Generic Modal Cut Elimination Applied to Conditional Logics

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Abstract. We develop a general criterion for cut elimination in sequent calculi for propositional modal logics, which rests on absorption of cut, contraction, weakening and inversion by the purely modal part of the rule system. Our criterion applies also to a wide variety of logics outside the realm of normal modal logic. We give extensive example instantiations of our framework to various conditional logics. For these, we obtain fully internalised calculi which are substantially simpler than those known in the literature, along with leaner proofs of cut elimination and complexity. In one case, conditional logic with modus ponens and conditional excluded middle, cut elimination and complexity are explicitly stated as open in the literature.

1 Introduction

Cut elimination, originally invented by Gentzen [5], is one of the core concepts of proof theory and plays a major role in particular for algorithmic aspects of logic, including the subformula property, the complexity of automated reasoning and, via interpolation, modularity issues. The large number of logical calculi that are currently in use, in particular in various areas of computer science, motivates efforts to define families of sequent calculi that cover a variety of logics and admit uniform proofs of cut elimination, enabled by suitable sufficient conditions. Here, we present such a method for modal sequent calculi that applies to possibly nonnormal normal modal logics, which appear, e.g. in concurrency and knowledge representation. We use a separation of the modal calculi into a fixed underlying propositional part and a modal part; the core of our criterion is absorption of cut by the modal rules. This concept generalises the notion of resolution closed rule set [9, 12], dropping the assumption that the logic at hand is rank-1, i.e. axiomatised by formulas in which the nesting depth of modal operators is uniformly equal to 1 (such as K).

Our method is reasonably simple and intuitive, and nevertheless applies to a wide range of modal logics. While we use normal modal logics such as K and T as running examples to illustrate our concepts at the time of introduction, our

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main applications are conditional logics, which have a binary modal operator read as a non-monotonic implication (unlike default logics, conditional logics allow nested non-monotonic implications). In particular, we prove cut-elimination (hence, since the generic systems under consideration are analytic, the subformula property) for the conditional logics CK, CKMP, CKCEM, and CKMPCEM using our generic procedure. An easy analysis of proof search in the arising cutfree calculi moreover establishes that the satisfiability problem of each of these logics is in *PSPACE*. This is a tight bound for CK and CKMP, whereas the provability problem in extensions of CKCEM can be solved in *coNP*, as we show by a slightly adapted algorithmic treatment of our calculus using a dynamic programming approach in the spirit of [13]. We point out that while (different) cut-free labelled sequent calculi for CK, CKMP, CKCEM, and some further conditional logics, as well as the ensuing upper complexity bounds, have previously been presented by Olivetti et al., the corresponding issues for CKMPCEM have explicitly been left as open problems [8].

Related work A set of sufficient conditions for a sequent calculus to admit cut elimination and a subsequent analysis of the complexity of cut elimination (not proof search) is presented in [10]. The range of application of this method is very wide and encompasses, e.g. first-order logic, the modal logic S4, linear logic, and intuitionistic propositional logic. This generality is reflected in the fact that the method as a whole is substantially more involved than ours. A simpler method for a different and comparatively restrictive class of calculi, so-called canonical calculi, is considered in [1]; this method does not apply to typical modal systems, as it considers only so-called *canonical* rules, i.e., left and right introduction rules for connectives which permit adding a common context simultaneously in the premise and the conclusion. (In fact, it might be regarded as the essence of modal logic that its rules fail to be canonical, e.g. the necessitation rule $A/\Box A$ does not generalise to $\Gamma, A/\Gamma, \Box A$ for a sequent Γ .) Moreover, the format of the rules in *op.cit.* does not allow for the introduction of more than one occurrence of a logical connective, which is necessary even for the most basic modal logics. The same applies to [4]. In [3], logical rules are treated on an individual basis, which precludes the treatment of cuts between two rule conclusions. Overall, our notion of absorption is substantially more general when compared to similar notions in the papers discussed above, which stipulate that cuts between left and right rules for the same connective are absorbed by structural rules. In our own earlier work [9], we have considered a special case of the method presented here in the restricted context of rank-1 logics; in particular, these results did not cover logics such as K4, CKMP, or CKMPCEM.

2 Preliminaries and Notation

A modal similarity type (or modal signature) is a set Λ of modal operators with associated arities that we keep fixed throughout the paper. Given a set V of propositional variables, the set $\mathcal{F}(\Lambda)$ of Λ -formulas is given by the grammar

$$\mathcal{F}(A) \ni A, B ::= \bot \mid p \mid \neg A \mid A \land B \mid \heartsuit(A_1, \dots, A_n)$$

where $p \in V$ and $\heartsuit \in \Lambda$ is *n*-ary. We use standard abbreviations of the other propositional connectives \top , \lor and \rightarrow . A *A*-sequent is a finite multiset of *A*formulas, and the set of *A*-sequents is denoted by $S(\Lambda)$. We write the multiset union of Γ and Δ as Γ, Δ and identify a formula $A \in \mathcal{F}(\Lambda)$ with the singleton sequent containing only *A*. If $S \subseteq \mathcal{F}(\Lambda)$ is a set of formulas, then an *S*-substitution is a mapping $\sigma: V \to S$. We denote the result of uniformly substituting $\sigma(p)$ for *p* in a formula *A* by $A\sigma$. This extends pointwise to *A*-sequents so that $\Gamma\sigma = A_1\sigma, \ldots, A_n\sigma$ if $\Gamma = A_1, \ldots, A_n$. If $S \subseteq \mathcal{F}(\Lambda)$ is a set of *A*-formulas and $A \in \mathcal{F}(\Lambda)$, we say that *A* is a propositional consequence of *S* if there exist $A_1, \ldots, A_n \in S$ such that $A_1 \land \cdots \land A_n \to A$ is a substitution instance of a propositional tautology. We write $S \vdash_{\mathsf{PL}} A$ if *A* is a propositional consequence of *S* and $A \vdash_{\mathsf{PL}} B$ for $\{A\} \vdash_{\mathsf{PL}} B$ for the case of single formulas.

3 Modal Deduction Systems

To facilitate the task of comparing the notion of provability in both Hilbert and Gentzen type proof systems, we introduce the following notion of a proof rule that can be used, without any modifications, in both systems.

Definition 1. A Λ -rule is of the form $\frac{\Gamma_1,\ldots,\Gamma_n}{\Gamma_0}$ where $n \geq 0$ and Γ_0,\ldots,Γ_n are Λ -sequents. The sequents Γ_1,\ldots,Γ_n are the premises of the rule and Γ_0 its conclusion. A rule $\overline{\Gamma_0}$ without premises is called a Λ -axiom, which we denote by just its conclusion, Γ_0 . A rule set is just a set of Λ -rules, and we say that a rule set R is substitution closed, if $\Gamma_1 \sigma \ldots \Gamma_n \sigma / \Gamma_0 \sigma \in \mathbb{R}$ whenever $\Gamma_1 \ldots \Gamma_n / \Gamma_0 \in \mathbb{R}$ and $\sigma : V \to \mathcal{F}(\Lambda)$ is a substitution.

In view of the sequent calculi that we introduce later, we read sequents disjunctively. Consequently, a rule $\Gamma_1, \ldots, \Gamma_n/\Gamma_0$ can be used to prove the disjunction Γ_0 , provided that $\bigvee \Gamma_i$ is provable, for all $1 \leq i \leq n$. We emphasise that a rule is an expression of the object language, i.e. it does not contain meta-linguistic variables. As such, it represents a specific deduction step rather than a family of possible deductions, which helps to economise on syntactic categories. In our examples, concrete rule sets are presented as instances of rule schemas.

Example 2. For the modal logics K, K4 and T, we fix the modal signature $\Lambda = \{\Box\}$ consisting of a single modal operator \Box with arity one. The language of conditional logic is given by the similarity type $\Lambda = \{\Rightarrow\}$ where the conditional arrow \Rightarrow has arity 2. We use infix notation and write $A \Rightarrow B$ instead of $\Rightarrow (A, B)$ for $A, B \in \mathcal{F}(\Lambda)$. Deduction over modal and conditional logics are governed by the following rule sets:

1. The rule set K associated to the modal logic K consists of all instances of the necessitation rule (N) and the distribution axiom (D) below.

$$(\mathsf{N})\frac{A}{\Box A} \qquad (\mathsf{D})\Box(A \to B) \to (\Box A \to \Box B) \qquad (\mathsf{4})\Box\Box A \to \Box A \qquad (\mathsf{R})\Box A \to A$$

The rule sets for T and K4 arise by extending this set with the reflexivity axiom (R) and the (4)-axiom, respectively. We reserve the name (T) for the reflexivity rule in a cut-free system.

2. Conditional logic, e.g. the system CK of [2] is axiomatised by the rule set that consists of all instances of (RCEA) on the left, and (RCK) on the right below:

$$\frac{A \leftrightarrow A'}{(A \Rightarrow B) \leftrightarrow (A' \Rightarrow B)} \qquad \frac{B_1 \wedge \dots \wedge B_n \rightarrow B}{(A \Rightarrow B_1) \wedge \dots \wedge (A \Rightarrow B_n) \rightarrow (A \Rightarrow B)}$$

As additional axioms, we consider

$$(\mathsf{ID})A \Rightarrow A \qquad (\mathsf{MP})(A \Rightarrow B) \to (A \to B) \qquad (\mathsf{CEM})(A \Rightarrow B) \lor (A \Rightarrow \neg B)$$

that induce extensions of CK that we denote by juxtaposition of the respective axioms, e.g. CKMPCEM contains the rules for CK and the axioms (MP) and (CEM).

Rules with more than one premise arise through saturation of a given rule set under cut that, e.g. leads to the rules (CK_q) and (MP_q) presented in Section 6.

The notion of deduction in modal Hilbert systems then takes the following form.

Definition 3. Suppose R is a set of rules. The set of R-derivable formulas in the Hilbert-system given by R is the least set of formulas that

- contains $A\sigma$ whenever A is a propositional tautology and σ is a substitution
- contains B whenever it contains A and $A \rightarrow B$
- contains $\bigvee \Gamma_0$ whenever it contains $\bigvee \Gamma_1, \ldots, \bigvee \Gamma_n$ and $\frac{\Gamma_1 \ldots \Gamma_n}{\Gamma_0} \in \mathsf{R}$.

We write $\mathsf{HR} \vdash A$ if A is R-derivable.

In other words, the set of derivable formulas is the least set that contains propositional tautologies, is closed under uniform substitution, modus ponens and application of rules. We will later consider Hilbert systems that induce the same provability predicate based on the following notion of admissibility.

Definition 4. A rule set R' is admissible in HR if $HR \vdash A \iff H(R \cup R') \vdash A$ for all formulas $A \in \mathcal{F}(A)$. Two rule sets R, R' are equivalent if R is admissible in HR' and R' is admissible in HR.

In words, R' is admissible in HR if adding the rules R' to those of R leaves the set of provable formulas unchanged. We note the following trivial, but useful consequence of admissibility.

Lemma 5. $\mathsf{HR} \vdash A$ iff $\mathsf{HR}' \vdash A$ if R and R' are equivalent and $A \in \mathcal{F}(\Lambda)$.

The next proposition is concerned with the structure of proofs in Hilbert systems and is the key for proving equivalence of Hilbert and Gentzen-type systems.

Proposition 6. The set $HT(R) = \{A \in \mathcal{F}(\Lambda) \mid HR \vdash A\}$ is the smallest set S of formulas that contains a formula $A \in \mathcal{F}(\Lambda)$ whenever there are rules $\Theta_1/\Gamma_1, \ldots, \Theta_n/\Gamma_n \in R$ and substitutions $\sigma_1, \ldots, \sigma_n : V \to \mathcal{F}(\Lambda)$ such that $\bigvee \Delta \sigma_i \in S$ for all $\Delta \in \Theta_i$ $(i = 1, \ldots, n)$ and $\{\bigvee \Gamma_1 \sigma, \ldots, \bigvee \Gamma_n \sigma\} \vdash_{PL} A$. In other words, in a modal Hilbert system, each provable formula is a propositional consequence of rule conclusions with provable premises. This result forms the basis of our comparison of Hilbert and Gentzen systems, and we show that cut elimination essentially amounts to the fact that – in the corresponding Hilbert system – each valid formula is a consequence of a *single* rule conclusion with provable premise.

We now set the stage for sequent systems that we are going to address in the remainder of the paper. The notion of derivability in the sequent calculus associated with a rule set R is formulated parametric in terms of a set X of additional rules that will later be instantiated with relativised versions of cut, weakening, contraction and inversion.

Definition 7. Suppose R and X are sets of Λ -rules. The set of RC + X-derivable sequents in the Gentzen-system given by R is the least set of sequents that

- contains $A, \neg A, \Gamma$ for all sequents $\Gamma \in \mathsf{S}(\Lambda)$ and formulas $A \in \mathcal{F}(\Lambda)$
- contains $\neg \bot$, Γ for all $\Gamma \in \mathsf{S}(\Lambda)$
- is closed under instances of the rule schemas

$$\frac{\varGamma, \neg A, \neg B}{\varGamma, \neg (A \land B)} \qquad \frac{\varGamma, A \quad \varGamma, B}{\varGamma, A \land B} \qquad \frac{\varGamma, A}{\varGamma, \neg \neg A}$$

where $A \in \mathcal{F}(\Lambda)$ ranges over formulas and $\Gamma \subseteq \mathcal{F}(\Lambda)$ over multisets of formulas. We call the above rules the propositional rules and the formula occurring in the conclusion but not in Γ principal in the respective rule.

• is closed under the rules in $\mathsf{R} \cup \mathsf{X}$, i.e. it contains Γ_0 whenever it contains $\Gamma_1, \ldots, \Gamma_n$ for $\frac{\Gamma_1 \ldots \Gamma_n}{\Gamma_0} \in \mathsf{R} \cup \mathsf{X}$.

We write $\mathsf{GR} + \mathsf{X} \vdash \Gamma$ if Γ can be derived in this way and $\mathsf{GR} \vdash \Gamma$ if $\mathsf{X} = \emptyset$.

The set X of extra rules will later be instantiated with a relativised version of the cut rule and additional axioms that locally capture the effect of weakening, contraction and inversion, applied to rule premises. This allows to formulate *local* conditions for the admissibility of cut that can be checked on a per-rule basis.

Many other formulations of sequent systems only permit axioms of the form $\Gamma, p, \neg p$ where $p \in V$ is a propositional atom. The reason for being more liberal here is that this makes it easier to prove admissibility of uniform substitution, at the expense of loosing depth-preserving admissibility of structural rules. We come back to this matter in Remark 12.

The following proposition is readily established by an induction on the provability predicate $\mathsf{RH} \vdash$.

Proposition 8. Suppose $\Gamma \in S(\Lambda)$ is a sequent. Then $\mathsf{RH} \vdash \bigvee \Gamma$ if $\mathsf{RG} \vdash \Gamma$.

The remainder of the paper is concerned with the converse of the above proposition, which relies on specific properties of the rule set R.

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In order to establish the converse of Proposition 8 we need to establish that the cut rule is admissible in the Gentzen system GR defined by the ruleset R. Clearly, we cannot expect that cut elimination holds in general: it is well known (and easy to check) that the sequent system arising from the rule set consisting of all instances of (N) and (D), presented in Example 2 does not enjoy cut elimination. In other words, we have to look for constructions that allow us to transform a given rule set into one for which cut elimination holds. The main result of our analysis is that cut elimination holds if the rule set under consideration satisfies four crucial requirements that are local in the sense that they can be checked on a per-rule basis without the need of carrying out a fully-fledged cut-elimination proof: absorption of weakening, contraction, inversion and cut.

The first three properties can be checked for each rule individually and amount to the admissibility of the respective principle, and the last requirement amounts to the possibility of eliminating cut between a pair of rule conclusions. We emphasise that these properties can be checked locally for the modal rules, and cut elimination will follow automatically. It is not particularly surprising that cut elimination holds under these assumptions. However, isolating the four conditions above provides us with means to convert a modal Hilbert system into an equivalent cut-free sequent calculus. We now introduce relativised versions of the structural rules that will be the main tool in the proof of cut elimination.

Definition 9. Suppose Γ is a Λ -sequent and let $A(\Gamma)$ consist of the axioms

- $\Gamma, A \text{ for all } A \in \mathcal{F}(\Lambda)$
- $\Delta, A \text{ if } \Gamma = \Delta, A, A \text{ for some } \Delta \in \mathsf{S}(\Lambda), A \in \mathcal{F}(\Lambda)$
- $\Delta, A \text{ if } \Gamma = \Delta, \neg \neg A \text{ for some } \Delta \in \mathsf{S}(\Lambda), A \in \mathcal{F}(\Lambda)$
- $\Delta, \neg A_1, \neg A_2$ if $\Gamma = \Delta, \neg (A_1 \land A_2)$ for some $\Delta \in \mathsf{S}(\Lambda), A_1, A_2 \in \mathcal{F}(\Lambda)$ Δ, A_i for i = 1, 2 if $\Gamma = \Delta, (A_1 \land A_2)$ for some $\Delta \in \mathsf{S}(\Lambda), A_1, A_2 \in \mathcal{F}(\Lambda)$

We say that a rule set R absorbs the structural rules if

$$\mathsf{GR} + \mathsf{A}(\Gamma_1) \cup \cdots \cup \mathsf{A}(\Gamma_n) \vdash \Gamma$$

for all $\frac{\Gamma_1 \dots \Gamma_n}{\Gamma_0} \in \mathsf{R}$ and all $\Gamma \in \mathsf{A}(\Gamma_0)$.

In other words, a deduction step that applies weakening, contraction or inversion to a rule conclusion can be replaced by a (possibly different) rule where the corresponding structural rules are applied to the premises. We discuss a number of standard examples before stating that absorption of the structural rules implies their admissibility.

Example 10. The rule sets containing all instances of either of the following rule schemas (K), (T) and (K4)

$$\frac{\neg A_1, \dots, \neg A_n, A_0}{\neg \Box A_1, \dots, \neg \Box A_n, \Box A_0, \Gamma} \quad \frac{\neg A, \neg \Box A, \Gamma}{\neg \Box A, \Gamma} \quad \frac{\neg A_1, \neg \Box A_1, \dots, \neg A_n, \neg \Box A_n, B}{\neg \Box A_1, \dots, \neg \Box A_n, \Box B, \Gamma}$$

absorbs the structural rules. We note that (K) absorbs weakening due to the presence of Γ in the conclusion, and the absorption of contraction in (T) and (K4) is a consequence of the presence of the negated \Box -formulas in the premise. The absorption of inversion in a consequence of the weakening context Γ in (K) and (K4) and implied by duplicating the context Γ in (T). On the other hand, the rule sets defined by

$$\frac{\neg A_1, \dots, \neg A_n, A_0}{\neg \Box A_1, \dots, \neg \Box A_n, \Box A_0} \qquad \qquad \frac{\neg A, \Gamma}{\neg \Box A, \Gamma}$$

fail to absorb the structural rules: the rule on the left fails to absorb weakening, whereas the right-hand rule does not absorb contraction.

It should be intuitively clear that absorption of structural rules implies their admissibility, which we establish next.

Proposition 11. Suppose R absorbs the structural rules. Then all instances of the rule schemas of weakening, contraction and inversion

$$\frac{\varGamma}{\varGamma,A} \qquad \frac{\varGamma,A,A}{\varGamma,A} \qquad \frac{\varGamma,\neg\neg A}{\varGamma,A} \qquad \frac{\varGamma,\neg(A_1 \wedge A_2)}{\varGamma,\neg A_1,\neg A_2} \qquad \frac{\varGamma,A_1 \wedge A_2}{\varGamma,A_i}(i=1,2)$$

where $\Gamma \in \mathsf{S}(\Lambda)$ and $A, A_1, A_2 \in \mathcal{F}(\Lambda)$ are admissible in GR .

Remark 12. 1. The main purpose for introducing the notion of absorption of structural rules (Definition 9) is to have a handy criterion that guarantees admissibility of the structural rules (Proposition 11). Our definition offers a compromise between generality and simplicity. In essence, a rule set absorbs structural rules, if an application of weakening, contraction or inversion can be pushed up one level of the proof tree. A weaker version of Definition 9 would require that an application of weakening, contraction or inversion to a rule conclusion can be replaced by a sequence of deduction steps where the structural rule in question can not only be applied to the premises of the rule, but also freely anywhere else, provided that these additional applications are smaller in a well-founded ordering. However, we are presently not aware of any examples where this extra generality would be necessary.

2. In many Sequent systems, the statement of Proposition 11 can be strengthened to say that weakening, contraction and inversion are depth-preserving admissible, i.e. does not increase the height of the proof tree. This is in general false for the systems considered here as axioms are of the form $A, \neg A, \Gamma$ for $A \in \mathcal{F}(\Lambda)$ and, for instance, $(A \wedge B), \neg (A \wedge B)$ is derivable with a proof of height one (being an axiom), but, e.g. $A \wedge B, \neg A, \neg B$ cannot be established by a proof of depth one (not being an axiom). It is easy to see that weakening, inversion and contraction are in fact depth-preserving admissible if only atomic axioms of the form $p, \neg p, \Gamma$ are allowed, for $p \in V$ a propositional variable. The more general form of axioms adopted in this paper allows us to simplify many constructions as we do not have to consider a congruence rule explicitly which would allow us to prove (rather than to assume as axioms) sequents of the form $\Box A, \neg \Box A, \Gamma$. Having dealt with the structural rules, we now address our main concern: the admissibility of the cut rule. In contrast to the absorption of structural rules, we need one additional degree of freedom in that we need to allow ourselves to apply cut to a structurally smaller formula.

Definition 13. The size of a formula $A \in \mathcal{F}(A)$ is given inductively by size(p) = $size(\bot) = 1$, $size(A \land B) = size(A \lor B) = 1 + size(A) + size(B)$ and, for the modal $\begin{aligned} case, \ \mathsf{size}(\heartsuit(A_1, \dots, A_n)) &= 1 + \mathsf{size}(A_1) + \dots + \mathsf{size}(A_n). \\ A \ ruleset \ \mathsf{R} \ \text{absorbs cut}, \ if \ for \ all \ rules \ (r_1) \frac{\Gamma_1, \dots, \Gamma_n}{A, \Gamma_0}, \ (r_2) \frac{\Delta_1, \dots, \Delta_k}{\neg A, \Delta_0} \in \mathsf{R} \end{aligned}$

 $\mathsf{GR} + \mathsf{Cut}(A, r_1, r_2) \vdash \Gamma_0, \Delta_0$

where $Cut(A, r_1, r_2)$ consists of all instances of the rule schemas

$$\frac{\varGamma, C \quad \Delta, \neg C}{\varGamma, \Delta} \quad \frac{\varGamma}{\varGamma, A} \qquad \frac{\varGamma, A, A}{\varGamma, A} \quad \frac{\varGamma, \neg \neg A}{\varGamma, A} \quad \frac{\varGamma, \neg (A_1 \land A_2)}{\varGamma, \neg A_1, \neg A_2} \quad \frac{\varGamma, A_1 \land A_2}{\varGamma, A_i}$$

where size(C) < size(A) in the leftmost rule and i = 1, 2 in the rightmost schema, together with the axioms $\Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_k$ and all sequents of the form Γ, Δ where $\Gamma, \Delta \in \mathsf{S}(\Lambda)$ and, for some $B \in \mathcal{F}(\Lambda)$,

- $\Gamma, B \text{ and } \Delta, \neg B \in \{\Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_k\}, \text{ or }$
- $\Gamma, B = \Gamma_0, A \text{ and } \Delta, \neg B \in \{\Delta_1, \dots, \Delta_k\}, \text{ or }$
- $\Gamma, B = \Delta_0, \neg A \text{ and } \Delta, \neg B \in \{\Gamma_1, \ldots, \Gamma_n\}.$

A rule set that absorbs structural rules and the cut rule is called absorbing.

The intuition behind the above definition is similar to that of absorption of structural rules, but we have two additional degrees of freedom: we can not only apply the cut rule to rule premises, but we can moreover freely use both cut on structurally smaller formulas and the structural rules. This allows us to use the standard argument, a double induction on the structure of the cut formula and the size (or height) of the proof tree, to establish cut elimination. This is carried out in the proof of the next theorem.

Theorem 14. Suppose R is absorbing. Then the cut rule

$$\frac{\varGamma, A \quad \Delta, \neg A}{\varGamma, \Delta}$$

is admissible in GR.

The proof proceeds by a double induction on the size of the cut formula and the size of the proof tree, and analyse all possible ways in which the cut rule can be applied. The case of cuts arising between conclusions of modal rules follows from absorption of cut. Cuts between conclusions of a modal and a propositional rule can be eliminated by using the absorption of structural rules.

We illustrate the preceding theorem by using it to derive the well-known fact that cut-elimination holds for the modal logics K, K4 and T and use it to derive cut-elimination for various conditional logics in Section 6.

Example 15. The rule sets K, K4 and T are absorbing. We have already seen that they absorb weakening, contraction and inversion in Example 10 so everything that remains to be seen is that they also absorb cut. For (K), we need to apply cut to a formula of smaller size. For the two instances

$$(r_1)\frac{\neg A_1,\ldots,\neg A_n,A_0}{\neg \Box A_1,\ldots,\neg \Box A_n,\Box A_0,\Gamma} \qquad (r_2)\frac{\neg B_1,\ldots,\neg B_k,B_0}{\neg \Box B_1,\ldots,\neg \Box B_k,\Box B_0,\Delta}$$

we need to consider, up to symmetry, the cases $A_i = B_0$, $\Box A_i \in \Delta$ and $\neg \Box A_0 \in \Delta$, for $i = 1, \ldots, n$. Here, we only treat the first case for i = 1 where we have to show that $\neg \Box A_2, \ldots, \neg \Box A_n, \Box A_0, \neg \Box B_1, \ldots, \neg \Box B_k, \Gamma, \Delta$ is derivable from $\mathsf{GR} + \mathsf{Cut}(\Box A_1, r_1, r_2)$, which follows as the latter system allows us to apply cut on $A_1 = B_0$. The case $\Box A_i \in \Delta$ and $\neg \Box A_0 \in \Delta$ are straight forward.

The argument to show that (K4) is absorbing is similar, and uses an additional (admissible) instance of cut on a formula of smaller size and contraction. For (T) we only consider instances of cut between two conclusions of

$$(r_1)\frac{\neg A, \neg \Box A, \Gamma}{\neg \Box A, \Gamma} \qquad (r_2)\frac{\neg B, \neg \Box B, \Delta}{\neg \Box B, \Delta}$$

of the T-rule. We only demonstrate the case $\Box A \in \Delta$. In this case, $\Delta = \Delta', \Box A$ and we have to show that $\neg \Box B, \Gamma, \Delta'$ can be derived in $\mathsf{Cut}(\Box A, r_1, r_2)$. The latter system allows us to cut $\neg \Box A$ between the conclusion of (T) on the left and the premise of the right hand rule, i.e., we have that $\mathsf{Cut}(\Box A, r_1, r_2) \vdash$ $\neg B, \neg \Box B, \Gamma, \Delta'$ and an application of (T) now gives derivability of $\neg \Box B, \Gamma, \Delta'$.

5 Equivalence of Hilbert and Gentzen Systems

We now investigate the relationship between provability in a Hilbert-system and provability in the associated Gentzen system. We note the following standard lemmas that we will use later on.

Lemma 16. Suppose $A \in \mathcal{F}(\Lambda)$ is a propositional tautology. Then $\mathsf{GR} \vdash A$. If moreover R is closed under substitution, then $\mathsf{GR} \vdash \Gamma\sigma$ whenever $\mathsf{GR} \vdash \Gamma$ for all $\Gamma \in \mathsf{S}(\Lambda)$.

Remark 17. Being able to prove the previous lemma is the main reason for formulating axioms as $A, \neg A, \Gamma$ where $A \in \mathcal{F}(\Lambda)$ rather than $p, \neg p, \Gamma$. Both formulations are equivalent if the modal congruence rule

$$\begin{array}{ccc} A_1 \leftrightarrow A'_1 & \dots & A_n \leftrightarrow A'_n \\ \heartsuit(A_1, \dots, A_n) \to \heartsuit(A'_1, \dots, A'_n) \end{array}$$

is admissible. However, Lemma 16 can be proved without the assumption that congruence is admissible using axioms of the form $A, \neg A, \Gamma$.

Theorem 18. Suppose R is absorbing and substitution closed. Then $\mathsf{GR} \vdash \Gamma \iff$ $\mathsf{HR} \vdash \bigvee \Gamma$ for all $\Gamma \in \mathsf{S}(\Lambda)$. *Proof (Sketch).* We only need to show the direction from right to left. Inductively assume that $\mathsf{HR} \vdash \bigvee \Gamma$ for $\Gamma \in \mathsf{S}(\Lambda)$. By Proposition 8 we have that there are rules Θ_i/Γ_i and substitutions σ_i , $i = 1, \ldots, n$ such that

- $\mathsf{HR} \vdash \Delta \sigma_i$ whenever $\Delta \in \Theta_i$ $(i = 1, \dots, n)$
- $\{\bigvee \Gamma_1 \sigma_1, \ldots, \bigvee \Gamma_n \sigma_n\} \vdash_{\mathsf{PL}} \bigvee \Gamma.$

By induction hypothesis, $\mathsf{GR} \vdash \Delta \sigma_i$ for all $i = 1, \ldots, n$ and $\Delta \in \Theta_i$. By Lemma 16 we have

$$\mathsf{GR} \vdash \bigvee \Gamma_1 \sigma_1 \wedge \cdots \wedge \bigvee \Gamma_n \sigma_n \to \bigvee \Gamma.$$

The claim follows by applying cut, contraction and inversion.

The construction of an absorbing rule set from a given set of axioms and rules essentially boils down to adding the missing instances of cut, weakening, contraction and inversion to a given rule set. The soundness of this process is witnessed by the following two simple lemmas, which we use in this section to derive an absorbing rule set for K and to establish cut-elimination for a large range of conditional logics in the next section.

Lemma 19. Suppose $\Gamma_1, \ldots, \Gamma_n / \neg A, \Gamma_0$ and $\Delta_1, \ldots, \Delta_k / A, \Delta_0 \in \mathbb{R}$. Then the rule $\Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_k / \Gamma_0, \Delta_0$ is admissible in HR.

The same applies to instances of the structural rules of weakening, contraction and inversion. As we are extending the rule set while leaving the provability predicate in the Hilbert calculus unchanged, the following formulation is handy for our purposes – in particular it implies the fact that we can freely use structural rules both in the premise and conclusion.

Lemma 20. Suppose that $\Gamma_1, \ldots, \Gamma_n/\Gamma_0 \in \mathsf{R}$. If $\Delta_0, \ldots, \Delta_1 \in \mathsf{S}(\Lambda)$ and both

$$\{\bigvee \Delta_1, \dots, \bigvee \Delta_k\} \vdash_{\mathsf{PL}} \bigvee \Gamma_i (1 \le i \le n) \text{ and } \bigvee \Gamma_0 \vdash_{\mathsf{PL}} \bigvee \Delta_0$$

then the rule $\Gamma_1, \ldots, \Gamma_k/\Gamma_0$ is admissible in HR.

This gives us a recipe for constructing rule sets that absorb contraction and cut: simply add more rules according to the lemmas above. This will not change the notion of provability in the Hilbert system, but when this process terminates, the ensuing rule set will be absorbing and gives rise to a cut free sequent calculus.

Example 21 (Modal Logic K). In a Hilbert-style calculus, the axiomatisation of K is usually described in terms of the distribution axiom (which we view as a rule with empty premise) and the necessitation rule:

$$(\mathsf{D}) \quad \Box(A \to B) \to \Box A \to \Box B \qquad (\mathsf{N})\frac{A}{\Box A}$$

We first apply Lemma 19 to break the propositional connectives in the distribution axiom. We have that the axiom $\neg \Box (A \rightarrow B), \neg \Box A, \Box B$ is admissible by Lemma 20, and applying Lemma 19 to this axiom and the instance $A \to B/\Box (A \to B)$ of the necessitation rule gives admissibility of the all instances of

$$\frac{\neg A, B}{\neg \Box A, \Box B}$$

with the help of (admissible) propositional reasoning in the premise. The same procedure, applied to the instances

$$\frac{\neg A, B \to C}{\neg \Box A, \Box (B \to C)} \qquad \neg \Box (B \to C), \neg \Box B, \Box C$$

gives admissibility of the left hand rule below,

$$\frac{\neg A, \neg B, C}{\neg \Box A, \neg \Box B, \Box C} \qquad \frac{\neg A_1, \dots, \neg A_n, A_0}{\neg \Box A_1, \dots, \neg \Box A_n, \Box A_0, \Gamma}$$

and continuing in this way and absorbing weakening, we obtain admissibility of the rule on the right, where $\Gamma \in S(\Lambda)$ is an arbitrary context. We have shown previously that this rule set is absorbing, and it is easy to see that it is equivalent to the rule set consisting of all instances of (N) and (D).

6 Applications: Sequent Calculi for Conditional Logics

After having seen how the construction of absorbing rule sets gives rise to cutelimination for a number of well-studied normal modal logics, in this section we construct a cut-free sequent calculus for a number of conditional logics.

Conditional logics [2] are extensions of propositional logic by a non-monotonic conditional $A \Rightarrow B$, read as "B holds under the condition that A". The conditional implication is non-monotonic in general, that is the validity of $A \Rightarrow B$ does not imply that also $(A \land C) \Rightarrow B$ is a valid statement.

Axiomatically, the first argument A of the conditional operation $A \Rightarrow B$ behaves like the \Box in neighbourhood frames and only supports replacement of equivalents, whereas the second argument B obeys the rules of K. We recall from Example 2 that CK CK is axiomatised by the rules (RCEA) and (RCK) that we augment with a subset of the following axioms:

$$(\mathsf{ID}) A \Rightarrow A \qquad (\mathsf{MP}) (A \Rightarrow B) \to A \to B \qquad (\mathsf{CEM}) (A \Rightarrow B) \lor (A \Rightarrow \neg B).$$

The first axiom embodies a form of identity in the sense that A holds under condition A and (MP) is a conditional form of modus ponens. The axiom (CEM) is the conditional excluded middle. We denote combination of rule sets by jux-taposition so that CKID comprises all instances of CK and ID.

6.1 Cut Elimination for Extensions of CK without CEM

We first treat extensions of the basic conditional logic CK with axioms ID and MP, but not including CEM and discuss CEM later, as the effect of adding CEM leads to a more general form of the CK rule. We start by introducing some notation that provides a shorthand for expressing the bi-implications in the premise of CK.

Notation 22. If $A_0, \ldots, A_n \in \mathcal{F}(\Lambda)$ are conditional formulas, we write $A_0 = \cdots = A_n$ for the sequence of sequents consisting of $\neg A_0, A_i$ and $\neg A_i, A_0$ for all $1 \leq i \leq n$.

If we absorb cuts using Lemmas 19 and 20 we see that all instances of

$$(\mathsf{CK}_g)\frac{A_0 = \dots = A_n \quad \neg B_1, \dots, \neg B_n, B_0}{\neg (A_1 \Rightarrow B_1), \dots, \neg (A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma}$$

are admissible in HCK. It is easy to see that the rule set CK_g is actually absorbing: **Theorem 23.** The rule set CK_g is absorbing and equivalent to CK . As a consequence, GCK_g has cut-elimination and $\mathsf{GCK}_g \vdash A$ iff $\mathsf{HCK} \vdash A$ whenever $A \in \mathcal{F}(\Lambda)$.

Proof. Using Lemmas 19 and Lemma 20 it is immediate that the rule set CK_g is admissible in HCK . The argument that show that CK_g is absorbing is analogous to that for the modal logic K (Example 15), and the result follows from Theorem 18.

The logic CKID arises form CK by adding the identity axiom $A \Rightarrow A$ to the rule set CK_H that axiomatises standard conditional logic. Applying Lemma 19 to the two rule instances on the left

$$\frac{\neg A, B \quad \neg B, A}{\neg (A \Rightarrow A), (A \Rightarrow B)} \qquad A \Rightarrow A \qquad (\mathsf{ID}_g) \frac{A = B}{A \Rightarrow B, \Gamma}$$

gives rise to the rule schema (ID_g) on the right where we have used Lemma 20 to absorb weakening. If we denote the rule set consisting of all instances of CK_g and ID_g by CKID_g , we obtain:

Proposition 24. The rule set $CKID_g$ is absorbing and equivalent to CKID.

Proof. It is easy to see that CKID_g absorbs the structural rules, and that CKID_g is equivalent to CKID_g . Cuts between conclusions of (ID_g) are readily seen to be absorbed, and absorption of cuts between an instance of CK_g and an instance of ID follows by construction.

The logic CKMP arises by augmenting the logic CK with the additional axiom $(A \Rightarrow B) \rightarrow (A \rightarrow B)$. The effect of adding (MP) is similar to that of enriching the modal logic K with the (T)-axiom. Adding the missing cuts to CK augmented with (MP) and absorbing the structural rules leads to the rule schema

$$(\mathsf{MP}_g)\frac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma}$$

and we denote the rule set consisting of all instances of CK_g and MP_g by CKMP_g . Our cut elimination theorem then takes the following form:

Proposition 25. The rule set CKMP_q is absorbing and equivalent to CKMP .

Proof. Again, it is easy to see that CKMP_g is admissible in HCKMP and the converse follows by construction. All we have to show is that CKMP_g is absorbing, where the absorption of structural rules is easy and left to the reader. For the absorption of cut, the argument is similar to cut elimination in the modal logic T .

6.2 Cut Elimination for Extensions of CKCEM

To construct an absorbing rule set for conditional logic plus the axiom

$$(\mathsf{CEM})(A \Rightarrow B) \lor (A \Rightarrow \neg B)$$

we start from the admissible rule set for CK and close under cuts that arise with (CEM). Repeated applications of Lemma 19 and Lemma 20 lead to the rule set

$$(\mathsf{CKCEM}_g)\frac{A_0 = \dots = A_n \quad B_0, \dots, B_j, \neg B_{j+1}, \neg B_n}{(A_0 \Rightarrow B_0), \dots, (A_j \Rightarrow B_j), \neg (A_{j+1} \Rightarrow B_{j+1}), \dots, \neg (A_n \Rightarrow B_n), \Gamma}$$

for $1 \leq j \leq n$.

Proposition 26. The rule set $CKCEM_a$ is absorbing and equivalent to CKCEM.

As a consequence, cut elimination holds in CKCEM_g . We can apply essentially the same argument to an extension of CK with both conditional modus ponens and conditional excluded middle, but have to take care of the cuts arising between MP_g and CKCEM_g , which leads to the new rule

$$(\mathsf{MPEM}_g)\frac{A, (A \Rightarrow B), \Gamma}{(A \Rightarrow B), \Gamma} \xrightarrow{B, (A \Rightarrow B), \Gamma}.$$

If we denote the extension of CKCEM_g with MP_g and MPEM_g by $\mathsf{CKCEMMP}_g,$ we obtain:

Proposition 27. $\mathsf{CKCEMMP}_g$ is absorbing and equivalent to $\mathsf{CKCEMMP}$.

We note that the latter theorem was left as an open problem for the sequent system presented in [8]. In summary, we obtain the following results about extensions of the conditional logic CK.

Theorem 28. Suppose that L is one of CK, CKID, CKMP, CKCEM or CKCEMMP. Then $GL_g \vdash A$ whenever $HL \vdash A$ for all $A \in \mathcal{F}(A)$. Moreover, cut elimination holds in GL.

The theorem follows, in each of the cases, from Theorem 14 and Theorem 18 together with the fact that the rule set L and L_g are equivalent and the latter is absorbing.

7 Complexity of Proof Search

It is comparatively straightforward to extract complexity bounds for provability of the logics considered above by analysing the complexity of proof search under suitable strategies in the cut-free sequent systems obtained. Clearly, in those cases where all modal rules peel off exactly one layer of modal operators, the depth of proofs is polynomial in the nesting depth of modal operators in the target formula, and therefore, proof search is in PSPACE under mild assumptions on the branching width of proofs [12, 9]. Besides reproving Ladner's classical result for K [7], we thus have

Theorem 29. Provability in CK and CKID is in PSPACE.

This reproves known complexity bounds originally shown in [8] (alternative short proofs using coalgebraic semantics are given in [11]). For CKCEM, the bound can be improved using dynamic programming in the same style as in [13]:

Theorem 30. Provability in CKCEM is in coNP.

More interesting are those cases where some of the modal operators from the conclusion remain in the premise, such as T, K4, CKMP, and CKCEM (where the difference between non-iterative logics, i.e. ones whose Hilbert-axiomatisation does not use nested modalities, such as T, CKMP, and CKMPCEM, and iterative logics such as K4 is surprisingly hard to spot in the sequent presentation). For K4, the standard approach is to consider proofs of minimal depth, which therefore never attempt to prove a sequent repeatedly, and analyse the maximal depth that a branch of a proof can have without repeating a sequent. For T, a different strategy is used, where the (T) rule is limited to be applied at most once to every formula of the form $\neg \Box A$ in between two applications of (K) [6]. A similar strategy works for the conditional logics CKMP and CKMPCEM, which we explain in some additional detail for CKMP.

We let CKMP_g^0 and CKMP_g^1 denote restricted sequent systems where in CKMP_q^0 , a formula $\neg(A \Rightarrow B)$ is marked on a branch as soon as the rule (MP_q) has been applied to it (backwards) and unmarked only at the next application of rule (CK_q) . Rule (MP_q) applies only to unmarked formulas. In CKMP_q^1 we instead impose a similar restriction where rule (MP_q) applies to a sequent $\neg(A \Rightarrow B), \Gamma$ only in case Γ does not contain a propositional descendant of either A or $\neg B$. Here, a sequent Δ is called a *propositional descendant* of a formula A if it can be generated from A by applying propositional sequent rules backwards (e.g. the propositional descendants of $(\neg(A \land B) \land C)$ are $\neg(A \land B)$; C; and $\neg A, \neg B$). It is easy to check that CKMP_q^1 -proofs can be converted into CKMP_q^0 proofs, i.e. CKMP_{q}^{1} is the most restrictive system. One shows that CKMP_{q}^{1} admits contraction and inversion by verifying that the corresponding proof transformations in CKMP_g preserve CKMP_q^1 -proofs. It is then clear that every application of the rule (MP_g) that violates the CKMP_q^1 -restriction can be replaced by inversion and contraction, so that CKMP_q^1 , and hence also CKMP_q^0 , proves the same formulas as CKMP_g . Proofs in CKMP_g^0 are easily seen to have at most polynomial depth. Essentially the same reasoning applies to CKMPCEM. Therefore, we have

Theorem 31. Provability in CKMP is in PSPACE; provability in CKMPCEM is in coNP.

We note that the complexity of CKMPCEM was explicitly left open in [8].

8 Conclusions

We have established a generic method of cut elimination in modal sequent system based on absorption of cut and structural rules by sets of modal rules. We have applied this method in particular to various conditional logics, thus obtaining cut-free unlabelled sequent calculi that complement recently introduced labelled calculi [8]. In at least one case, the conditional logic CKMPCEM with modus ponens and conditional excluded middle, our calculus seems to be the first cut-free calculus in the literature, as cut elimination for the corresponding calculus in [8] was explicitly left open. We have applied these calculi to obtain complexity bounds on proof search in conditional logics; in particular we have reproved known upper complexity bounds for CK, CKID, CKMP [8] and improved the bound for CKCEM from PSPACE to coNP using dynamic programming techniques following [13]. Moreover, we have obtained an upper bound coNP for CKMPCEM, for which no bound has previously been published. We conjecture that our general method can also be applied to other base logics, e.g. intuitionistic propositional logic or first-order logic, which is subject to further investigations.

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