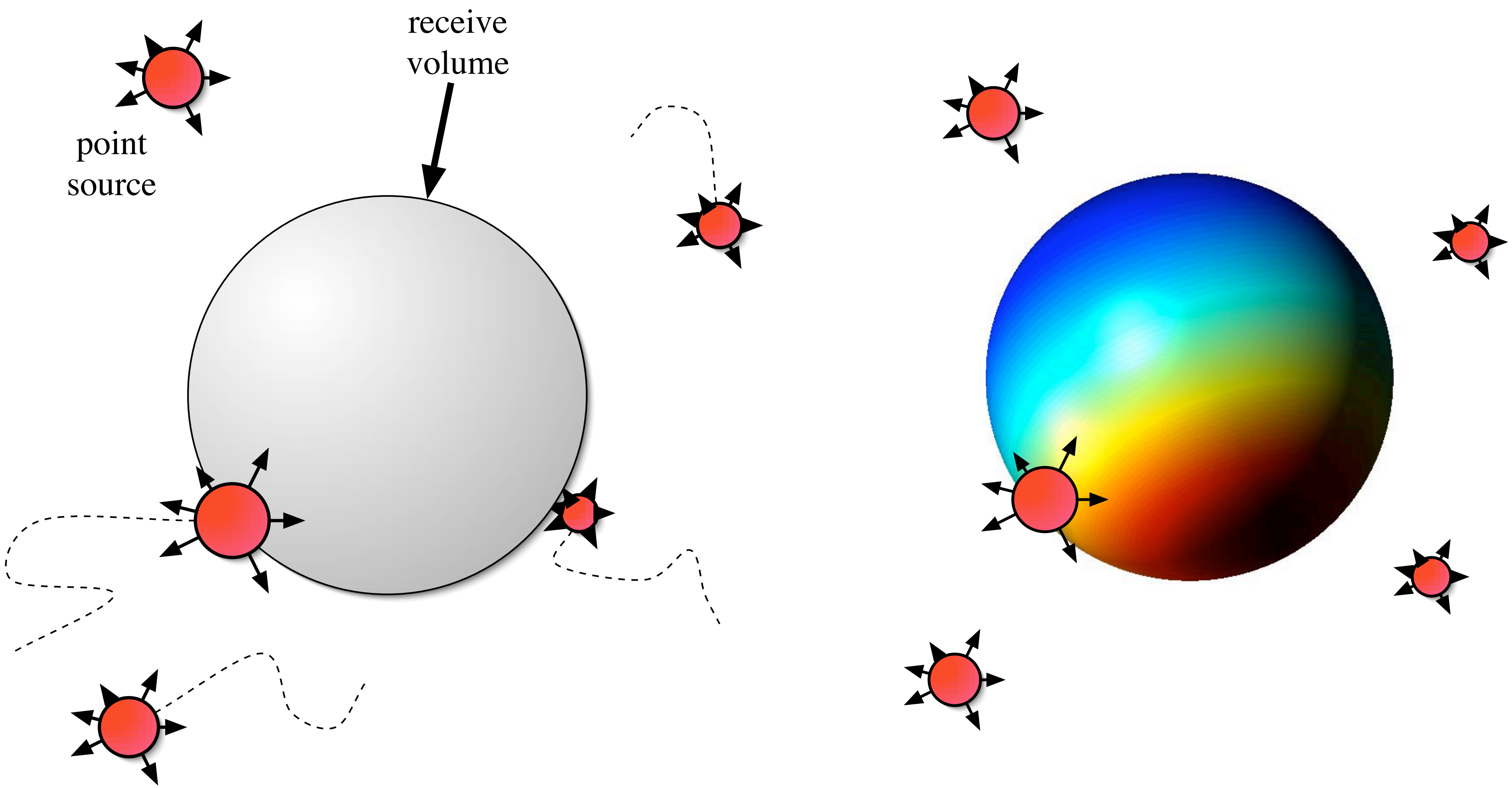
A simple question?

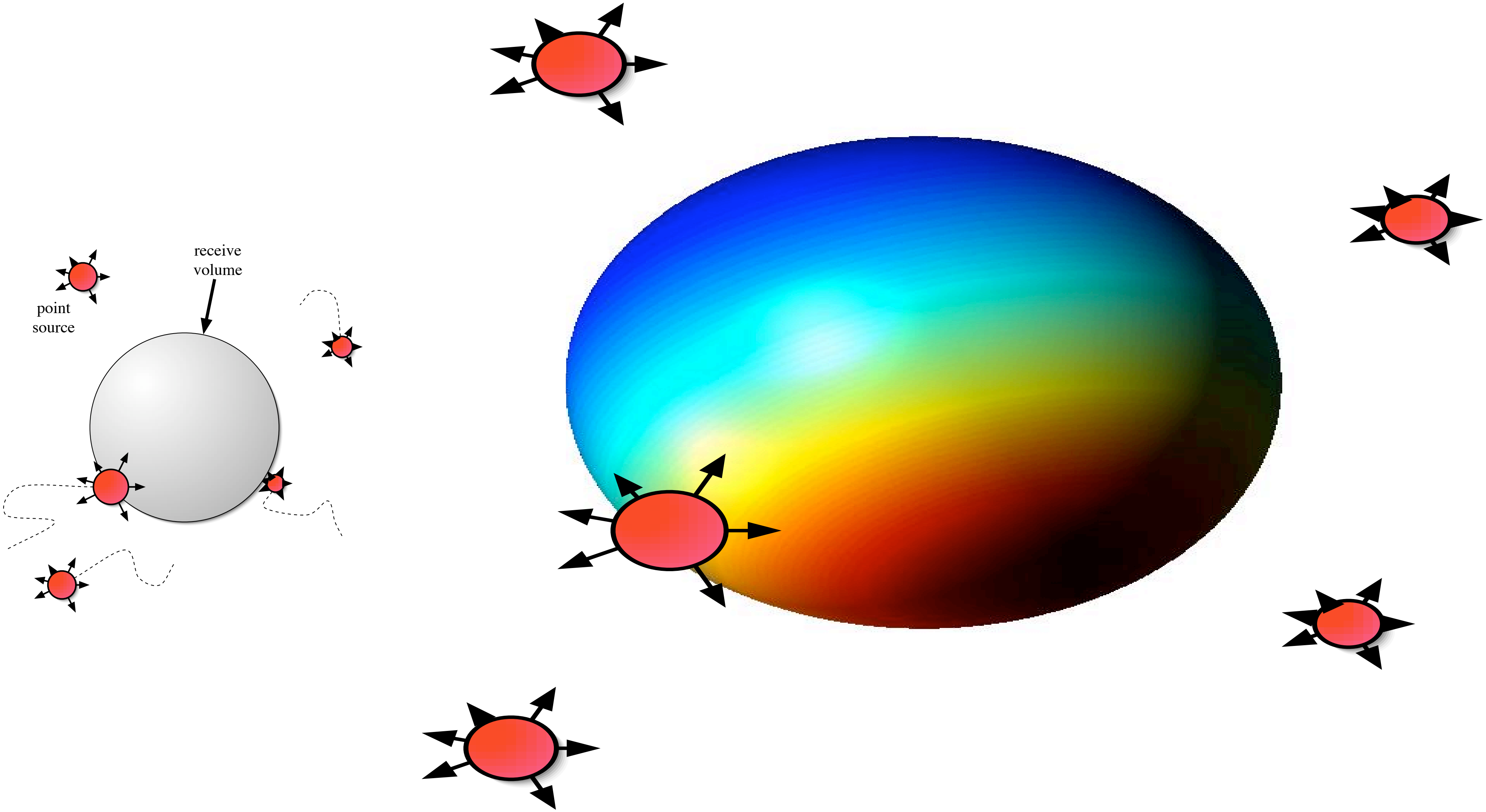
How much information can be stored in a volume of space?
How small can the Library of Congress get?
How much information can be transmitted between two volumes?

Is the answer to these, respectively ∞, 0 and ∞?

[Sza04] David Szabados. Seagate sees the future: Storing the U.S. library of congress on a 50-cent coin. www.seagate.com, 2004.

And?




receive
volume
point
source

Course Overview

- The background
  - Crash Course: Functional Analysis for Comms
  - Crash Course: Information Theory
  - Application: MIMO from the ground up.
- The idea
  - Operators for Comms
  - Numerical techniques
  - Info Theory for Operator Comms

Why?

Beyond taunting, why would we care?

- Objective: provide formal background for new work

House keeping

- This is not undergraduate work anymore
- No exam: Work will be in several blocks. Collaboration acceptable with appropriate credit. Informal: we can skip through sections and/or focus heavily on sections as needed. However....
- Options for assessment:
  - Review of literature
  - Presentation
  - New work – eg. paper submission
  - Open: *If you want a particular form of assessment let me know ASAP.*
  - *40 hours of assessment.*
- Attendance is not compulsory. Participation encouraged

Structure of course

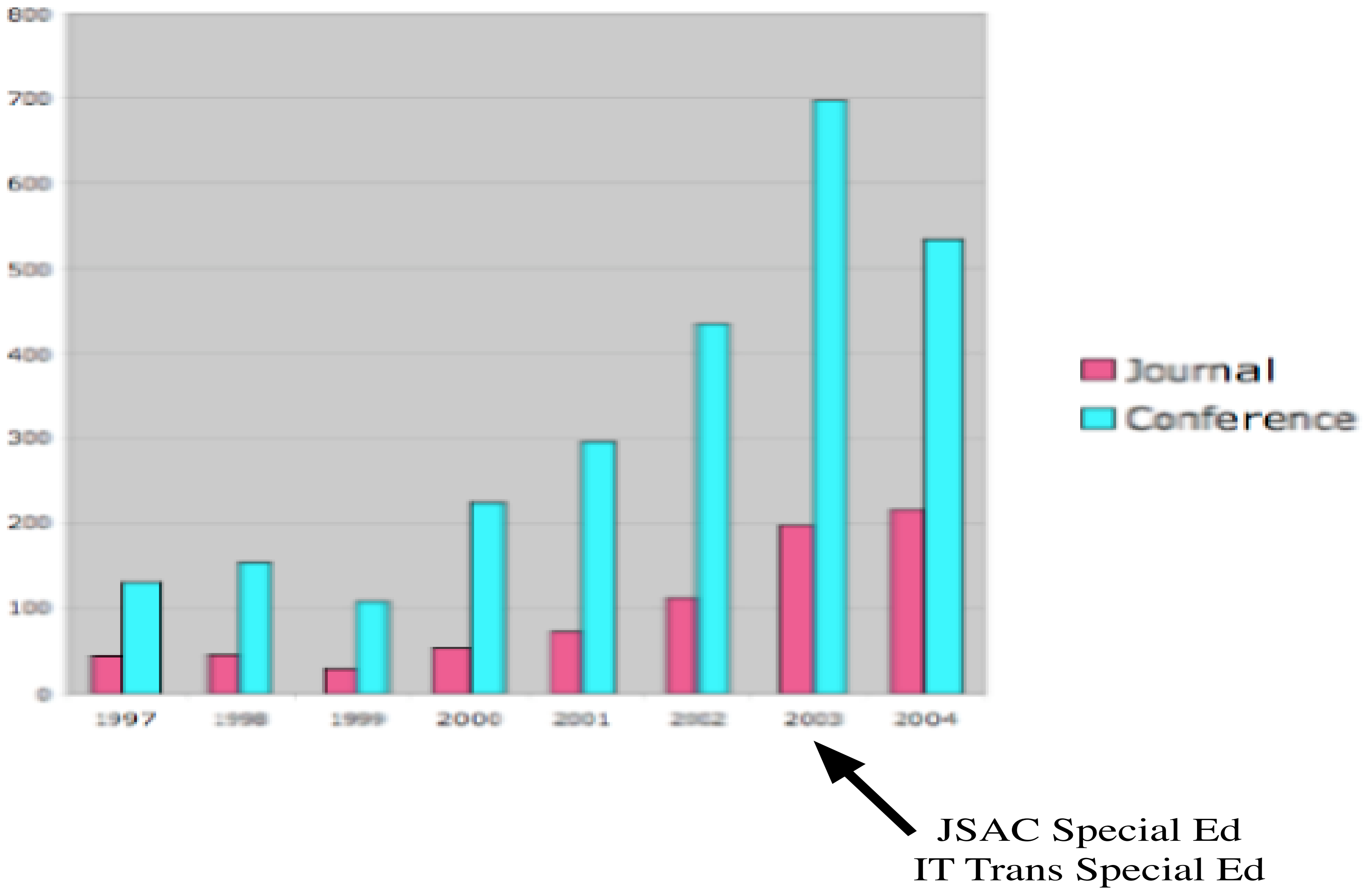
- Focus on new work, in Wireless Signal Processing group.
- Reading brick available. Brick $\neq$ literature survey
- Recommended Texts:
  - Kreysig, *Introductory Functional Analysis with Applications*
  - Telatar, *Capacity of Multi-antenna Gaussian Channels*
  - Gallager, *Information Theory and Reliable Communication*
- Many others.

[Kre78] Erwin Kreysig. *Introductory functional analysis with applications*. John Wiley & Sons, New York, USA, 1978.

[Tel99] I. Emre Telatar. Capacity of multi-antenna Gaussian channels. *Euro. Trans. Telecomm.*, 10(6):585–595, November 1999.

[Gal68] Robert G. Gallager. *Information Theory and Reliable Communication*. John Wiley & Sons, New York, USA, 1968.

World-wide interest in MIMO



History of Continuous Spatial Channels

- **1928** Harry Nyquist: temporal sampling as side-issue in Telegraph ISI channel
- **1948** Claude Shannon: discrete and continuous time channels (section 10). Section 10 largely overlooked in subsequent rush.
- **1962–1968** Robert Gallager: Bandlimited filter channel. $2WT$ rule-of-thumb. Vector Channel.
- **1964–1978** Slepian, et.al: precise theory of bandlimited functions
Theory is widely confused as artifact of Fourier Transform.
- **1995** Telatar: Renewed interest in vector channel.
Widely inaccurate application by immature MIMO audience.

[Tel95] I. E. Telatar. Capacity of multi-antenna Gaussian channels. Technical Report # BL0112170-950615-07TM, AT & T Bell Laboratories, 1995.

[Pet63] M. Petrich. On the number of orthogonal signals which can be placed in a WT-product. *J. Soc. Indust. Appl. Math.*, 11(4):936–940, December 1963.

[Nyq28] H. Nyquist. Certain topics in telegraph transmission theory. *Transactions of the AIEE*, pages 617–644, February 1928.

[Sha48] Claude E. Shannon. A mathematical theory of communication. *Bell System Tech. J.*, 27:379–423, 623–656, July 1948.

History of Continuous Spatial Channels

David Miller

Emre Telatar

- **2000** Miller: Optical channel as eigenmode problem
- **2001** Telatar: $1/N$ factor for MIMO conditioning *Early concerns about perpetual growth*
- **2003** Kennedy et.al: Operator forms for Helmholtz Balls

[Mil00] David A. Miller. Communicating with waves between volumes: evaluating orthogonal spatial channels and limits on coupling strengths. *Applied Optics*, 39(11):1681–1699, April 2000.

[CRT01] N. Chiurtu, B. Rimoldi, and E. Telatar. Dense multiple antenna systems. In *IEEE Information Theory Workshop*, pages 108–109, Cairns, Australia, September 2–7 2001.

[KA03] Rodney A. Kennedy and Thushara D. Abhayapala. Spatial concentration of wave-fields: Towards spatial information content in arbitrary multipath scattering. In *4th Aust. Commun. Theory Workshop, AusCTW*, pages 38–45, Melbourne, Australia, February 4–5 2003.

[FH03] Minyue Fu and Leif W. Hanlen. Capacity of MIMO channels: A volumetric approach. In *Proc. IEEE Intl. Conf. Commun., ICC*, pages 2673–2677, Anchorage, Alaska, May 11–15 2003.

Trouble...

Many authors have *attempted* to show a spatial limit to capacity

- **2000** Moustakas et. al. *Science* article on limiting density
- **2001** Hui et al. ISIT claim for scattering matrices
- **2002** Gesbert et.al. "Dense mobile devices"

Unfortunately, while it is easy to say "wireless signals obey Maxwell's equations", it is not clear how to proceed beyond a simple linear model. The results have an intuitive appeal, but little more.... So, are any of the first questions easier to solve?

[MBB+00] A. L. Moustakas, H. U. Baranger, L. Balents, A. M. Sengupta, and S. H. Simon. Communication through a diffusive medium: coherence and capacity. *Science*, 287:287–290, January 1 2000.

[HBS01] J.Y. Hui, Chunyu B., and Hongxia S. Spatial communication capacity based on electromagnetic wave equations. In *Proc. IEEE Intl. Symp. Inform. Theory, ISIT*, page 342, Washington USA, June 24–29 2001.

[GEC02] D. Gesbert, T. Ekman, and N. Christophersen. Capacity limits of dense palm-sized MIMO arrays. In *Proc. IEEE Globecom*, volume 2, pages 1187 – 1191, November 17–21 2002.

[BK03] Jacob D. Bekenstein and Alfred T. Kamajian. Information in the holographic universe. *Sci. Am.*, pages 48–55, August 2003.

A simple question?

How much information can be stored in a volume of space?
What is the minimum volume for a fixed amount of information?
How much information can be transmitted between two volumes?

Some suspicion might suggest the answer is not ∞, 0 and ∞ respectively....

Or it's going to be a long 10 weeks....

What to do?

James Clerk Maxwell. 1831–1879. Developed axiomatic approach to EM.

Hermann Ludwig Ferdinand von Helmholtz. 1821–1894

- All wireless transmissions obey Maxwell's Laws.

$$\underbrace{\nabla f(\mathbf{r},t) + kf(\mathbf{r},t)}_{\textit{receive fn}} = \underbrace{g(\mathbf{r},t)}_{\textit{driving fn}}$$

- By itself this is rather impressive, but slightly useless
  - We DO NOT want to model EM fields
- Next try: Observe that for $g(\mathbf{r}) = 0$ (we'll return to this later) the field is a received field only. Satisfies Helmholtz equation:

$$\nabla f(\mathbf{r}) + kf(\mathbf{r}) = 0$$

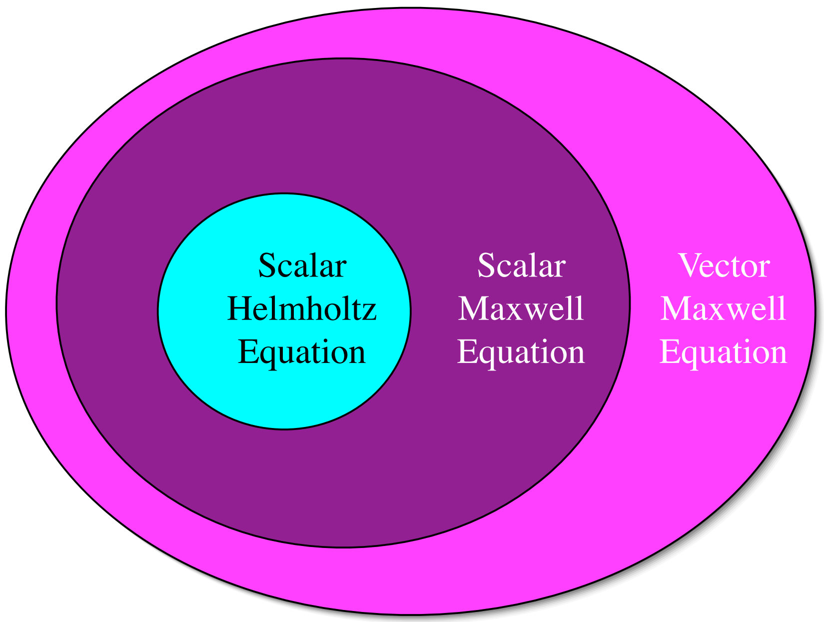
What to do?

- All wireless transmissions obey Maxwell's Laws.

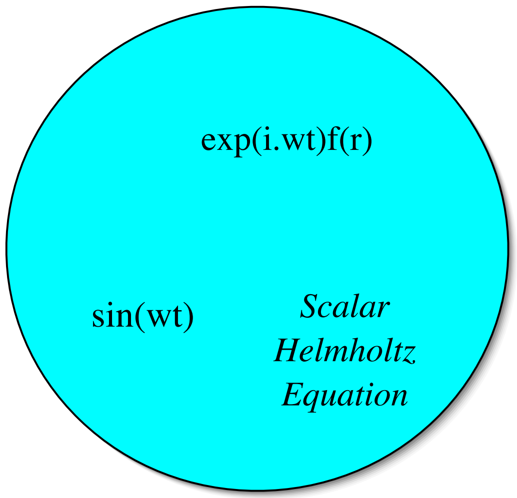
$$\underbrace{\nabla f(\mathbf{r},t) + k f(\mathbf{r},t)}_{\textit{receive fn}} = \underbrace{g(\mathbf{r},t)}_{\textit{driving fn}} = 0$$

- Some obey additional constraints

Helmholtz



What to do?



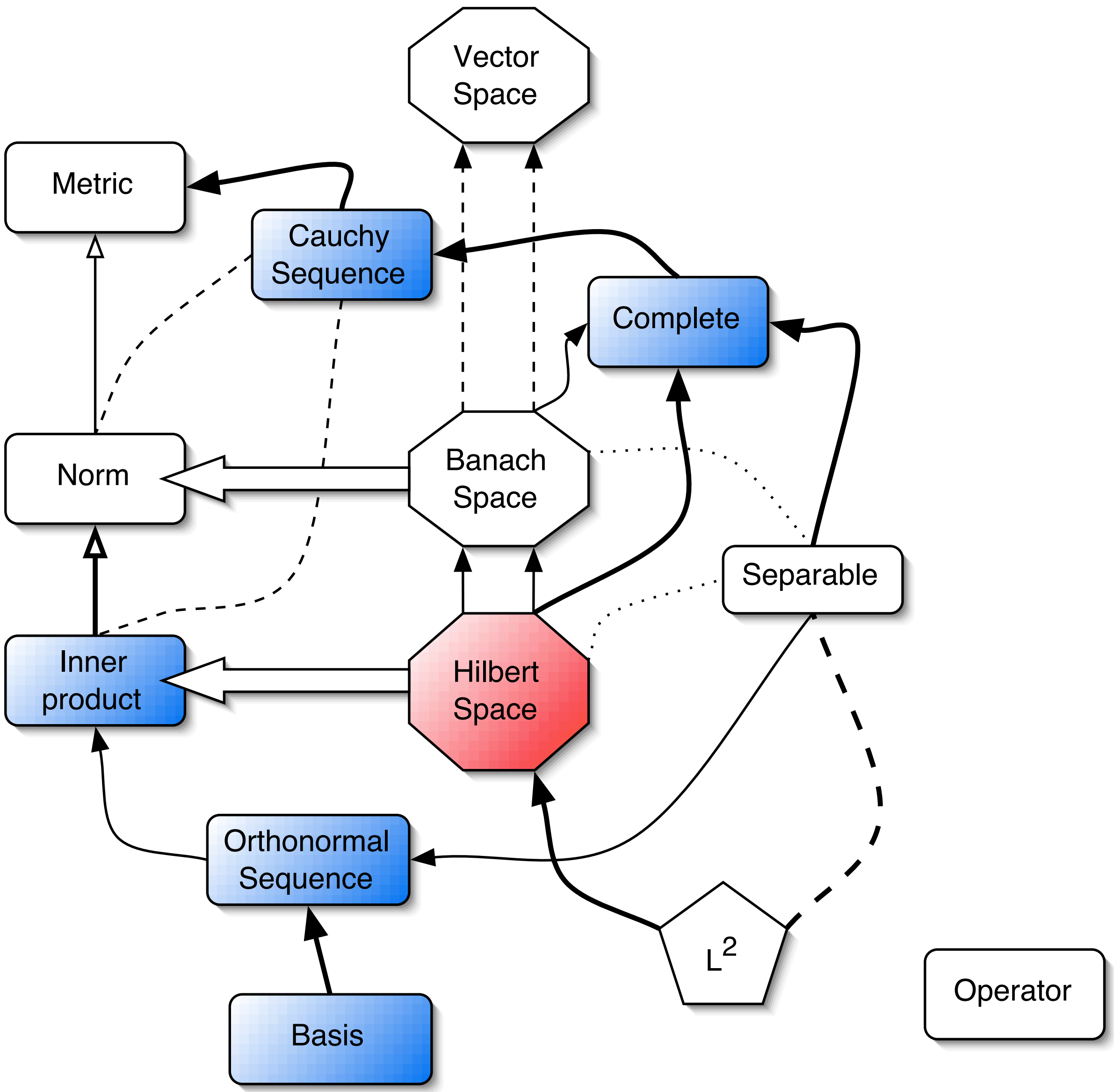
- Only certain functions obey scalar Helmholtz equation
- Consider bag of objects
- Math-speak: A space of functions
  – Don't care if they're functions or not.
  – A space of vectors

- Question: Do we need to investigate every function in the blue ball?
- We are rapidly approaching Functional Analysis
- Our aim: Use *some* Hilbert Space techniques to overcome functional difficulties

- It is possible to generate most of the results with vague, hand-waves toward functional representation.
- It is not possible to develop new results without a thorough understanding of the terminology
- See introduction and section 3 of Miller [Mil00]

[Mil00] David A. Miller. Communicating with waves between volumes: evaluating orthogonal spatial channels and limits on coupling strengths. *Applied Optics*, 39(11):1681–1699, April 2000.

Concept Map


Vector Space
Metric
Cauchy Sequence
Complete
Norm
Banach Space
Separable
Inner product
Hilbert Space
Orthonormal Sequence
L2
Basis
Operator

Functional Analysis for the Impatient

David Hilbert

Rod Kennedy

- We will cover basic topics in 1–2 lectures.
  - Geometrical viewpoint
- Recall early vector space classes
  - What is a vector space?
  - What is a vector?
- Hilbert spaces arose from need:
  - Fourier series representation
  - How “big” is ∞?
  - What is our need?

[Ken02] Rodney A. Kennedy. Hilbert spaces with applications. Research School of Information Science & Engineering, Australian National University, Australia, 2002. Lecture notes.

[Hil00] David Hilbert. Mathematical problems. Technical report, International Congress of Mathematicians, Paris, France, 1900. `http://aleph0.clarku.edu/~djoyce/hilbert/problems.html`.

[DM99] L. Debnath and P. Mikusińksi. *Introduction to Hilbert Spaces with Applications*. Academic Press, San Deigo, CA, USA, 1999.

[Kre78] Erwin Kreysig. *Introductory functional analysis with applications*. John Wiley & Sons, New York, USA, 1978.

Objective

- Given a received signal in space, want to know how to represent it

- How close is it to “another” “similar” signal?

- How many “different” signals can we measure?

- Before we can represent objects we need concepts:

  1. Distance between objects (why?)
  2. Completeness

Vector space

DEFINITION 1 (VECTOR $\mathbf{x}$).
Atomic mathematical unit. Points $\mathbf{x}$ in an abstract setting.

- $\mathbf{x} = [1, 2, 3]$
- $\mathbf{x} = [a, b, \ldots]$
- Beware “Matlab” view.
- How can we combine two vectors? $\mathbf{x}$, $\mathbf{y}$

Vector space

DEFINITION 2 (VECTOR SPACE $\mathcal{X}$).
A collection of vectors, whose elements $\mathbf{x} \in \mathcal{X}$ obey a common rule. Space acts over a field $\mathbb{F}$ of scalars.

Example: $\mathcal{X} = \{\mathbf{x} \in \mathcal{X} : \mathbf{x} = [\alpha, \beta]\}$ and $\mathbb{F} = \mathbb{R}$

Basic Properties

- for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ and any $\alpha, \beta \in \mathbb{F}$ then $\alpha\mathbf{x} + \beta\mathbf{y} = \mathbf{z} \in \mathcal{X}$
- $\alpha\left(\mathbf{x} + \mathbf{y}\right) + \beta\mathbf{x} = \left(\alpha + \beta\right)\mathbf{x} + \alpha\mathbf{y} \in \mathcal{X}$

Metric

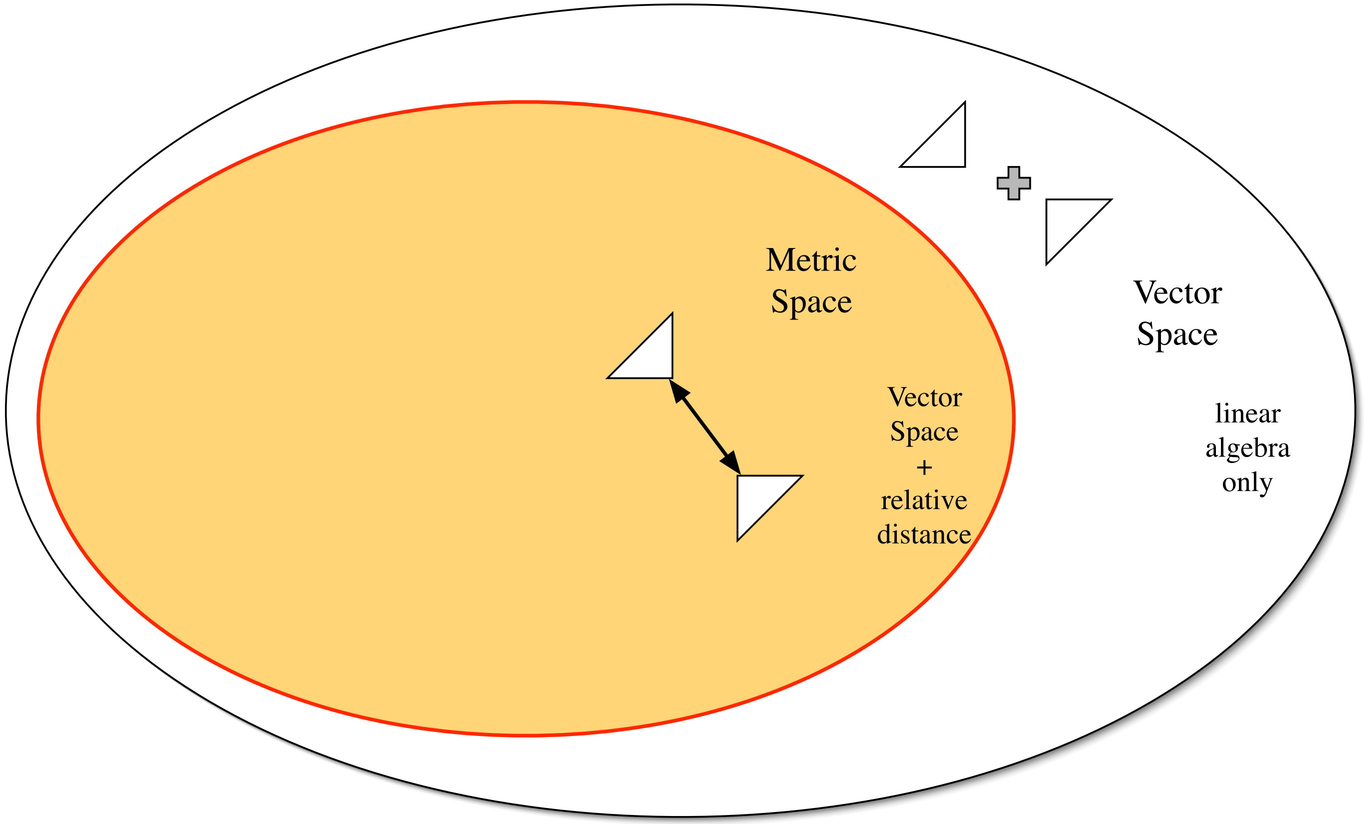
- How "far apart" are two vectors? (eg. $\mathbf{x}$ and $\mathbf{y}$)
  - For Vector Spaces we can't answer this!

DEFINITION 3 (METRIC $d(\mathbf{x},\mathbf{y})$).
Real-valued, with domain $\mathcal{X} \times \mathcal{X}$ and for $\mathbf{x},\mathbf{y} \in \mathcal{X}$:

1. $d(\cdot,\cdot)$ *is real-valued, non-negative, finite.*
2. $d(\mathbf{x},\mathbf{y}) = 0$ *if and only if* $\mathbf{x} = \mathbf{y}$ *Be careful here....*
3. $d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x})$ *Symmetric*
4. $d(\mathbf{x},\mathbf{y}) \leq d(\mathbf{x},\mathbf{z}) + d(\mathbf{z},\mathbf{y})$ *Triangle*

Metric Space

- A Vector Space with a metric attached. $\{\mathcal{X}, d(\cdot,\cdot)\}$
- How "far apart" are two vectors (eg. $\mathbf{x}$ and $\mathbf{y}$) : use metric.
- We can tell how far apart two objects are without knowing how big they are



Norm

- Introduces magnitude: How "big" is $\mathbf{x}$?

DEFINITION 4 (NORM $\|\mathbf{x}\|$).
A norm is a real-valued function whose domain is a vector space $\mathcal{V}$ and whose value at $\mathbf{x} \in \mathcal{V}$ satisfies

$$\|\mathbf{x}\| : \{\mathbf{x} \in \mathcal{V} \mapsto \mathbb{R}^+\}$$

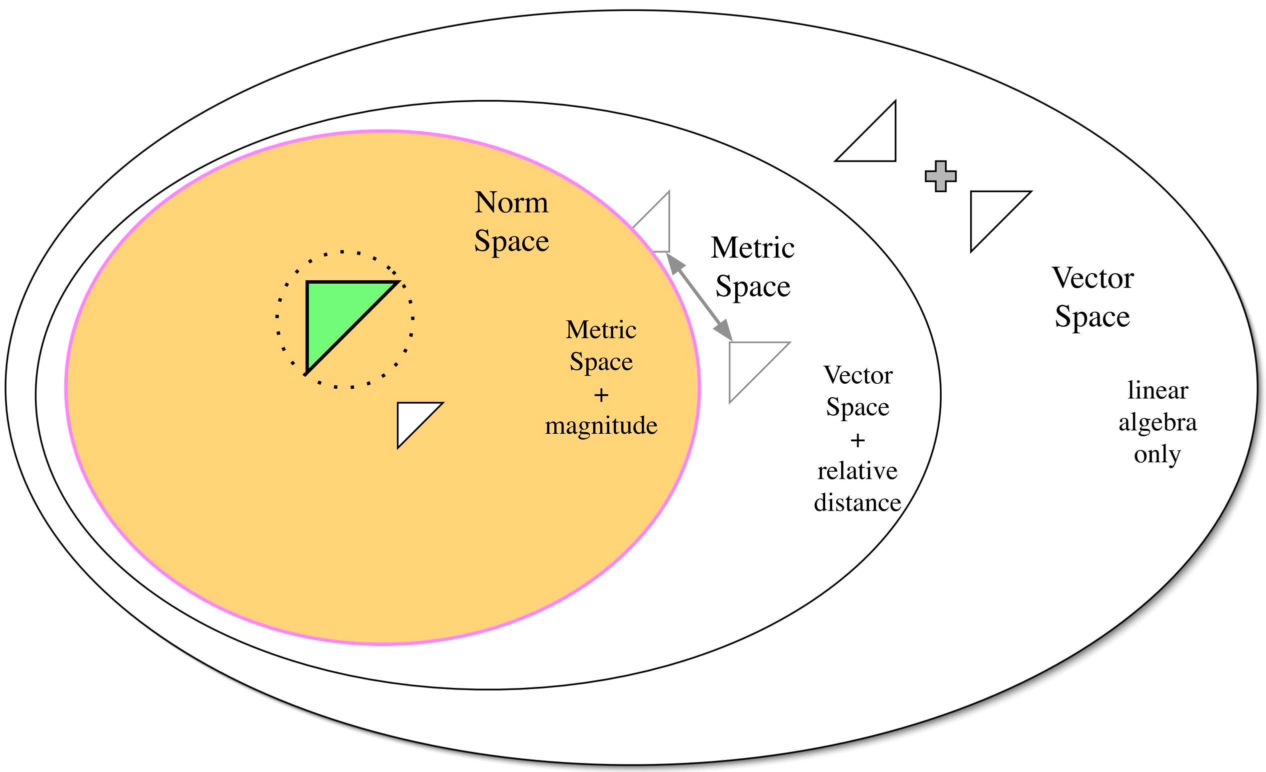
For $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $\alpha \in \mathbb{F}$ a norm satisfies

1. $\|\mathbf{x}\| \geq 0$ *(Positive)*
2. $\|\mathbf{x}\| = 0$ *if, and only if* $\mathbf{x} = \mathbf{0}$ *Note this is* $\mathbf{0}$ *the vector!*
3. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ *(Linear)*
4. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *(Triangle inequality)*

Normed Space

- A Vector Space with a *norm* attached. $\{\mathcal{X}, \|\cdot\|\}$

- How "far apart" are two vectors (eg. $\mathbf{x}$ and $\mathbf{y}$) in the normed space?



Inner Product

- Introduces angle
- Project $\mathbf{x}$ onto $\mathbf{y}$

DEFINITION 5 (INNER PRODUCT $\langle \mathbf{x} | \, \mathbf{y} \rangle$).
Is a real-valued function of pairs of vectors $\mathbf{x}$ and $\mathbf{y}$ such that for all vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and scalars α,

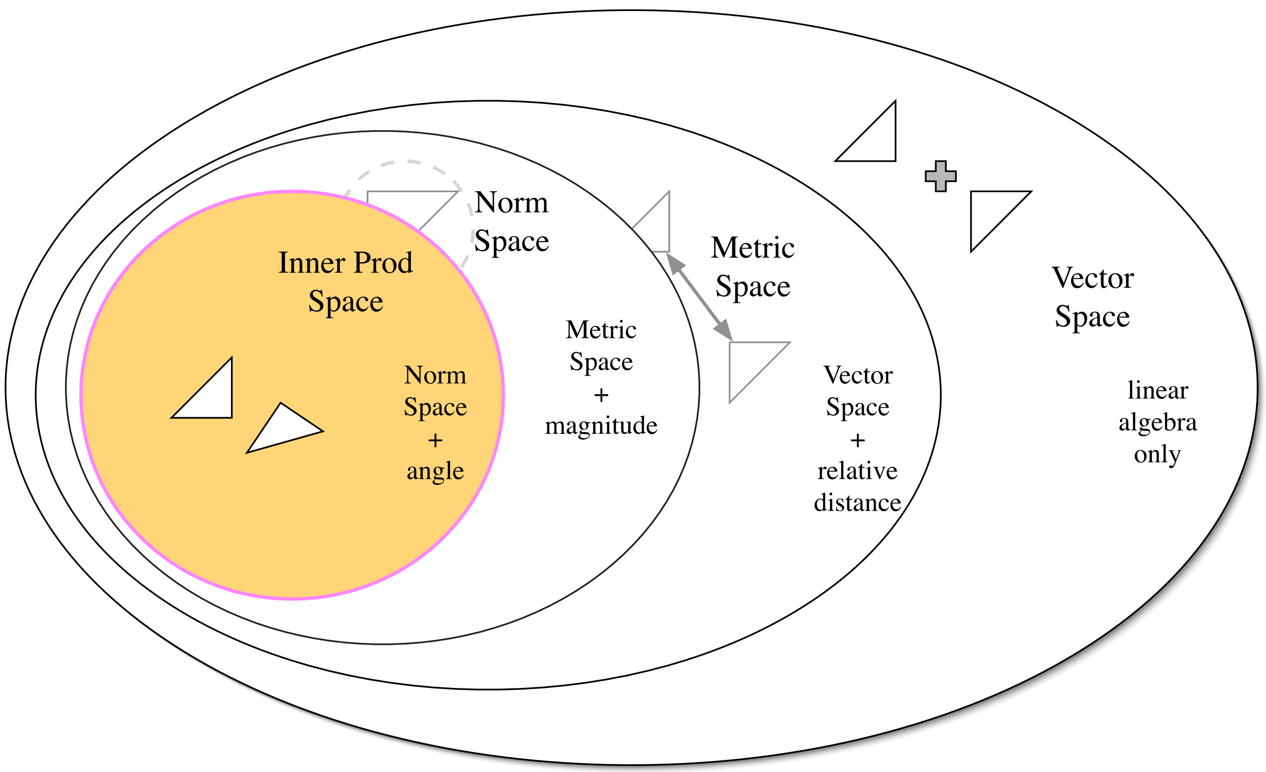
1. $\langle (\mathbf{x} + \mathbf{y}) | \, \mathbf{z} \rangle = \langle \mathbf{x} | \, \mathbf{y} \rangle + \langle \mathbf{x} | \, \mathbf{z} \rangle$
2. $\langle \alpha \mathbf{x} | \, \mathbf{y} \rangle = \alpha \langle \mathbf{x} | \, \mathbf{y} \rangle$
3. $\langle \mathbf{x} | \, \mathbf{y} \rangle = \overline{\langle \mathbf{y} | \, \mathbf{x} \rangle}$
4. $\langle \mathbf{x} | \, \mathbf{x} \rangle \geq 0$
5. $\langle \mathbf{x} | \, \mathbf{x} \rangle = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$.

Inner Product Examples

- $\int f(x)g(x)$
- $\sum_k f_k g_k$
- Vector Dot Product

Inner Product Space

- A Vector Space with an *inner product* attached. $\{\mathcal{X}, \langle \cdot | \cdot \rangle\}$
- How "far apart" are two vectors (eg. $\mathbf{x}$ and $\mathbf{y}$) in the inner product space?



Inner Product Space

- Very nice properties.
- Parallelogram Inequality Holds:

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

- Schwarz inequality: $|\langle x|\, y\rangle| \leq \|x\|\|y\|$
- Examples:

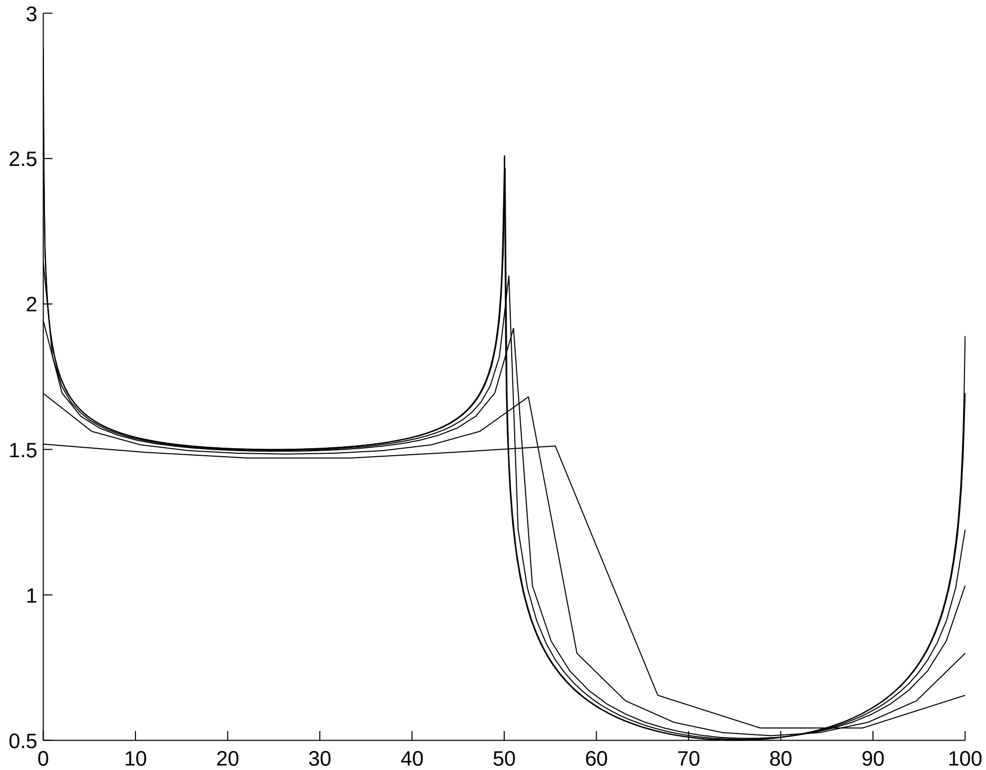
  l^2 is the space of all sequences of complex numbers with finite square-sum $\sum_{k=0}^{\infty} |x_k|^2 < \infty$

  L^2 is the space of all Lebesgue integrable functions, with finite mean square integral $\int |f(r)|^2 \, dr < \infty$

- Non-examples: $l^p, \quad p \in \mathbb{Z}, p \neq 2$

How do I represent an object?

- Recall vector spaces: we want something like a basis. why?
- We need a set $\mathcal{B}$ which is dense in the space.
- Most definitions of "dense" are somewhat circular. So think of successive approximations of Fourier Series.



Transfinite: slightly bigger than finite....

- Finite Vector analysis: matrices
  - what is the dimension of a given vector space?

- To get our basis concept, we need sequences.

Transfinite: slightly bigger than finite....

- Finite Vector analysis: matrices
  - what is the dimension of a given vector space?

- To get our basis concept, we need sequences.

> **DEFINITION 6 (SEQUENCE).**
> *A sequence is a countably infinite ordered **set**.*
> *The set $\{x, y, z, \ldots\}$ may be made into a (sub-) sequence by defining an ordering on the elements.*

- The integers $\{1, 2, 3, 4, \ldots, k\}$ are a sub-sequence for $k < \infty$

- The set of positive integers $\mathbb{Z}^+ = \{i\}_{i=1}^{\infty}$ is a sequence.

- What about the rationals $\mathbb{Q}$?

Pop quiz

1. Can negative integers be ordered? $\mathbb{Z} = \left\{ \{-i\}_{i=1}^{\infty}, 0, \{i\}_{i=1}^{\infty} \right\}$
2. which of $\mathbb{Z}$ and $\mathbb{Z}^{+}$ has more elements?
3. Can rationals be ordered? $\frac{q}{r}, \qquad q, r \in \mathbb{Z}$
4. Are there more rationals than integers?
5. what about reals?
6. What does this matter?

Pop quiz answers

Georg Cantor
1845–1918

- The integers have cardinality $\aleph_0$ pronounced "aleph-zero" from Hebrew.
- $\aleph_0 + k = \aleph_0 \quad \aleph_0 + \aleph_0 = \aleph_0 \quad \aleph_0 \cdot k = \aleph_0 \quad \aleph_0 \cdot \aleph_0 = \aleph_0$
- Countably infinite concept
- The reals are uncountably infinite.
- Cantor introduced concept of equal cardinality: if we can place a set in 1-to-1 relation with a set of cardinality c then they both have equal cardinality
- Nasty quiz question: Is the cardinality of a square of integers equal to the cardinality of a line of integers?

Transfinite interlude continued

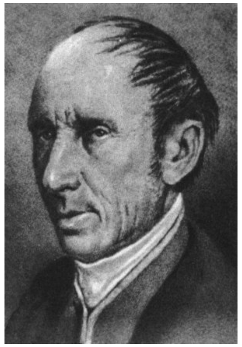
DEFINITION 7 (CONVERGENT SEQUENCE).
Consider a sequence $X = \{x_1, x_2, \ldots, x_k, \ldots, x_n\}$ *in a metric space* $\mathcal{M}$ *with metric* $d(x, y)$.
The sequence X ***converges*** *if it contains a value* $x^\star \in X$ *so that the limit:*

$$\lim_{n\to\infty} d(x^\star, x_n) = 0$$

The value $x^\star$ *is called the limit of the sequence.*

- Note $x^\star \in X$.
- Why do we need a metric space?
- Does the sequence $\{1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000} \ldots\}$ converge

Transfinite interlude continued



Augustin Cauchy
1789 - 1857

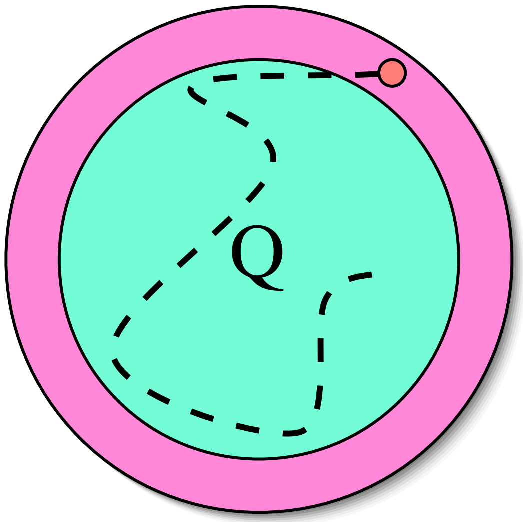
DEFINITION 8 (CAUCHY SEQUENCE).
Consider a sequence $X = \{x_1, x_2, \ldots, x_k, \ldots, x_n\}$ *in a metric space* $\mathcal{M}$ *with metric* $d(x, y)$. *If, for every scalar* $\epsilon > 0$, *there is a scalar* N *such that*

$$d(x_i, x_j) < \epsilon, \quad \forall i, j > N$$

the sequence X *is a* ***Cauchy sequence.***

- What's the difference?
- Hint: Cauchy sequences are what we (as engineers) like to think of as limits

Completeness



Definition 9 (Completeness).
*cf. [**?**, pp. 28] A (metric) space $\mathcal{X}$ is **complete** if every Cauchy sequence Y in $\mathcal{X}$ converges in $\mathcal{X}$.* ie. *in $\mathcal{X}$ every Cauchy sequence Y is also convergent.*

- The rationals $\mathbb{Q}$ are not complete
- We can complete the rationals, by adding in all the limits of all the sequences.
  - This gives the reals (continuum)

Completeness for normed spaces

Stefan Banach

1892–1945

David Hilbert

1862–1943

- Spaces can be completed too.
  - Complete normed spaces are Banach Spaces
  - Complete inner product spaces are Hilbert Spaces

Completeness for normed spaces

- Recall the Fourier series earlier.

- Dangerous statement: Consider an infinite sequence of points (functions) in a normed space.

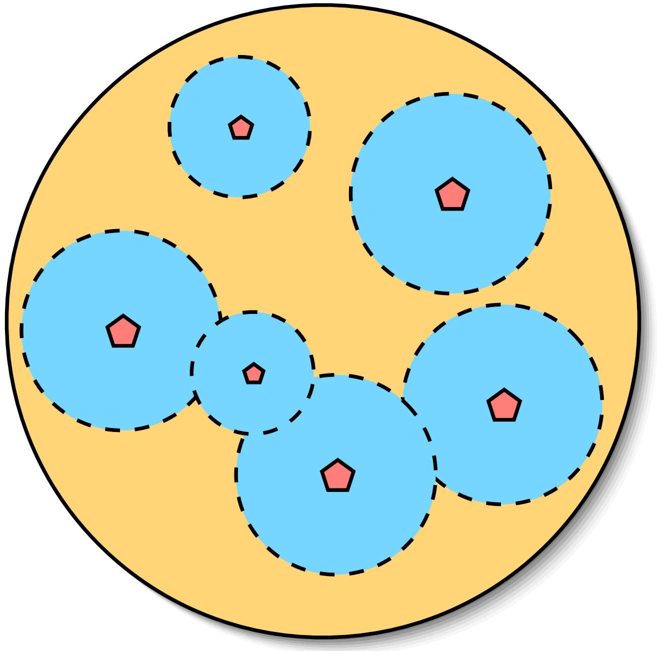
$$f_k(x) = \begin{cases} -1 & x \leq -\frac{1}{k} \\ x & -\frac{1}{k} < x < \frac{1}{k} \\ 1 & x \geq \frac{1}{k} \end{cases}$$

- What do the functions in this sequence look like as $k \to \infty$?

- Is this sequence Cauchy?

- Does this sequence converge?

Density



DEFINITION 10 (DENSE SET).
A subset $M \subseteq X$ is dense in X if, for every point $\mathbf{x} \in X$, and $\epsilon > 0$ there is a point $\mathbf{p} \in M$ such that $d(p, x) < \epsilon$

Every ball $B \in X$ contains elements from M

DEFINITION 11 (SEPARABLE).
A set is separable if it contains a countably dense subset.

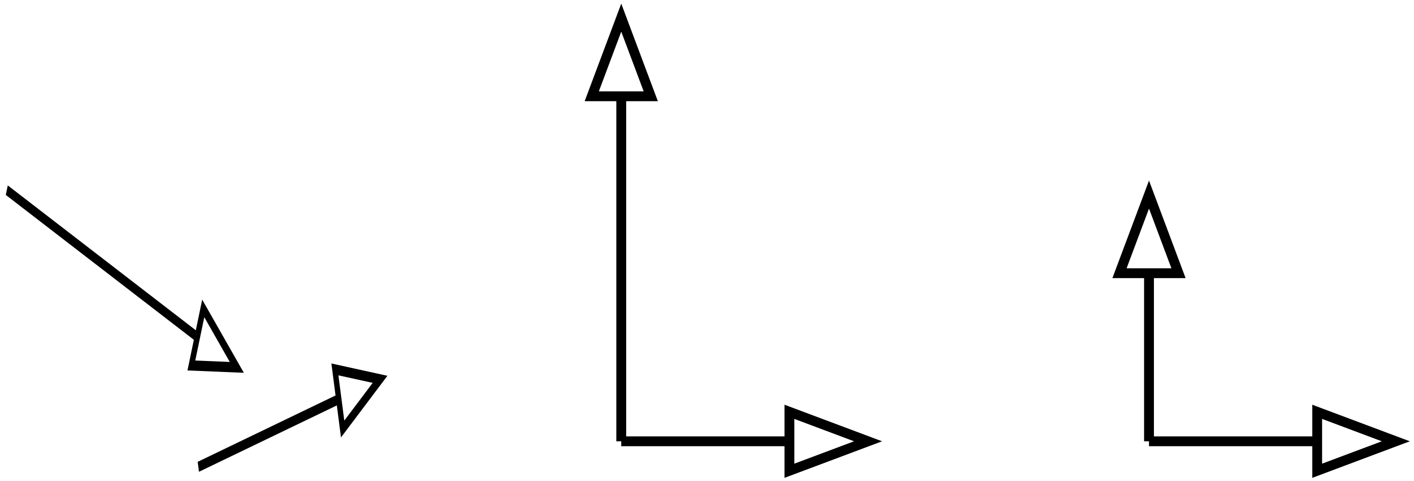
- Countable dense sets act as back-bones for spaces.
- We will use them as basis-like objects shortly.

Our tools

Given a (separable) Hilbert Space $\mathcal{H}$

- Inner product $\langle \mathbf{x} | \, \mathbf{y} \rangle$
- Completeness
- Dense
- what else do we need?

Orthonormal



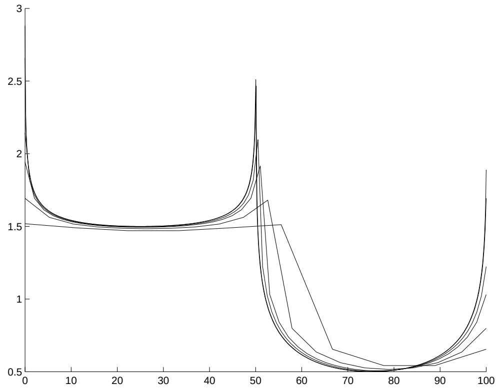
(a) Vectors

(b) Ortho(gonal) Vectors

(c) Ortho(gonal), Normal(ised) Vectors: orthonormal

Vectors, Orthogonal vectors and Orthonormal vectors.

Complete Orthonormal Sequences



- When will Fourier series converge?
- We need to use the norm as an error measure

$$\left\| f(x) - \sum_{k}^{N} \varphi(x) \cdot \alpha_k \right\|$$

- is $N \to \infty$ enough?

Orthonormal Sequences in $\mathcal{H}$

DEFINITION 12 (ORTHONORMAL SET).
A set of vectors S is Orthonormal if for all $\mathbf{x}, \mathbf{y} \in S$:

$$\langle \mathbf{x} | \, \mathbf{y} \rangle = \begin{cases} 0 & \mathbf{x} \neq \mathbf{y} \\ 1 & \textit{else} \end{cases}$$

and, if we can enumerate each vector

DEFINITION 13 (ORTHONORMAL SEQUENCE).
A sequence of vectors $S = \{\mathbf{x}_k\}_{k=0}^{\infty}$ is Orthonormal if $\langle \mathbf{x}_i | \, \mathbf{x}_j \rangle = \delta_{ij}$ for all $i, j \in \mathbb{Z}^*$

Complete Orthonormal Sequences in $\mathcal{H}$

DEFINITION 14 (EQUIVALENCE).
For a complete orthonormal sequence $\{\mathbf{e}_i\}_{i=1}^{\infty}$ in a separable Hilbert Space $\mathcal{H}$, the following statements are equivalent

1. $f = \sum_{i=1}^{\infty} \langle f | \mathbf{e}_i \rangle \mathbf{e}_i \quad \forall f \in \mathcal{H}$, *Fourier Expansion*

2. $\forall \epsilon > 0, \quad \exists N_0 < \infty$ *such that* $\|f - \sum_{i=1}^{n} \langle f | \mathbf{e}_i \rangle \mathbf{e}_i\| < \epsilon \quad \forall n > N_0$ *and* $\forall f \in \mathcal{H}$

3. *No function* $g \neq 0$ *in* $\mathcal{H}$ *which is orthogonal to the set* $\{\mathbf{e}_i\}_{i=0}^{\infty}$,

4. $\langle f | g \rangle = \sum_{i=1}^{\infty} \langle f | \mathbf{e}_i \rangle \overline{\langle g | \mathbf{e}_i \rangle} \quad \forall f, g \in \mathcal{H}$, *and setting* $f = g$ *gives*

5. $\|f\|^2 = \sum_{i=1}^{\infty} |\langle f | \mathbf{e}_i \rangle|^2 \quad \forall f \in \mathcal{H}$

What does $=$ mean for vectors?

- This was a major stumbling block for Fourier analysis – analysts mistakenly believed $f(x) = g(x)$ required equality at all points x.

- Item 1 $f = \sum_{i=1}^{\infty} \langle f | \mathbf{e}_i \rangle \mathbf{e}_i$ is read as:

$$\lim_{N \to \infty} \left\| \mathbf{f} - \sum_{i=1}^{\infty} \langle f | \mathbf{e}_i \rangle \mathbf{e}_i \right\| = 0$$

- Convergence in mean

Do it yourself orthonormal sequences

Jorgen Gram

1850–1916

Erhart Schmidt

1876–1959

Gram-Schmidt Orthonormalisation.

set of vectors $\{\mathbf{x}_m\}_{m=0}^{N}$

1.
$$\mathbf{y}_0 \leftarrow \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|}$$

2. Remove projection of $\mathbf{x}_0$ from next vector $\mathbf{x}_1$
$$\mathbf{y}_k = \mathbf{x}_k - \sum_{i=0}^{k} \langle \mathbf{x}_i | \, \mathbf{x}_k \rangle \mathbf{x}_i$$

3. normalise result
$$\mathbf{y}_k \leftarrow \frac{\mathbf{y}_k}{\|\mathbf{y}_k\|}$$

4. repeat

For next week

- We will cover pieces of chaps. 2, 4, 8 of Gallager [Gal68]
- Read Miller [Mil00]
  - Intro,
  - Section 3, A
  - Conclusions
  - We will cover the rest in 2-3 weeks.
- Read sections 1, 3 of Telatar [Tel99] + first half of section 4

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[Mil00] David A. Miller. Communicating with waves between volumes: evaluating orthogonal spatial channels and limits on coupling strengths. *Applied Optics*, 39(11):1681–1699, April 2000.

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Exercises

1. Is the space of continuous functions complete?

2. Consider the space of vectors, with

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