Implementation of Critical Path Heuristics for SAT

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Abstract. Recent work has shown that SAT can be theoretically more powerful than heuristic search provided the heuristic used by search is implemented as a set of clauses on which unit propagation simulates the evaluation of the heuristic. The $h^{\max}$ heuristic has been shown to be implemented trivially by the empty set of clauses. This paper presents an implementation of h^m, a generalization of $h^{\max}$.

1 Introduction

Heuristic search and reduction to SAT are two major approaches to planning that can be viewed as complementary: On the one hand, pruning heuristics used in search are generally not available to SAT solvers. On the other, logical reasoning methods used by SAT solvers, such as unit propagation and clause learning, are not directly applicable in heuristic search. Interestingly, with their own strengths and weaknesses, the two approaches appear to exhibit comparable performance overall on the types of problems they both handle [5].

One may logically expect, therefore, that if either approach is to overcome one of its weaknesses, the balance may be tipped in its favor. Indeed, the notion of *implementing* pruning heuristics for SAT has been proposed [4], where a heuristic is encoded into a set of clauses such that unit propagation will derive a contradiction whenever the corresponding branch of a native search would be pruned by the heuristic. Furthermore, the same work shows that a SAT search can simulate (forward state-space) heuristic search in polynomial time as long as the former efficiently implements the same heuristic used by the latter. Where $h^{\max}$ [1] is the heuristic used, it also shows that the reverse does not hold: There are problems that are exponentially harder for heuristic search than for SAT.

It is understood that the $h^{\max}$ heuristic is implicit in the basic SAT encoding of planning [4]. This heuristic is based on taking the cost of achieving a set of goals to be that of achieving the costliest goal in the set, and applying the idea recursively in regression.

The h^m family of heuristics [3], also known as *critical path* heuristics, generalize and strengthen $h^{\max}$ by considering the costliest subset of m goals (hence $h^{\max} = h^1$). This paper presents an implementation of h^m for SAT. The size of the implementation is exponential in m, which is consistent with the fact that computing h^m in the native search space requires time exponential in m.

2 Background

As in [4], we consider a sequential SAT encoding of STRIPS planning problems based on explanatory axioms. Our planning problem P consists of a set of *facts* F and a set of *actions* A, and each action $a \in A$ has a set of *preconditions* $\text{pre}(a)$, *add effects* $\text{add}(a)$, and *delete effects* $\text{del}(a)$, which are all sets of facts $\subseteq F$. A problem instance specifies, in addition, an initial state s_I as the set of facts that initially hold and a set of goals $G \subseteq F$ to achieve. The objective is to find a plan the contains the fewest actions.

In a SAT encoding, all facts and actions become Boolean variables, and there is a fresh copy of these variables for each time step, up to a fixed horizon T. Specifically, we have $f@t$ for each $f \in F$ and $t \in \{0, \cdots, T\}$, and $a@t$ for each $a \in A$ and $t \in \{0, \cdots, T-1\}$. Each action a is encoded by the following set of clauses for all $t \in \{0, \cdots, T-1\}$: $\overline{a@t} \vee f@t,$ for all $f \in \text{pre}(a)$; $\overline{a@t} \vee f@(t+1),$ for all $f \in \text{add}(a)$; $\overline{a@t} \vee \overline{f@(t+1)},$ for all $f \in \text{del}(a)$. For each fact f and time $t \in \{0, \cdots, T-1\}$, we have the following *frame axioms* stating that each fact remains unchanged unless some action capable of changing it occurs: $f@t \vee \overline{f@(t+1)} \vee a_{k_1}@t \vee \cdots \vee a_{k_m}@t$, $\overline{f@t} \vee f@(t+1) \vee a_{n_1}@t \vee \cdots \vee a_{n_s}@t$, where $a_{k_1}, \cdots, a_{k_m}$ are all the actions whose add effects include f, and $a_{n_1}, \cdots, a_{n_s}$ are all the actions whose delete effects include f. To ensure that at most one action occurs at a time, we have the following for all $t \in \{0, \cdots, T-1\}$: $\overline{a@t} \vee \overline{a'@t},$ for all distinct $a, a' \in A$. Finally, the goals G are encoded by a set of unit clauses: $f@T,$ for all $f \in G$.

For a given natural number T, we denote the collection of the clauses described above by H_T. For a state s, again as a set of facts, and time t, we write $s@t$ as shorthand for $\{f@t \mid f \in s\} \cup \{\overline{f@t} \mid f \in F \backslash s\}$. The optimal planning problem is then to determine the smallest T such that $s_I@0 \cup H_T$ is satisfiable, where such a T exists.

We write $\Delta \vdash_{UP} \beta$ to denote that unit propagation on clauses Δ results in a set of clauses that include β (a special case is when $\beta = \bot$, a contradiction). Let h be a heuristic such that $h_s(G)$ gives an estimate on the number of actions required to achieve the goals G from the state s. The following definition generalizes the one in [4]:

Definition 1 *A set of clauses χ_T* implements *the heuristic h, for a given horizon T, if $s@t \cup H_T \cup \chi_T \vdash_{UP} \bot$ for all states s and times $t \in \{0, \cdots, T\}$ such that $(T - t) < h_s(G)$.*

3 Implementation of h^m

In our present context, the h^m estimate for achieving a set of facts $\psi \in F$ from a state s may be defined as follows:

$$h^m_s(\psi) = \begin{cases} 0 & \text{if } \psi \subseteq s \\ \min_a h^m_s(\text{R}(\psi, a)) + 1 & \text{if } |\psi| \leq m \\ \max_{\psi' \subset \psi, |\psi'| = m} h^m_s(\psi') & \text{if } |\psi| > m \end{cases} \qquad (1)$$

where $\text{R}(\psi, a)$ is the *regression operator*, defined as

$$\text{R}(\psi, a) = (\psi \setminus \text{add}(a)) \cup \text{pre}(a)$$

if $\text{add}(a) \cap \psi \neq \emptyset$ and $\text{del}(a) \cap \psi = \emptyset$, and undefined otherwise, and the min ranges over all actions a such that $\text{R}(\psi, a)$ is defined [3].

We now describe our implementation χ^m_T of h^m, followed by a proof of its correctness.

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Because unit propagation will only produce consequences regarding the truth of single variables, to simulate h^m, we need Boolean variables that correspond to sets of facts. Specifically, for each set of facts $\phi \subseteq F$ such that $2 \leq |\phi| \leq m$, we create a *meta-fact* f_ϕ, which implies the truth of all facts in ϕ, at each time point $t \in \{0, \cdots, T\}$:

$$\overline{f_\phi @t} \vee f@t, \text{ for all } f \in \phi \tag{2}$$

For succinctness of presentation, in what follows we will take f_ϕ to be (an alias of) f in case $\phi = \{f\}$; thus f_ϕ is defined for all $\phi \subseteq F$ such that $1 \leq |\phi| \leq m$.

The *frame axiom* for each meta-fact f_ϕ, $2 \leq |\phi| \leq m$, gives the condition under which the value of f_ϕ can possibly change from *false* to *true*. It consists of the following clauses for all $t \in \{0, \cdots, T-1\}$:

$$f_\phi @t \vee \overline{f_\phi @(t+1)} \vee x_1 \vee \cdots \vee x_p \tag{3}$$

$$\overline{x_i} \vee f_{\phi'} @t, \text{ for all } \phi' \subseteq \mathrm{R}(\phi, a_{k_i}) \neq \emptyset, \tag{4}$$

$$|\phi'| = \min(m, |\mathrm{R}(\phi, a_{k_i})|), \text{and for all } i \in \{1, \cdots, p\}$$

where $a_{k_1}, \cdots, a_{k_p}$ are all the actions for which $\mathrm{R}(\phi, a_{k_i})$ is defined, and $x_1, \cdots, x_p$ are a set of (not necessarily distinct) auxiliary variables such that x_i and x_j are the same variable iff $\mathrm{R}(\phi, a_{k_i}) = \mathrm{R}(\phi, a_{k_j})$ (which implies that clauses 4 for i are identical to those for j and hence only one set will actually appear in χ_T^m). This sharing of variables and clauses applies across all frame axioms for the same time step t. To avoid doubt, we also note that whenever $\mathrm{R}(\phi, a_{k_i}) = \emptyset$, the set of clauses (4) is empty but the literal x_i appears in clause (3) regardless (this corresponds to cases where the set of facts ϕ can be achieved by an action that has no preconditions).

Finally, the following unit clauses assert the achievement of goals in terms of meta-facts:

$$f_\phi @T, \text{ for all } \phi \subseteq G, 2 \leq |\phi| \leq m \tag{5}$$

The clauses (2–5) given above make up our implementation χ_T^m of h^m. That it does not rule out valid plans is implied by the following:

Theorem 2 *For any satisfying assignment π for $s_I@0 \cup H_T$, there is an assignment π' for the meta-fact and auxiliary variables in χ_T^m such that $\pi \cup \pi'$ satisfies χ_T^m.*

To show that it correctly implements the heuristic, we first prove a more general theorem:

Theorem 3 *Let t, i be natural numbers such that $t + i \leq T$. For all $\psi \subseteq F$ and states s such that $h_s^m(\psi) > i$, $s@t \cup H_T \cup \chi_T^m \vdash_{UP} \overline{f_\phi @(t+i)}$ for some $\phi \subseteq \psi, |\phi| = \min(m, |\psi|)$.*

Proof: The proof is by induction on i. If $i = 0$, then the definition (1) of h^m implies that $\psi \not\subseteq s$. Hence there is an $f \in \psi$ such that $s@t \vdash_{UP} \overline{f@t}$, which, in conjunction with (2), implies that $s@t \cup \chi_t^m \vdash_{UP} \overline{f_\phi @t}$ for all ϕ, $1 \leq |\phi| \leq m$, such that $f \in \phi \subseteq F$. Choose any such ϕ of size $\min(m, |\psi|)$ such that $\phi \subseteq \psi$, and the statement is proved.

Assume that the statement holds for $i = n, 0 \leq n < T - t$. Consider any $\psi \subseteq F$ and state s such that $h_s^m(\psi) > n + 1$. The inductive step consists in a case analysis on $|\psi|$.

Suppose $|\psi| \leq m$. Since $h_s^m(\psi) > n + 1 > n$, by the induction hypothesis,

$$s@t \cup H_T \cup \chi_T^m \vdash_{UP} \overline{f_\phi @(t+n)} \tag{6}$$

for some $\phi \subseteq \psi, |\phi| = \min(m, |\psi|)$. But $|\psi| \leq m$; therefore $\phi = \psi$. Hence $n+1 < h_s^m(\psi) = h_s^m(\phi) = \min_a h_s^m(\mathrm{R}(\phi, a)) + 1$, which implies that $h_s^m(\mathrm{R}(\phi, a)) > n$ for all a where $\mathrm{R}(\phi, a)$ is defined. By the induction hypothesis, $s@t \cup H_T \cup \chi_T^m \vdash_{UP} \overline{f_{\phi'} @(t+n)}$ for some $\phi' \subseteq \mathrm{R}(\phi, a), |\phi'| = \min(m, |\mathrm{R}(\phi, a)|)$. In conjunction with the frame axiom (4) for f_ϕ, this implies that $s@t \cup H_T \cup \chi_T^m \vdash_{UP} \overline{x_i}$ for all x_i in the frame axiom. In conjunction with (3) and (6), this implies that

$$s@t \cup H_T \cup \chi_T^m \vdash_{UP} \overline{f_\phi @(t+n+1)}. \tag{7}$$

In other words, the statement holds for $i = n + 1$.

If $|\psi| > m$, then $h_s^m(\psi) = \max_{\psi' \subset \psi, |\psi'| = m} h_s^m(\psi')$. The preceding argument applies to the ψ' that attains the maximum in this equation, and shows that (7) holds for $\phi = \psi' \subset \psi$, where $|\phi| = |\psi'| = m$, which completes the proof. □

Letting $\psi = G$ and $i = T - t$ in Theorem 3, it readily follows that χ_T^m correctly implements h^m according to Definition 1:

Corollary 4 *$s@t \cup H_T \cup \chi_T^m \vdash_{UP} \perp$ for all states s and times $t \in \{0, \cdots, T\}$ such that $(T - t) < h_s^m(G)$.*

4 Discussion

Recent work [2] has shown that the h^m heuristic for the planning problem P can be formulated as the $h^{\max}$ for a new problem P^m. The latter features the same set of meta-facts as χ_T^m, and a set of new *meta-actions* $\alpha_{a,\phi}$ each representing the execution of action a while "preserving" the truth of all facts ϕ. This work implies that, in principle, a basic SAT encoding of P^m, when suitably linked into that of P, would function as an implementation of h^m for P. Such an implementation, however, would include action definition and one-action-at-a-time clauses for the meta-actions, and frame axioms preventing *true* meta-facts from becoming *false*, which are absent from χ_T^m. On the other hand, χ_T^m includes clauses (4), which may be regarded as an economical way of encoding any relevant information that would be represented by the meta-actions.

Although the details are omitted, we note that it is not difficult to adapt our implementation of h^m to the parallel setting, where compatible actions are allowed to take place in the same time step, and an optimal plan is one with the fewest steps (after the basic SAT encoding and the definition of h^m have both been adapted accordingly).

We conclude the paper by considering two theoretical questions that arise from the presented work: (i) Can the addition of χ_T^m allow resolution to derive the empty clause in exponentially fewer steps? (ii) Can the addition of the meta-fact variables, along with the clauses (2 and 5, e.g.) defining them, alone allow resolution to derive the empty clause in exponentially fewer steps? Answers to these questions will offer important insights into the power and limitations of SAT-based planning by determining to what extent a pruning heuristic can possibly be made redundant by suitable resolution strategies.

REFERENCES

[1] Blai Bonet and Hector Geffner, 'Planning as heuristic search', *Artificial Intelligence*, **129**(1-2), 5–33, (2001).

[2] Patrik Haslum, '$h^m(P) = h^1(P^m)$: Alternative characterisations of the generalisation from $h^{\max}$ to h^m', in *Proceedings of the 19th International Conference on Automated Planning and Scheduling (ICAPS)*, (2009).

[3] Patrik Haslum and Hector Geffner, 'Admissible heuristics for optimal planning', in *Proceedings of the Fifth International Conference on Artificial Intelligence Planning Systems (AIPS)*, pp. 140–149, (2000).

[4] Jussi Rintanen, 'Planning with SAT, admissible heuristics and A*', in *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, pp. 2015–2020, (2011).

[5] Jussi Rintanen, 'Planning with specialized SAT solvers', in *Proceedings of the 25th AAAI Conference on Artificial Intelligence (AAAI)*, (2011).