Semantical Principles in the Modal Logic of Coalgebras

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Abstract. Coalgebras for a functor on the category of sets subsume many formulations of the notion of transition system, including labelled transition systems, Kripke models, Kripke frames and many types of automata. This paper presents a multimodal language which is bisimulation invariant and (under a natural completeness condition) expressive enough to characterise elements of the underlying state space up to bisimulation. Like Moss' coalgebraic logic, the theory can be applied to an arbitrary signature functor on the category of sets. Also, an upper bound for the size of conjunctions and disjunctions needed to obtain characteristic formulas is given.

1 Introduction

Rutten [19] demonstrates that coalgebras for a functor generalise many notions of transition systems. It was then probably Moss [16] who first realised that modal logic constitutes a natural way to formulate bisimulation-invariant properties on the state spaces of coalgebras.

Given an arbitrary signature functor on the category of sets, the syntax of his coalgebraic logic is obtained via an initial algebra construction, where the application of the signature functor is used to construct formulas. This has the advantage of being very general (few restrictions on the signature functor), but the language is abstract in the sense that it lacks the usual modal operators $\Box$ and $\Diamond$.

Other approaches [9, 10, 13, 17, 18] devise multimodal languages, given by a set of modal operators and a set of atomic propositions, which are based on the syntactic analysis of the signature functor (and therefore only work for a restricted class of transition signatures).

This paper aims at combining both methods by exhibiting the underlying semantical structures which give rise to (the interpretation of) modal operators with respect to coalgebras for arbitrary signature functors.

After a brief introduction to the general theory of coalgebras (Section 2), we look at examples of modal logics for two different signature functors in Section 3. The analysis of the semantical structures, which permit to use modalities to formulate properties on the state space of coalgebras, reveals that modal

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operators arise through a special type of natural transformation, which we chose to call “natural relation”.

Abstracting away from the examples, Section 4 presents a concrete multi-modal language which arises through a set of natural relations and can be used to formulate predicates on the state space of coalgebras for arbitrary signature functors. On the semantical side, it is shown that a set of natural relations induces a translation from coalgebras to Kripke models, and that the semantics of formulas wrt. coalgebras coincides with the semantics of formulas wrt. the induced Kripke model. We then prove in Section 5 that the interpretation of the language is indeed invariant under (coalgebraic) bisimulation. In the last section we characterise the expressive power of the language, and prove that under a natural completeness condition, every point of the state space can be characterised up to bisimulation. We also give an upper bound for the size of conjunctions and disjunctions needed to obtain characteristic formulas.

2 Transition Systems and Coalgebras

Given an endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ on the category of sets and functions, a T-*coalgebra* is a pair (C, γ) where C is a set (the state space or carrier set of the coalgebra) and $\gamma : C \to TC$ is a function. Using this definition, which dualises the categorical formulation of algebras, many notions of automata and transition systems can be treated in a uniform framework. We only sketch the fundamental definitions and refer the reader to [11, 19] for a more detailed account.

Example 1 (Labelled Transition Systems). Suppose L is a set of labels. Labelled transition systems, commonly used to formulate operational semantics of process calculi such as CCS, arise as coalgebras for the functor $TX = \mathcal{P}(L \times X)$. Indeed, given a set C of states and a transition relation R_l for each label $l \in L$, we obtain a T-coalgebra (C, γ) where $\gamma(c) = \{(l, \hat{c}) \in L \times C \mid c \; R_l \; \hat{c}\}$. Conversely, every coalgebra structure $\gamma : C \to TC$ gives rise to a family of transition relations $(R_l)_{l \in L}$ via $c \; R_l \; c'$ iff $(l, c') \in \gamma(c)$.

Many types of automata can also be viewed as coalgebras for an appropriate type of signature functor on the category of sets:

Example 2 (Deterministic Automata). Let $TX = (O \times X)^I + E$ and $(C, \gamma : C \to TC)$ be a T-coalgebra. Given an element of the state space $c \in C$, the result $\gamma(c)$ of applying the transition function is either an error condition $e \in E$ or a function $f : I \to O \times C \in (O \times C)^I$. Supplying an input token $i \in I$, the result $f(i)$ of evaluating f gives us an output token $o \in O$ and a new state $c' \in C$.

Morphisms of coalgebras are functions between the corresponding state spaces, which are compatible with the respective transition structures. Dualising the categorical formulation of algebra morphisms, a *coalgebra morphism* between two T-coalgebras (C, γ) and (D, δ) is a function $f : C \to D$ such that $Tf \circ \gamma = \delta \circ f$.

Diagrammatically, f must make the diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\gamma"] & D \arrow[d, "\delta"] \\ TC \arrow[r, "Tf"] & TD \end{tikzcd}$$

commutative. The reader may wish to convince himself that in the case of labelled transition systems above, a coalgebra morphism is a functional bisimulation in the sense of Milner [15]. It is an easy exercise to show that coalgebras for a functor T, together with their morphisms, constitute a category.

One important feature of the functional (ie. coalgebraic) formulation of transition systems is that every signature functor comes with a built in notion of bisimulation. Following Aczel and Mendler [1], a *bisimulation* between two coalgebras (C, γ) and (D, δ) is a relation $B \subseteq C \times D$, that can be equipped with a transition structure $\beta : B \to TB$, which is compatible with the projections $\pi_C : B \to C$ and $\pi_D : B \to D$. More precisely, $B \subseteq C \times D$ is a bisimulation, if there exists $\beta : B \to TB$ such that

$$\begin{tikzcd} C \arrow[d, "\gamma"] & B \arrow[l, "\pi_C"'] \arrow[r, "\pi_D"] \arrow[d, "\beta"] & D \arrow[d, "\delta"] \\ TC & TB \arrow[l, "T\pi_C"'] \arrow[r, "T\pi_D"] & TD \end{tikzcd}$$

commutes. Again, the reader may wish to convince himself that in the case of labelled transition systems, coalgebraic bisimulations, as just defined, are indeed bisimulations of labelled transition systems.

3 Modal Logic for Coalgebras: Examples

We exemplify the connection between modal logics and coalgebras for a functor by means of the examples given in the previous section. In both examples we observe that the modalities and atomic propositions of the respective languages arise via special types of natural transformation, the "natural relations" already mentioned in the introduction.

The general theory developed in the subsequent sections is based on this observation in that it shows, that every set of natural relations induces a multimodal language which allows to formulate bisimulation invariant properties on the state spaces of coalgebras for an arbitrary signature functor.

3.1 Labelled Transition Systems

Consider the functor $TX = \mathcal{P}(L \times X)$ on the category of sets and functions. We have demonstrated in Example 1, that T-coalgebras are labelled transition systems over the set L of labels. It is well known that Hennessy-Milner logic [8] (also discussed in [22]) is an expressive, bisimulation invariant language, which allows to formulate predicate on the state space of labelled transition systems.

Consider the set $\mathcal{L}$ of formulas built up from the atomic propositions $\mathtt{tt}$, $\mathtt{ff}$, conjunctions, disjunctions and a pair of modal operators $\Box_l$ and $\Diamond_l$ for every $l \in L$. Given a T-coalgebra (labelled transition system) (C, γ) and a formula $\phi \in \mathcal{L}$, we write $[\![\phi]\!]_{(C,\gamma)}$ for the set $\{c \in C \mid (c, \gamma) \models \phi\}$ of points $c \in C$, which satisfy the formula ϕ with respect to the transition structure γ, and drop the subscript (C, γ) if the transition structure is clear from the context. Omitting the straightforward interpretation of atomic propositions, conjunctions and disjunctions, the interpretation of the formula $\Box_l \phi$ is given by

$$[\![\Box_l \phi]\!]_{(C,\gamma)} = \{c \in C \mid \forall c' \in C.(l, c') \in \gamma(c) \implies c' \in [\![\phi]\!]_{(C,\gamma)}\} \tag{1}$$

for any $l \in L$.

Note that the same definition can be used for any carrier set and transition structure. This leads us to define, given $l \in L$, a parameterised relation $\mu_l(A) \subseteq TA \times A$, given by

$$\mathfrak{a}\ \mu_l(A)\ a \quad \iff \quad (l, a) \in \mathfrak{a} \tag{2}$$

for an arbitrary set A, $\mathfrak{a} \in TA$ and $a \in A$.

Using this definition, we can now reformulate (1) as

$$[\![\Box_l \phi]\!]_{(C,\gamma)} = \{c \in C \mid \forall c' \in C.\gamma(c)\ \mu_l(C)\ c' \implies c' \in [\![\phi]\!]_{(C,\gamma)}\} \tag{3}$$

and obtain the interpretation of the existential modality via

$$[\![\Diamond_l \phi]\!]_{(C,\gamma)} = \{c \in C \mid \exists c' \in C.\gamma(c)\ \mu_l(C)\ c' \wedge c' \in [\![\phi]\!]_{(C,\gamma)}\}. \tag{4}$$

The fact that (2) is a canonical definition, which works for any set A, is witnessed by the following universal property: For any function $f : A \to B$, the diagram of sets and relations

$$\begin{tikzcd} TA \arrow[r, "\mu_l(A)", "\shortmid"{marking}] \arrow[d, "\mathrm{G}(Tf)", "\shortmid"{marking}] & A \arrow[d, "\mathrm{G}(f)", "\shortmid"{marking}] \\ TB \arrow[r, "\mu_l(B)", "\shortmid"{marking}] & B \end{tikzcd} \tag{5}$$

commutes (where we write $R : A \nrightarrow B$ for a relation $R \subseteq A \times B$ and $\mathrm{G}(f)$ for the graph of a function; composition of the arrows in the diagram is relational composition).

Parameterised relations, which satisfy condition (5) will be called natural relations in the sequel. Thus summing up, one can say that *natural relations give rise to the interpretation of modalities.*

3.2 Input/Output Automata

In Example 2 we have seen that deterministic input/output automata are coalgebras for the functor $TX = (O \times X)^I + E$. We now go on to demonstrate that the modalities needed to describe properties of these automata also arise via parameterised relations, that is, relations which satisfy the naturality condition (5).

Given a T coalgebra $(C, \gamma : C \to TC)$ and a state $c \in C$, the modality of interest here describes the behaviour of a (possible) successor state, which arises after supplying an input token, if the result $\gamma(c)$ of applying the transition function does not yield an error condition $e \in E$.

Given $i \in I$ and an arbitrary set A, we consider the relation $\mu_i(A) \subseteq TA \times A$, given by

$$\mathfrak{a}\ \mu_i(A)\ a \quad \text{iff} \quad \exists f : I \to (O \times A) \in (O \times A)^I . \mathfrak{a} = \mathrm{inl}(f) \wedge \pi_A \circ f(i) = a,$$

where $\mathrm{inl} : (O \times A)^I \to (O \times A)^I + E$ is the canonical injection and π_A denotes the projection function $O \times A \to A$. Note that this parameterised relation also satisfies the naturality condition (5) and allows us to define a pair of modalities $\Box_i$ and $\Diamond_i$ using equations (3) and (4).

In order to obtain a language which allows to specify the behaviour of a state $c \in C$, we furthermore need atomic propositions to be able to formulate that the application $\gamma(c)$ of the transition function yields an error condition $e \in E$ and that – in case $\gamma(c) \in (O \times C)^I$ – supplying an input token $i \in I$ yields an output token $o \in O$.

This is taken care of by a set of atomic propositions $\{p_e \mid e \in E\} \cup \{p_{(i,o)} \mid (i, o) \in I \times O\}$. The interpretation of the atomic propositions in this example is straightforward:

$$[\![p_e]\!]_{(C,\gamma)} = \{c \in C \mid \gamma(c) = \mathrm{inr}(e)\} \tag{6}$$

and

$$[\![p_{(i,o)}]\!]_{(C,\gamma)} = \{c \in C \mid \exists f \in (O \times C)^I . \gamma(c) = \mathrm{inl}(f) \wedge \pi_O \circ f(i) = o\}, \tag{7}$$

where $\mathrm{inr} : E \to (O \times C)^I + E$ is again the canonical injection and $\pi_O : O \times C \to O$ denotes the projection function.

In both cases it deserves to be mentioned that the atomic propositions arise as subsets of the set $T1$ (where we write $1 = \{*\}$ for the terminal object in the category of sets and $!_C : C \to 1$ for the unique morphism).

To be more precise, consider the sets

$$p_e|_{T1} = \{\mathrm{inr}(e) \mid e \in E\} \tag{8}$$

and

$$p_{(i,o)}|_{T1} = \{\mathrm{inl}(f) \mid f \in (O \times 1)^I \wedge \pi_O \circ f(i) = o\}, \tag{9}$$

where in this case $\mathrm{inr} : E \to (O \times 1)^I + E$ and $\mathrm{inl} : (O \times 1)^I \to (O \times 1)^I + E$.

Using the subsets defined by (8) and (9), we now recover the interpretation of the atomic propositions, originally given by (6) and (7) as

$$[\![p_e]\!]_{(C,\gamma)} = (T!_C \circ \gamma)^{-1}(p_e|_{T1})$$

and

$$[\![p_{(i,o)}]\!]_{(C,\gamma)} = (T!_C \circ \gamma)^{-1}(p_{(i,o)}|_{T1}),$$

respectively.

Thus one can say that *atomic propositions in modal logics for T-coalgebras arise as subsets of the set $T1$.*

4 From Natural Relation to Modal Logics

If $T : \mathsf{Set} \to \mathsf{Set}$ is an endofunctor, the examples in the previous section suggest, that modal logics for coalgebras of a functor are induced by a set of natural relations for T and a set of predicates on $T1$. The remainder of the paper is devoted to showing that this is indeed the case. We start by exhibiting the modal language which arises from a set of natural relations and a set of atomic propositions and show in the subsequent sections, that the language presented is bisimulation invariant and (under a completeness condition on the set of relations) strong enough to distinguish non-bisimilar points.

4.1 Natural Relations

Categorically speaking, natural relations are natural transformations between functors mapping from the category Set of sets and functions to the category Rel of sets and relations. This is captured in

Definition 1 (Natural Relations). *Suppose T is an endofunctor on the category of sets. A* natural relation *for T is a natural transformation*

$$\mathcal{I} \circ T \to \mathcal{I},$$

where $\mathcal{I} : \mathsf{Set} \to \mathsf{Rel}$ is the identity on sets and sends every function to the relation given by its graph.

Unravelling the definition of natural transformations we obtain the formulation of the naturality requirement, which is present in the examples:

Proposition 1 (Universal Property of Natural Relations). *Suppose $T : \mathsf{Set} \to \mathsf{Set}$ is a functor and $\mu(A) \subseteq TA \times A$ is a relation for every set A. Then μ is a natural relation, iff for every function $f : A \to B$, the diagram of sets and relations*

$$\begin{tikzcd} TA \arrow[r, "\mu(A)", "+"{marking}] \arrow[d, "\mathrm{G}(Tf)", "+"{marking}] & A \arrow[d, "\mathrm{G}(f)", "+"{marking}] \\ TB \arrow[r, "\mu(B)", "+"{marking}] & B \end{tikzcd}$$

commutes with respect to relational composition.

Regarding examples of natural relations, we refer the reader to the examples discussed in Section 2.

By moving from a relation $R : A \not\to B$ to a function $\mathbb{S}_R : A \to \mathcal{P}(B)$ given by $\mathbb{S}_R(a) = \{b \in B \mid a \; R \; b\}$, we can also view natural relations $\mathcal{I} \circ T \to \mathcal{I}$ as natural transformations (we write $\mathcal{P}$ for the covariant powerset functor):

Proposition 2 (Natural Relations as Natural Transformations). *Suppose T is an endofunctor on the category of sets. Then there is a one-to-one correspondence between the set $\mathrm{Nat}(\mathcal{I} \circ T, \mathcal{I})$ of natural transformations $\mathcal{I} \circ T \to \mathcal{I}$ and the set $\mathrm{Nat}(T, \mathcal{P})$ of natural transformations $T \to \mathcal{P}$ given by*

$$\mathrm{Nat}(\mathcal{I} \circ T, \mathcal{I}) \ni \mu \mapsto \mathbb{S}_\mu \in \mathrm{Nat}(T, \mathcal{P})$$

where

$$\mathbb{S}_\mu(X) = \mathbb{S}_{\mu(X)}$$

for all sets X.

The above proposition is as an instance of a far more general fact[1]. Note that the category Rel of sets and relations can be equivalently described as the Kleisli category of the powerset monad $(\mathcal{P}, \{\cdot\}, \cup)$ on Set, see for instance [14], Example T2. One then recovers the inclusion functor $\mathcal{I}$ as the functor which sends every function $f : X \to Y$ to the composition $\{\cdot\} \circ f : X \to \mathcal{P}(Y)$ of f with the singleton map $\{\cdot\}$, considered as map $\mathcal{I}(f) : X \to Y$ in the Kleisli category of the powerset monad.

In the same spirit one defines a canonical functor $\mathcal{I} : \mathbb{C} \to \mathrm{Kl}(\mathsf{M})$, embedding $\mathbb{C}$ in the Kleisli category $\mathrm{Kl}(\mathsf{M})$ of M, for any monad M on a category $\mathbb{C}$. Now suppose $T : \mathbb{C} \to \mathbb{C}$ is any endofunctor, M is the underlying functor of the monad M and $\rho_X : TX \to MX$ is a family of morphisms in $\mathbb{C}$, indexed by the objects of $\mathbb{C}$. Then an easy diagram chase shows that the naturality of $\rho : T \dot{\to} M$ is equivalent to the naturality of $\hat{\rho} : \mathcal{I} \circ T \dot{\to} \mathcal{I}$, where the components of $\hat{\rho}$ are the components of ρ, considered as morphisms in the Kleisli category $\mathrm{Kl}(\mathsf{M})$.

It should also be pointed out that every set $\mathcal{A}$ of subsets of $T1$ gives rise to a natural transformation $\mathbb{P}_\mathcal{A} : T \to \mathcal{P}(\mathcal{A})$, where $\mathcal{P}(\mathcal{A})$ is the constant functor which sends every set to $\mathcal{P}(\mathcal{A})$.

Proposition 3. *Suppose T is an endofunctor on the category of sets and $\mathcal{A}$ is a set of subsets of $T1$. Define a function $\mathbb{P}_{\mathcal{A},X} : TX \to \mathcal{P}(A)$ for any set X by*

$$\mathbb{P}_{\mathcal{A},X}(x) = \{a \in \mathcal{A} \mid T!_X(x) \in a\}.$$

Then $\mathbb{P}_\mathcal{A} : T \to \mathcal{P}(\mathcal{A})$ is a natural transformation.

For the remainder of this section we assume, that $T : \mathsf{Set} \to \mathsf{Set}$ is an endofunctor on the category of sets and functions, $\mathcal{M}$ is a set of natural relations for T, $\mathcal{A}$ is a set of subsets of $T1$ and κ is a cardinal number.

[1] We would like to thank one of the anonymous referees for pointing this out

4.2 Syntax and Semantics of $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$

As it is often the case with modal languages, we sometimes need infinitary constructs in the language to obtain enough expressive power. In order to be able to deal with the general case later, we fix a cardinal number κ, which serves as upper bound for the size of conjunctions and disjunctions.

The language $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ induced by the set $\mathcal{M}$ of natural relations and $\mathcal{A}$ of atomic propositions is given by the least set of formulas containing

- An atomic proposition p_a for every $a \in \mathcal{A}$
- The formulas $\bigwedge \Phi$ and $\bigvee \Phi$, if Φ is a set of formulas of cardinality less than or equal to κ, and
- The formulas $\Diamond_\mu \phi$ and $\Box_\mu \phi$ for every $\mu \in \mathcal{M}$ and every formula ϕ of $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$.

Note that $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ contains as a special case the formulas $\bigwedge \emptyset$ and $\bigvee \emptyset$, which we shall abbreviate to $\mathtt{tt}$ and $\mathtt{ff}$, respectively.

In order to simplify the exposition of the semantics of $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$, we introduce an easy bit of notation.

Definition 2. *Suppose $R \subseteq A \times B$ is a relation. Then R induces two operations, which we denote by $\Box_R$ and $\Diamond_R$, both mapping $\mathcal{P}(B) \to \mathcal{P}(A)$, whose definitions are given by*

$$\Diamond_R(\mathfrak{b} \subseteq B) = \{a \in A \mid \exists b \in B. a \; R \; b \wedge b \in \mathfrak{b}\}$$

and

$$\Box_R(\mathfrak{b} \subseteq B) = \{a \in A \mid \forall b \in B. a \; R \; b \implies b \in \mathfrak{b}\},$$

respectively. Following Moss [16], we introduce a further operator $\triangle_R : \mathcal{P}(B) \to \mathcal{P}(A)$ defined by

$$\triangle_R(\mathfrak{b} \subseteq B) = \Box_R(\mathfrak{b}) \cap \bigcap_{b \in \mathfrak{b}} \Diamond_R(\{b\}),$$

which we will use later.

The semantics of $[\![\phi]\!]_{(C,\gamma)}$ of a formula $\phi \in \mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ can now be inductively defined:

- $[\![p_a]\!]_{(C,\gamma)} = (T!_C \circ \gamma)^{-1}(a)$ for atomic propositions p_a given by $a \in \mathcal{A}$
- $[\![\bigwedge \Phi]\!]_{(C,\gamma)} = \bigcap_{\phi \in \Phi} [\![\phi]\!]_{(C,\gamma)}$ and $[\![\bigvee \Phi]\!]_{(C,\gamma)} = \bigcup_{\phi \in \Phi} [\![\phi]\!]_{(C,\gamma)}$ for conjunctions and disjunctions (following standard conventions, we set $[\![\mathtt{tt}]\!]_{(C,\gamma)} = C$ and $[\![\mathtt{ff}]\!]_{(C,\gamma)} = \emptyset$), and
- $[\![\Box_\mu \phi]\!]_{(C,\gamma)} = \Box_{\mu(C) \circ \mathrm{G}(\gamma)}([\![\phi]\!]_{(C,\gamma)})$ and $[\![\Diamond_\mu \phi]\!]_{(C,\gamma)} = \Diamond_{\mu(C) \circ \mathrm{G}(\gamma)}([\![\phi]\!]_{(C,\gamma)})$ for the modal operators.

If the transition structure is clear from the context, we sometimes abbreviate $[\![\phi]\!]_{(C,\gamma)}$ to $[\![\phi]\!]_C$ (and sometimes even to $[\![\phi]\!]$). In case we want to emphasise that a formula ϕ holds at a specific point $c \in C$ of the underlying set, we also write $c \models_\gamma \phi$ for $c \in [\![\phi]\!]_{(C,\gamma)}$.

4.3 Translation to Kripke Models

We show that, given a set of natural relations $\mathcal{M}$ and a set of atomic propositions (subsets of $T1$) $\mathcal{A}$, we obtain a translation of T-coalgebras to Kripke models. This translation is canonical in the sense that the semantics of formulas wrt. T-coalgebras coincides with the standard textbook semantics of modal logic wrt. the induced Kripke model.

Suppose $\mathcal{A}$ is a set of atomic propositions and $\mathcal{M}$ a set determining the modalities. Kripke models over $\mathcal{A}$ and $\mathcal{M}$, which support the interpretation of multimodal languages, are generally given by triples $(C, (R_\mu)_{\mu\in\mathcal{M}}, V)$ where C is the carrier set of the model (the set of possible worlds), $R_\mu \subseteq C \times C$ is an endorelation (transition relation) on C for every $\mu \in \mathcal{M}$ and $V : C \to \mathcal{P}(\mathcal{A})$ is a (valuation) function, which assigns to every world $c \in C$ the set $V(c) \subseteq \mathcal{A}$ of propositions valid in world $c \in C$.

Suppose $M = (C, (R_\mu)_{\mu\in\mathcal{M}}, V)$ is a Kripke model. The standard textbook semantics (as described for the finitary case for instance in [7]) $(\!|\phi|\!)_M$ of a formula $\phi \in \mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ is inductively defined by the clauses

- $(\!|p_a|\!)_M = V^{-1}(\{p\}) = \{c \in C \mid p \in V(C)\}$ for atoms p_a (where $a \in \mathcal{A}$)
- $(\!|\bigwedge \Phi|\!)_M = \bigcap_{\phi\in\Phi} (\!|\phi|\!)_M$ and $(\!|\bigvee \Phi|\!)_M = \bigcup_{\phi\in\Phi} (\!|\phi|\!)_M$ for conjunctions and disjunctions, and
- $(\!|\Box_\mu \phi|\!)_M = \Box_{R_\mu}((\!|\phi|\!)_M)$ and $(\!|\Diamond_\mu \phi|\!)_M = \Diamond_{R_\mu}((\!|\phi|\!)_M)$ for the modal operators.

Now suppose that T is an endofunctor on Set, $\mathcal{M}$ a set of natural relations for T and $\mathcal{A}$ is a set of subsets of $T1$. With every T-coalgebra we associate a Kripke model $\mathbb{K}(C, \gamma)$ as follows:

$$\mathbb{K}(C, \gamma) = (C, (R_\mu)_{\mu\in\mathcal{M}}, V)$$

where $R_\mu = \mu(C) \circ \mathrm{G}(\gamma)$ and $V(c) = \{a \in \mathcal{A} \mid T!_C \circ \gamma(c) \in a\}$.

Proposition 4 ($\mathcal{L}(\mathcal{A}, \mathcal{M}, \kappa)$ is canonical). *Suppose (C, γ) is a T-coalgebra. Then*

$$[\![\phi]\!]_{(C,\gamma)} = (\!|\phi|\!)_{\mathbb{K}(C,\gamma)}$$

for all formulas $\phi \in \mathcal{L}(\mathcal{A}, \mathcal{M}, \kappa)$.

Proof. Follows from the respective definitions by induction on the structure of the formulas. □

5 Invariance Properties of $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$

In this section, we demonstrate that $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ is an adequate logic for T-coalgebras. We do this by proving that the semantics of formulas is invariant under coalgebra morphisms and that bisimilar elements of the state space of coalgebras satisfy the same set of formulas.

For the whole section assume that T is an endofunctor on Set, $\mathcal{M}$ is a set of natural relations for T, $\mathcal{A}$ is a set of subsets of $T1$ and κ is a cardinal number.

Theorem 1 (Morphisms preserver semantics). *Suppose $f : (C, \gamma) \to (D, \delta)$ is a morphism of coalgebras. Then*

$$[\![\phi]\!]_C = f^{-1}([\![\phi]\!]_D)$$

for all formulas ϕ of $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$.

Before we embark on the proof of of the theorem, we isolate some properties of the relational operators introduced in Definition 2. One readily proves the following

Lemma 1 (Simple Properties of $\Box_R$, $\Diamond_R$). *Suppose $R : A \nrightarrow B$ and $S : B \nrightarrow C$ are relations and $f : A \to B$ is a function.*

1. $\Box_{S \circ R} = \Box_R \circ \Box_S$
2. $\Diamond_{S \circ R} = \Diamond_R \circ \Diamond_S$
3. *For every $\mathfrak{b} \subseteq b$, $\Box_{\mathrm{G}(f)}(\mathfrak{b}) = \Diamond_{\mathrm{G}(f)}(\mathfrak{b}) = f^{-1}(\mathfrak{b})$.*

We are now ready for the

Proof (of Theorem 1). We proceed by induction on the structure of the formulas. If $a \in \mathcal{A}$ (thus p_a is an atomic proposition), the claim follows from considering the diagram

$$\begin{tikzcd}
C \arrow[r, "\gamma"] \arrow[d, "f"] & TC \arrow[d, "Tf"] \arrow[dr, "T!_C"] & \\
D \arrow[r, "\delta"] & TD \arrow[r, "T!_D"] & T1
\end{tikzcd}$$

which commutes by functoriality of T.

For the case of modal operators, suppose $\phi \in \mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ with $[\![\phi]\!]_C = f^{-1}[\![\phi]\!]_D$ and $\mu \in \mathcal{M}$ and consider the diagram

$$\begin{tikzcd}
C \arrow[r, "\mathrm{G}(\gamma)", "\shortmid"{marking}] \arrow[d, "\mathrm{G}(f)", "\shortmid"{marking}] & TC \arrow[r, "\mu(C)", "\shortmid"{marking}] \arrow[d, "\mathrm{G}(Tf)", "\shortmid"{marking}] & C \arrow[d, "\mathrm{G}(f)", "\shortmid"{marking}] \\
D \arrow[r, "\mathrm{G}(\delta)", "\shortmid"{marking}] & TD \arrow[r, "\mu(D)", "\shortmid"{marking}] & D
\end{tikzcd}$$

of sets and relations and apply Lemma 1.

It is immediate that the semantics is stable under conjunctions and disjunctions. $\Box$

We have an easy and immediate

Corollary 1. *Suppose $f : (C, \gamma) \to (D, \delta)$ is a morphism of coalgebras and $c \in C$. Then*

$$c \models_\gamma \phi \quad \textit{iff} \quad f(c) \models_\delta \phi$$

for all formulas $\phi \in \mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$.

We now turn to the second invariance property mentioned at the beginning of this chapter and show that bisimilar points satisfy the same sets of formulas. Although this essentially follows from Theorem 1, its importance warrants to state it as

Theorem 2 (Bisimilarity implies logical equivalence). *Suppose (C, γ) and (D, δ) are T coalgebras and the points $c \in C$ and $d \in D$ are related by a bisimulation. Then*

$$c \models_\gamma \phi \quad \textit{iff} \quad d \models_\delta \phi$$

for all formulas $\phi \in \mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$.

6 Expressivity

This section shows, that the language $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ also satisfies an abstractness condition in the sense, that under a natural completeness condition on the pair $(\mathcal{M}, \mathcal{A})$, non-bisimilar points of the carrier set of coalgebras can be distinguished by formulas of $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$.

For the proof we assume the existence of a terminal coalgebra, that is, of a greatest fixed point for the signature functor T. We represent the greatest fixed point of the signature functor T as limit of the so-called terminal sequence, which makes the succession of state transitions explicit. The categorical dual of terminal sequences is commonly used to construct initial algebras, see [2, 21]. We use Theorem 2 of Adámek and Koubek [3], which states that in presence of a terminal coalgebra, the latter can be represented as a fixed point of the terminal sequence. Suppose for the remainder of this section, that T is an endofunctor on the category of sets, $\mathcal{M}$ is a set of natural relations for T and $\mathcal{A}$ is a set of subsets of $T1$.

6.1 Complete Pairs

It is obvious that we cannot in general guarantee that the language $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ is strong enough to actually distinguish non-bisimilar points, since the set $\mathcal{M}$ might not contain enough relations or we do not have enough atomic propositions. We start by giving a completeness criterion on the sets $\mathcal{M}$ and $\mathcal{A}$, which ensures that this does not happen.

We use the same notation as in Proposition 2 and 3 and write $\mathbb{S}_R(a) = \{b \in B \mid a \; R \; b\}$ if $R : A \leftrightarrow B$ is a relation and $a \in A$. We also write $\mathbb{P}_{\mathcal{A},X}(x) = \{a \in \mathcal{A} \mid T!_X(x) \in a\}$ if $x \in X$ for the set of atomic propositions $a \in \mathcal{A}$ satisfied by $x \in TX$. We shall abbreviate $\mathbb{P}_{\mathcal{A},X}$ to $\mathbb{P}_\mathcal{A}$ (or even to $\mathbb{P}$) in the sequel.

Definition 3 (Completeness of $(\mathcal{M}, \mathcal{A})$). *We call the pair $(\mathcal{M}, \mathcal{A})$* complete, *if*

$$\{x\} = \bigcap_{\mu \in \mathcal{M}} \{x' \in TX \mid \mathbb{S}_{\mu(X)}(x') = \mathbb{S}_{\mu(X)}(x)\} \cap \bigcap_{a \in \mathbb{P}_\mathcal{A}(x)} (T!)^{-1}(a)$$

for all sets X and all elements $x \in TX$.

Intuitively, the pair $(\mathcal{M}, \mathcal{A})$ is complete, if, given any set X, every element $x \in TX$ is determined by its $\mu(X)$-successors and the atomic propositions which are satisfied by x.

Using the natural transformations defined in Proposition 2 and 3, we can give a necessary (and under a natural assumption on the set of atomic propositions) sufficient criterion for completeness of a pair $(\mathcal{M}, \mathcal{A})$.

Proposition 5 (Characterisation of Completeness). *Suppose η denotes the natural transformation*

$$\eta : T \to \prod_{\mu \in \mathcal{M}} \mathcal{P} \times \mathcal{P}(\mathcal{A})$$

induced by $(\mathcal{M}, \mathcal{A})$.

1. *If $(\mathcal{M}, \mathcal{A})$ is complete, then η is monic (in the functor category $[\mathsf{Set}, \mathsf{Set}]$).*
2. *Suppose $\mathcal{A}$ separates points (that is, for every $x \in T1$ we have $\bigcap\{a \in \mathcal{A} \mid x \in a\} = \{x\}$). Then $(\mathcal{M}, \mathcal{A})$ is complete iff η is monic.*

The next example shows, that the assumption of separating points cannot be dropped from the premises of Proposition 5.

Example 3. Suppose $L = \{l_0, l_1\}$ and $TX = L \times X$. Define the relation $\mu_X : L \times X \not\leftrightarrow X$ for every set X by $\mu_X = \{((l, x), x) \mid l \in L \wedge x \in X\}$ and consider the set $\mathcal{A}$ of subsets of $T1 = L \times \{*\}$ consisting of $a_0 = \{(l_0, *)\}$ and $a_1 = \{(l_0, *), (l_1, *)\}$. Then the induced transformation $\eta : T \to \mathcal{P} \times \mathcal{P}(\mathcal{A})$ is injective although $(\mathcal{M}, \mathcal{A})$ is not complete.

We briefly note that the natural relations and atomic propositions defined in Section 3 give rise to complete pairs:

Example 4 (Complete pairs).

1. Consider the signature functor $TX = \mathcal{P}(L \times X)$. If $\mathcal{M} = \{\mu_l \mid l \in L\}$ is the set of natural relations defined in Section 3.1 and $\mathcal{A} = \emptyset$, then $(\mathcal{M}, \mathcal{A})$ is complete.
2. Suppose $TX = (O \times X)^I + E$ as in Section 3.2 and let $\mathcal{M} = \{\mu_i \mid i \in I\}$ and $\mathcal{A} = \{p_e|_{T1} \mid e \in E\} \cup \{p_{(i,o)}|_{T1} \mid (i, o) \in I \times O\}$ be the set of natural relations and atomic propositions defined there, respectively. Then $(\mathcal{M}, \mathcal{A})$ is complete.

6.2 Existence of Complete Pairs

It seems very hard to find a semantical characterisation of functors which admit a complete pair $(\mathcal{M}, \mathcal{A})$ of natural relations and subsets of $T1$. However, we can give some closure properties of the class of functors, which admit a complete pair. These closure properties are summarised in the next theorem, which is easy, but tedious to prove.

Theorem 3 (Closure Properties). *The class of functors which admit a complete pair contains*

- *the identity functor*
- *all constant functors*
- *the (covariant) powerset functor*

and is closed under

- *small limits*
- *small coproducts.*

Note that the class of functors admitting a complete pair is not closed under composition:

Example 5. Let $T = \mathcal{P} \circ \mathcal{P}$. Then T does not admit a complete pair.

For a proof, assume the contrary, and suppose $(\mathcal{M}, \mathcal{A})$ is a complete pair for T. Then Proposition 5 provides us with a monic natural transformation

$$\eta : \mathcal{P} \circ \mathcal{P} \stackrel{.}{\to} \prod_{\alpha < \kappa} \mathcal{P} \times 4,$$

where κ is the cardinality of the set $\mathcal{M}$ of natural relations (evidently, $\kappa \geq 1$). Thus, for every set X of cardinality greater than one, we obtain an injection

$$i_X : \mathcal{P}\mathcal{P}X \to \mathcal{P}(X)^{\kappa'}$$

where $\kappa' = \kappa + 1$. Now take $X = \prod_{n \in \omega} \kappa'$. Then $X \cong \kappa' \times X$ and $\mathcal{P}(X)^{\kappa'} \cong \mathcal{P}(\kappa' \times X) \cong \mathcal{P}(X)$, and i_X gives rise to an injection $\mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$, since $|X| \geq \kappa' = \kappa + 1 \geq 2$, which is clearly impossible.

We would like to close the section by discussing an example of an endofunctor T, which admits a complete pair (and thus an expressive logic, as shown in the next section). Although the structure of T is fairly simple, we would like to point out that it does not fall within the scope of of functors discussed in [10, 13, 17], since an inductive type is used.

Example 6. Denote by $\mathrm{List}(X)$ the set of all finite lists of elements of X. Clearly List extends to a functor on Set, whose action on a function is given pointwise applying f to every element of the list (the "map" as in functional programming languages). Now let

$$TX = L \times \mathrm{List}(X)$$

for a set L of labels. The functor T then models finitely branching, but infinite trees. It now follows from Theorem 3, that T admits a complete pair, which can also be seen directly: For each $l \in L$, let

$$a_l = \{(l, x) \mid x \in List(1)\}$$

and put $\mathcal{A} = \{a_l \mid l \in L\}$. For the set $\mathcal{M}$ of natural relations, consider $\mu_k(X) \subseteq TX \times X$ given by

$$(x_n)_{n<j} \ \mu_k(X) \ x \quad \iff \quad k < j \wedge x_k = x,$$

where we denote a list of elements in X by the finite sequence $(x_n)_{n<j}$ of its elements. Now let $\mathcal{M} = \{\mu_k \mid k \in \omega\}$. Then $(\mathcal{M}, \mathcal{A})$ is complete.

6.3 The Expressivity Theorem

This section proves that $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ is expressive enough to distinguish non-bisimilar points, subject to the completeness of $(\mathcal{M}, \mathcal{A})$ and the size of κ. The cardinality of conjunctions and disjunctions needed to obtain expressivity is given in terms of the cardinality of the final coalgebra and the convergence of the terminal sequence.

Before we state the expressiveness theorem, we briefly review the construction of greatest fixed points for set functors using terminal sequences. We only give a brief exposition, for details see the original paper by Adámek and Koubek [3] (or Worell [25] for a more categorical treatment).

The *terminal sequence* of an endofunctor T on the category of sets is an ordinal-indexed sequence Z_α of sets together with functions $f_{\alpha,\beta} : Z_\alpha \to Z_\beta$ for all ordinals $\beta \leq \alpha$ such that $Z_0 = \{*\}$, $Z_{\alpha+1} = T(Z_\alpha)$ and $Z_\lambda = \mathrm{Lim}_{\ \alpha<\lambda} Z_\alpha$. It can be seen as the continuation of the sequence

$$1 \xleftarrow{!_{T1}} T1 \xleftarrow{T_{T1}} T^2 1 \xleftarrow{T^2(!_{T1})} T^3 1 \cdots$$

through the class of all ordinal numbers.

Note that the terminal sequence generalises the construction of initial algebras and terminal coalgebras to functors, which do not preserve ω-colimits (resp. ω^{op}-limits). It has been shown in [3], Theorem 2, that in presence of a final T coalgebra, the terminal sequence converges (ie. there exists a (limit) ordinal α such that $f_{\alpha+1,\alpha}$ is an isomorphism) to the terminal coalgebra $(Z_\alpha, f^{-1}_{\alpha+1,\alpha})$. If $f_{\alpha+1,\alpha}$ is an isomorphism, we say that the terminal sequence *stabilises* at α.

We are now ready to state the expressiveness theorem:

Theorem 4 ($\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ has characteristic formulas). *Suppose $(\mathcal{M}, \mathcal{A})$ is a complete pair, T admits a terminal coalgebra $(Z, \zeta : Z \to TZ)$ and κ is a cardinal such that*

- *$\kappa \geq |\mathbb{S}_{\mu(Z)}(z)|$ for all $z \in TZ$ and $\mu \in \mathcal{M}$*
- *$\kappa \geq |\mathcal{M}|$ and $\kappa \geq |\mathcal{A}|$ and*
- *The terminal sequence for T stabilises at κ.*

Then there is a formula $\phi^z \in \mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ such that $[\![\phi^z]\!]_{(Z,\zeta)} = \{z\}$ for all $z \in Z$.

Given $z \in Z$, the proof defines a formula $\phi^z(\alpha)$ for each ordinal $\alpha < \kappa$ and $z \in Z$ with the property $[\![\phi^z(\alpha)]\!]_Z = f^{-1}_{\kappa,\alpha}(\{f_{\kappa,\alpha}(z)\})$ by "induction along the terminal sequence" $(Z_\alpha, f_{\alpha,\beta})$ for T. The formula $\phi^z = \phi^z(\kappa)$ then characterises z.

Before embarking on the proof of Theorem 4, we collect some simple observations:

Lemma 2. *Suppose $(Z_\alpha, f_{\alpha,\beta})$ is the terminal sequence for T and λ is a limit ordinal. Then, for any $p \in Z_\lambda$,*

$$\{p\} = \bigcap_{\alpha<\lambda} (f_{\lambda,\alpha})^{-1}(\{f_{\lambda,\alpha}(p)\}).$$

Lemma 3. *Suppose $R : A \leftrightarrow B$ is a relation. Then*

$$\triangle_R(\mathbb{S}(a)) = \{a' \in A \mid \mathbb{S}(a) = \mathbb{S}(a')\}$$

for all $a \in A$.

Invoking Definition 3 of completeness, the previous lemma admits an immediate corollary, which will then be used in the proof of Theorem 4.

Corollary 2. *Suppose $(\mathcal{M}, \mathcal{A})$ is complete and X is a set. Then*

$$\{x\} = \bigcap_{\mu \in M} (\triangle_{\mu(X)}(\mathbb{S}_{\mu(X)}(x))) \cap \bigcap_{a \in \mathbb{P}(x)} (T!_X)^{-1}(a)$$

for all $x \in TX$.

We are now ready for the

Proof (of Theorem 4). Suppose $(Z_\alpha, f_{\alpha,\beta})$ is the terminal sequence for T (thus $(Z_\kappa, f^{-1}_{\kappa+1,\kappa})$ is the terminal T-coalgebra, which we denote by (Z, ζ)).

Let $z \in Z$. For each ordinal $\alpha < \kappa$ we define a formula $\phi^z(\alpha)$ such that $[\![\phi^z(\alpha)]\!]_Z = f^{-1}_{\kappa,\alpha}(\{f_{\kappa,\alpha}(z)\ \})$. To simplify notation, we use the shorthand $\triangle_\mu \Phi = \Box_\mu \bigvee_{\phi \in \Phi} \wedge \bigwedge_{\phi \in \Phi} \Diamond_\mu \phi$ if Φ is a set of formulas of $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ of cardinality less than or equal to κ.

For $\alpha = 0$ let $\phi^z(\alpha) = \mathtt{tt}$. Clearly $f^{-1}_{\kappa,0}(\{f_{\kappa,0}(z)\}) = f^{-1}_{\kappa,0}(\{*\}) = Z = (T!_Z \circ f^{-1}_{\kappa+1,\kappa})^{-1}(T1) = [\![\mathtt{tt}]\!]_Z$.

For a successor ordinal $\alpha + 1$ let $\Phi^z_\mu(\alpha) = \{\phi^y(\alpha) \mid y \in \mathbb{S}_{\mu(Z_\alpha)}(f_{\kappa,\alpha+1}(z))\}$ (note that $\mu(Z_\alpha) : Z_{\alpha+1} = TZ_\alpha \leftrightarrow Z_\alpha$) and define

$$\phi^z(\alpha + 1) = \bigwedge_{\mu \in M} \triangle_\mu \Phi^z_\mu(\alpha) \wedge \bigwedge_{a \in \mathbb{P}(f_{\kappa,\alpha+1}(z))} p_a.$$

The defining property of natural relations together with the fact that the terminal sequence for T stabilises at κ (ie. T preserves the limit $Z_\kappa = \mathrm{Lim}\ _{\beta<\kappa} Z_\beta$) yields the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\mathrm{G}(\zeta)"] \arrow[dr, "\mathrm{G}(f_{\kappa,\alpha+1})"'] & TZ \arrow[r, "\mu(Z)"] \arrow[d, "\mathrm{G}(Tf_{\kappa,\alpha})"] & Z \arrow[d, "\mathrm{G}(f_{\kappa,\alpha})"] \\ & TZ_\alpha \arrow[r, "\mu(Z_\alpha)"] & Z_\alpha \end{tikzcd} \tag{10}$$

of sets and relations. Using (10), we derive

1. $|\mathbb{S}_{\mu(Z_\alpha)}(f_{\kappa,\alpha+1}(z))| \leq \kappa$, ie. $\triangle_\mu \Phi^z_\mu(\alpha)$ is indeed a formula of $\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$,
2. $[\![\triangle_\mu \Phi^z_\mu(\alpha)]\!] = f^{-1}_{\kappa,\alpha+1}(\triangle_{\mu(Z_\alpha)}(\mathbb{S}_{\mu(Z_\alpha)}(z)))$, and
3. $[\![p_a]\!] = f^{-1}_{\kappa,\alpha+1}(T!)^{-1}(a)$ (where $! : Z_\alpha \to 1$).

Putting the last two together, we obtain

$$[\![\phi^z(\alpha+1)]\!] = f^{-1}_{\kappa,\alpha+1}(\{f_{\kappa,\alpha+1}(z)\}),$$

by Corollary 2, which proves the claim for successor ordinals $\alpha + 1$.

If $\lambda \leq \kappa$ is a limit ordinal, we let $\phi^z(\lambda) = \bigwedge_{\beta<\lambda} \phi^z(\beta)$ and conclude $[\![\phi^z(\lambda)]\!] = f^{-1}_{\kappa,\lambda}(\{f_{\kappa,\lambda}(z)\})$ using Lemma 2. Defining $\phi^z = \phi^z(\kappa)$ finally proves the theorem. □

Some remarks concerning the conditions on the cardinal κ in Theorem 4 are in order. Clearly, we need conjunctions and disjunctions over possibly all atomic propositions and modalities. The third condition is also very natural, since we build the characteristic formula step by step, until we reach the terminal coalgebra, ie. the index, where the terminal sequence stabilises. The only unintuitive condition is the first, giving a lower bound for κ in terms of the final coalgebra. When looking at examples, one however notices that the restriction on the size of successors is very often already implicit in the signature functor T. One can for example show, that all polynomial functors T admit a set of natural relations $\mathcal{M}$, such that for all sets X and all $t \in TX$, the cardinality of the set of successors $\mathbb{S}_{\mu(X}(x)$ is at most one. Also, since we require T to have a terminal coalgebra, T cannot contain an unbounded powerset construction, hence the signature has to determine an upper bound of the set of successors also in this case.

As a corollary we conclude that in presence of a terminal coalgebra, any two bisimilar points satisfy the same sets of formulas. Note that for the corollary to work, we need the signature functor T to preserve weak pullbacks, since otherwise also non-bisimilar points are identified in the terminal coalgebra. Since in cases, where the signature functor does not preserve weak pullbacks, bisimulation fails to capture the notion of behavioural equivalence, we do not consider the restriction to weak pullback preserving functors as a defect of our theory.

In cases where the signature functor does not preserve weak pullbacks, Kurz argues in [12], that observable equivalence is not captured by bisimulation as defined by Aczel and Mendler [1], and – in presence of a final coalgebra – one should consider two state bisimilar, when they are identified in the final coalgebra, a notion, which can be equivalently described using co-congruences.

Corollary 3 ($\mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ is adequate). *Suppose T preserves weak pullbacks and the hypothesis of Theorem 4. If (C, γ) is a T-coalgebra and $c \in C$, there exists a formula $\phi^c \in \mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa)$ such that*

$$[\![\phi^c]\!]_{(C,\gamma)} = \{c' \in C \mid c \leftrightarrows c'\}$$

(where $c \leftrightarrows d$ iff there is a bisimulation $R \subseteq C \times D$ such that $c\,R\,d$).

Theorem 4 also allows us to derive a characterisation of coalgebraic bisimulation in logical terms. To this end, we denote by $\mathrm{Th}(c) = \{\phi \in \mathcal{L}(\mathcal{M}, \mathcal{A}, \kappa) \mid c \models_\gamma \phi\}$ the set of formulas satisfied by a point $c \in C$ for a T-coalgebra (C, γ).

Corollary 4 (Bisimulation is logical equivalence). *Suppose T preserves weak pullbacks and the hypothesis of Theorem 4. If (C, γ) and (D, δ) are T-coalgebras and $(c, d) \in C \times D$, then*

$$\mathrm{Th}(c) = \mathrm{Th}(d) \quad \Longleftrightarrow \quad c \leftrightarrows d$$

(where again $c \leftrightarrows d$ iff there is a bisimulation $R \subseteq C \times D$ with $c\ R\ d$).

We conclude by re-examining Example 6 and check the conditions of Theorem 4 for the functor $TX = L \times \mathrm{List}(X)$.

Example 7. Consider the functor $TX = L \times \mathrm{List}(X)$ form Example 6 and suppose that $\mathcal{M} = \{\mu_k \mid k \in \omega\}$ and $\mathcal{A} = \{a_l \mid l \in L\}$ are the sets of natural relations and atomic propositions defined there. We check the conditions on the cardinal κ given in Theorem 4 for this example.

First note, that for any T-coalgebra (C, γ) and any $t \in TC$, the set $\mathbb{S}_{\mu_k(C)}(t) = \{c \in C \mid (t, c) \in \mu_\kappa(C)\}$ of μ_k-successors of t has cardinality at most one, for every $k \in \omega$. If the set L of labels is at most countable, then the second condition of Theorem 4 holds for all cardinals $\kappa \geq \omega$. Regarding the third condition, Proposition 2.2 of [5] shows, that T is ω-bicontinuous, so the third condition is also satisfied if $\kappa \geq \omega$. Hence $\mathcal{L}(\mathcal{M}, \mathcal{A}, \omega)$ distinguishes non-bisimilar points of T-coalgebras, if $|L| \leq \omega$.

7 Conclusions and Related Work

We have exhibited two semantical principles which allow to use multimodal logics to specify bisimulation invariant properties of coalgebras for an arbitrary signature functor T. The same issue has been addressed in [4, 13, 16, 17]. We briefly compare the results presented in this paper to the contributions just mentioned.

Regarding the work of Moss [16], it has already been pointed out that the construction of the language used to formalise properties on state spaces of coalgebras is very general, and imposes few restrictions on the signature functor T. Since the construction of the language is carried out in the category of classes and set-continuous functions, T has to be set-based (ie. the action of T on classes has to be defined by its action on sets). In order to obtain a characterisation result, the signature functor T is also assumed to be uniform, a condition, which also appears (in slightly different form) in [24, 23]. Note that the defining property of uniformity (taken from [24], section 5.5) is the existence of a natural transformation $\rho : \bar{T} \to \mathcal{P} \circ W$, where $\bar{T}$ is the extension of T to the category of classes, $\mathcal{P}$ is the powerset functor and W maps a class C to the carrier of the $\mathcal{P}$ algebra free over C. Hence T can be embedded into a powerset construction, but it in general this does not seem to imply that T can be embedded into a product $\prod_{\alpha<\kappa} \mathcal{P}$ of the power set functor for a *fixed* cardinal κ. It remains as open question, whether in presence of an accessibility condition on T, such an embedding can be obtained, which would also lead to a better semantical characterisation of the class of functors, which admit complete pairs.

We turn to the work of Baltag [4], where a logical characterisation of simulation is given by extending a set functor T to a relator, that is, to an endofunctor $\mathsf{Rel}(T) : \mathsf{Rel} \to \mathsf{Rel}$ on the category of sets and relations. Baltag argues, that different extensions of T to a relator give rise to different notions of simulation, including bisimulation, which is captured by extending T to a strong relator. The logical language used to obtain a characterisation of (various notions of) simulation is similar to that used in [16]. One of the main goals of the present paper was to obtain languages, which (only) characterise bisimulation. In case the signature functor T preserves weak pullbacks, it is shown in [6] (which is also used in [20] giving – to the authors knowledge – the first characterisation of bisimulation in terms of relators) that T can be uniquely extended to a strong relator $\mathsf{Rel}(T)$. In this case, natural relations can be equivalently described as natural transformations $\mathsf{Rel}(T) \circ \mathcal{I} \stackrel{\cdot}{\to} \mathcal{I}$, where $\mathcal{I} : \mathsf{Set} \to \mathsf{Rel}$ is the canonical embedding. While this reformulation does not seem to simplify our treatment of coalgebraic modal logic, it would be interesting to see, whether replacing the strong relator $\mathsf{Rel}(T)$ by a different extension of T to a relator, the languages constructed in this paper give also rise to a characterisation of the different forms of simulation as discussed in [4].

The work of [10, 13, 17] focuses on an inductively defined class of functors, and the languages considered there are built by induction on the structure of the signature functor. We have shown in Theorem 3, that most of the functors considered in these approaches admit a complete pair. The notable exception are functors which contain more than one "occurrence" of the powerset functor $\mathcal{P}$, for example $TX = \mathcal{P}(A \times \mathcal{P}(B))$. The logic described in [17] admits a characterisation result even for those functors, but at the expense of a language constructed by an iteration of inductive definitions. That is, at every "occurrence" of the powerset functor, one has to close the language constructed so far under propositional connectives and modalities and uses the set thus obtained as the base case for a new inductive definition. This technique could be mimicked in the framework of natural relations by considering a chain of relations $T = T_k \stackrel{\mu_k}{\to} T_{k-1} \stackrel{\mu_{k-1}}{\to} \cdots \stackrel{\mu_1}{\to} T_0 = \mathrm{Id}$, where each set of relations $T_j \to T_{j-1}$ enjoys a completeness property. Looking at examples, the approach seems promising, but we have not yet worked out the details which then would lead to a more general theory.

Finally, we would like to comment on the predicate liftings used in [10]. By an easy inductive argument, one can see, that the "paths to identity" used in loc. cit. in order to obtain modal operators give rise to natural relations $T \leftrightarrow \mathrm{Id}$. On the other hand, every natural relation μ determines a pair of predicate liftings $\exists_\mu$ and $\forall_\mu$. Here we use the term "predicate lifting" in the general sense, indicating a natural transformation $2 \stackrel{\cdot}{\to} 2 \circ T$ (2 denotes the contravariant powerset functor) in contrast to [10], where one associates a fixed predicate lifting to each functor T by induction on its syntactical structure. It should also be noted that from a logical perspective, the interpretation of the modal operator associated to the predicate liftings $\exists_\mu$ and $\forall_\mu$ coincides with the interpretation of the existential and universal modality $\Diamond_\mu$ and $\Box_\mu$ induced by a natural relation $\mu : T \leftrightarrow \mathrm{Id}$.

It thus seems, that predicate liftings also give rise to logics for coalgebras, but expressiveness results are probably more difficult to obtain, since one can not argue in terms of successors any more (as we did in the proof of Theorem 4).

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