PSPACE Bounds for Rank-1 Modal Logics

Lutz Schröder
Department of Computer Science, Universität Bremen
and DFKI-Lab Bremen

Dirk Pattinson
Department of Computer Science
University of Leicester

Abstract

For lack of general algorithmic methods that apply to wide classes of logics, establishing a complexity bound for a given modal logic is often a laborious task. The present work is a step towards a general theory of the complexity of modal logics. Our main result is that all rank-1 logics enjoy a shallow model property and thus are, under mild assumptions on the format of their axiomatization, in $PSPACE$. This leads not only to a unified derivation of (known) tight $PSPACE$-bounds for a number of logics including K, coalition logic, and graded modal logic (and to a new algorithm in the latter case), but also to a previously unknown tight $PSPACE$-bound for probabilistic modal logic, with rational probabilities coded in binary. This generality is made possible by a coalgebraic semantics, which conveniently abstracts from the details of a given model class and thus allows covering a broad range of logics in a uniform way.

1. Intoduction

Modal logics are attractive from a computational point of view, as they often combine expressiveness with decidability. For many modal logics not involving dynamic features, satisfiability is known to be in $PSPACE$. This is typically proved for one logic at a time, e.g. by modifications of the witness algorithm for the modal logic K [4], but also using markedly different methods such as in the constraint-based $PSPACE$-algorithm for graded modal logic [31]. A first glimpse of a generalisable method was given in [34], where various epistemic logics, equipped with a neighborhood frame semantics, were shown to be in NP and $PSPACE$, respectively (with the K axiom being responsible for $PSPACE$-hardness; recent work [11] shows that negative introspection brings the complexity back down to NP). Nevertheless, there is to date no generally applicable theorem that allows establishing $PSPACE$-bounds for large classes of modal logics in a uniform way.

Here, we generalise the methods of [34] to obtain $PSPACE$ bounds for rank-1 modal logics (those axiomatisable by formulae whose nesting depth of modalities equals one) in a systematic way. Although limited to rank 1, our approach covers numerous relevant and non-trivial examples. E.g., our results recover known $PSPACE$ bounds for standard modal logics such as K and also for a range of non-normal modal logics such as graded modal logic [31] and coalition logic [23]. Moreover, our method goes beyond re-proving known results in a uniform fashion: we obtain a previously unknown $PSPACE$-bound for probabilistic modal logic [17, 13], with rational probabilities coded in binary. These logics are far from exotic: graded modal logic plays a role e.g. in decision support and knowledge representation [33, 19], and probabilistic modal logic has appeared in connection with model checking [17] and in modelling economic behaviour [13].

The key to the generality is to parametrise the theory over the type of systems defining the semantics, using coalgebraic methods. Coalgebra conveniently abstracts from the details of a concrete class of models by encoding it as an endofunctor on the category of sets. As specific instances, one obtains e.g. Kripke frames, (monotone) neighbourhood frames [12], game frames [23], probabilistic transition systems and automata [24, 3], weighted automata, linear automata [6], and multigraphs [9]. Despite the broad range of systems covered by the coalgebraic approach, a substantial body of concepts and non-trivial results has emerged, encompassing e.g. generic notions of bisimilarity and coinduction [2], corecursion [32], duality, and ultrafilter extensions [15]. Coalgebraic modal logic features in actual specification languages such as the object oriented specification language CCSL [26] and CoCasl [18].

The coalgebraic study of computational aspects of modal logic was initiated in [30], where the finite model property and associated $NEXPTIME$-bounds were proved. Here, we push this further by proving a coalgebraic shallow model property. Our $PSPACE$-algorithm traverses a shallow model, stripping off one layer of modalities in every step. This requires converting the axiomatisation of a given logic into a set of logical rules that obeys a specific closure condition, and a general construction to perform this conversion is provided. The algorithm runs in $PSPACE$, provided the induced set of rules has a polynomial bound on matchings, which is the case for all examples we are aware of.

2. Coalgebraic Modal Logic

We briefly recapitulate the basics of the coalgebraic interpretation of modal logic.

Definition 2.1. [27] Let $T : \mathbf{Set} \to \mathbf{Set}$ be a functor, referred to as the *signature functor*, where $\mathbf{Set}$ is the category of sets. A T*-coalgebra* $A = (X, \xi)$ is a pair (X, ξ) where X is a set (of *states*) and $\xi : X \to TX$ is a function called the *transition* function.

We view coalgebras as generalised transition systems: the transition function delivers a structured set of successors and observations for a state. Mutatis mutandis, we can in fact allow T to take proper classes as values, as we never iterate T or otherwise assume that TX is a set; details are left implicit. This allows us to treat more examples, in particular Pauly's coalition logic (Example 2.5.7 below).

Assumption 2.2. We can assume w.l.o.g. that T preserves injective maps [1]. For convenience of notation, we will in fact sometimes assume that $TX \subseteq TY$ in case $X \subseteq Y$. Moreover, we assume w.l.o.g. that T is non-trivial, i.e. $TX = \emptyset \implies X = \emptyset$ (otherwise, $TX = \emptyset$ for all X).

Modal logic in the form considered here has been introduced as a specification logic for coalgebraically modelled reactive systems in [22], generalising previous results [14, 25, 16, 20]. The coalgebraic semantics is based on predicate liftings, which abstract from the concrete interpretation of modal operators in the same way that the signature functor abstracts from a concrete class of models.

Definition 2.3. A *predicate lifting* for a functor T is a natural transformation $\lambda : 2 \to 2 \circ T^{op}$, where 2 denotes the contravariant powerset functor $\mathbf{Set}^{op} \to \mathbf{Set}$.

A coalgebraic semantics for a modal logic consists of a signature functor and an assignment of a predicate lifting to every modal operator; we write $[\lambda]$ for a modal operator that is interpreted using the lifting λ. Thus, a set Λ of predicate liftings for T determines the syntax of a modal logic $\mathcal{L}(\Lambda)$. Formulae $\phi, \psi \in \mathcal{L}(\Lambda)$ are defined by the grammar

$$\phi ::= \bot \mid \phi \wedge \psi \mid \neg\phi \mid [\lambda]\,\phi,$$

where λ ranges over Λ. Disjunctions $\phi \vee \psi$, truth $\top$, and other boolean operations are defined as usual. The *rank* of a formula is its maximal nesting depth of modal operators; note however that the notion of *rank*-1 *axiom* employed in [21, 8, 15, 30], replaced below by the notion of one-step rule, is stricter than 'formula of rank 1' in that it disallows non-trivial subformulae of rank 0.

The satisfaction relation $\models_C$ between states x of a T-coalgebra $C = (X, \xi)$ and $\mathcal{L}(\Lambda)$-formulae is defined inductively, with the usual clauses for the boolean operations. The clause for the modal operator $[\lambda]$ is

$$x \models_C [\lambda]\phi \iff \xi(x) \in \lambda_C(\llbracket\phi\rrbracket_C),$$

where $\llbracket\phi\rrbracket_C = \{x \in X \mid x \models_C \phi\}$. We drop the subscripts C when C is clear from the context.

From a coalgebraic perspective, the logics $\mathcal{L}(\Lambda)$ have pleasant properties. Behaviourally equivalent states (i.e. states identified by some pair of morphisms) have the same theory [22], and we can – in case T is accessible – always find enough (polyadic) modal operators to distinguish non-equivalent states [29]. In the interest of readability, we restrict our attention to unary modalities for the purpose of this work. However, we remark that our treatment extends to the polyadic case in a straightforward manner. Our main interest is in the (local) *satisfiability problem* for $\mathcal{L}(\Lambda)$:

Definition 2.4. An $\mathcal{L}(\Lambda)$-formula ϕ is *satisfiable* if there exist a T-coalgebra C and a state x in C such that $x \models_C \phi$.

For a more detailed discussion of global and local consequence and weak and strong completeness in a coalgebraic context see [30]. Many modal logics (including probabilistic modal logic and graded modal logic) fail to be compact and hence do not admit finitary *strongly* complete proof systems. The following examples show that the coalgebraic approach subsumes a large class of modal logics.

Example 2.5. [22, 8, 30]

1. Let $\mathcal{P}$ be the covariant powerset functor. Then $\mathcal{P}$-coalgebras are graphs, thought of as transition systems or indeed Kripke frames. The predicate lifting λ defined by

$$\lambda_X(A) = \{B \in \mathcal{P}(X) \mid B \subset A\}$$

gives rise to the standard box modality $\Box = [\lambda]$. This translates verbatim to the finitely branching case, captured by the (accessible) finite powerset functor $\mathcal{P}_{\mathit{fin}}$.

2. Coalgebras for the functor $N = 2 \circ 2^{op}$ (composition of the contravariant powerset functor with itself) are neighbourhood frames, the canonical semantic domain of non-normal logics [7]. The coalgebraic semantics induced by the predicate lifting λ defined by

$$\lambda_X(A) = \{\alpha \in N(X) \mid A \in \alpha\}$$

is just the neighbourhood semantics for $\Box = [\lambda]$.

3. Similarly, coalgebras for the subfunctor of N given by the upwards closed subsets of 2^X are monotone neighbourhood frames [12]. Putting $\Box = [\lambda]$, with λ defined as above, gives the standard interpretation of the $\Box$-modality of monotone modal logic.

4. It is straightforward to extend a given coalgebraic modal logic for T with a set U of *propositional symbols*. This is captured by passing to the functor $T'X = TX \times \mathcal{P}(U)$ and extending the set of predicate liftings by the liftings λ^a, $a \in U$, defined by

$$\lambda^a_X(A) = \{(t, B) \in TX \times \mathcal{P}(U) \mid a \in B\}.$$

Since λ^a is independent of its argument, the induced modal 'operator' can be written as just the propositional symbol a, with the expected meaning.

5. The *finite multiset* (or *bag*) functor $\mathcal{B}$ maps a set X to the set of maps $b : X \to \mathbb{N}$ with finite support. The action on morphisms $f : X \to Y$ is given by $\mathcal{B}f : \mathcal{B}X \to \mathcal{B}Y, b \mapsto \lambda y. \sum_{f(x)=y} b(x)$. Coalgebras for $\mathcal{B}$ are directed graphs with $\mathbb{N}$-weighted edges, often referred to as *multigraphs* [9], and provide a coalgebraic semantics for *graded modal logic* (GML): One defines a set of predicate liftings $\{\lambda^k \mid k \in \mathbb{N}\}$ by

$$\lambda^k_X(A) = \{b : X \to \mathbb{N} \in \mathcal{B}(X) \mid \textstyle\sum_{a \in A} b(a) > k\}.$$

The arising modal operators are precisely the modalities $\Diamond_k$ of GML [9], i.e. $x \vDash \Diamond_k \phi$ iff ϕ holds for more than k successor states of x, taking into account multiplicities. (GML is more standardly interpreted over Kripke frames, where $\Diamond_k \phi$ reads 'there are more than k successors satisfying ϕ'. Both interpretations induce the same notion of satisfiability [30].) Note that $\Diamond_k$ is monotone, but fails to be normal unless $k = 0$. (Recall that a diamond-like operator $\Diamond$ is called *monotone* if it satisfies $\Diamond(a \wedge b) \to \Diamond a$, and *normal* if it satisfies $\Diamond(a \vee b) \to \Diamond a \vee \Diamond b$). A non-monotone variation of GML arises when negative multiplicities are admitted.

6. The *finite distribution functor* D_ω maps a set X to the set of probability distributions on X with finite support. Coalgebras for the functor $T = D_\omega \times \mathcal{P}(U)$, where U is a set of propositional symbols, are probabilistic transition systems (also called *probabilistic type spaces* [13]) with finite branching degree.
The natural predicate liftings for T are the propositional symbols (Item 4 above) and the liftings λ^p defined by

$$\lambda^p(A) = \{P \in D_\omega X \mid PA \geq p\},$$

where $p \in [0,1] \cap \mathbb{Q}$. This yields the modalities $L_p = [\lambda^p]$ of *probabilistic modal logic (PML)* [17, 13], where $L_p \phi$ reads 'ϕ holds in the next step with probability at least p'. (In general [13], probabilistic type spaces can have arbitrary branching degree, but since PML has the finite model property, this has no bearing on satisfiability.) PML is non-normal ($L_p(a \vee b) \to L_p a \vee L_p b$ is not valid for $p > 0$).

7. A coalgebraic semantics for *coalition logic* [23] is given by the class-valued signature functor T defined by

$$TX = \{(S_1, \ldots, S_n, f) \mid \emptyset \neq S_i \in \mathbf{Set}, f : \prod_{i \in N} S_i \to X\}$$

where $N = \{1, \ldots, n\}$ is a fixed set of *agents*. Thus, the elements of TX are *strategic games* with set X of states, i.e. tuples consisting of nonempty sets S_i of *strategies* for all agents i, and an *outcome function* $(\prod S_i) \to X$. Then, a T-coalgebra is a *game frame* [23]. Subsets $C \subseteq N$ are called *coalitions*. We denote the set $\prod_{i \in C} S_i$ by S_C, and for $\sigma_C \in S_C, \sigma_{\bar{C}} \in S_{\bar{C}}$, where $\bar{C} = N - C$, $(\sigma_C, \sigma_{\bar{C}})$ denotes the obvious element of $\prod_{i \in N} S_i$. The modalities $[C]$ of coalition logic are captured as $[C] = [\lambda^C]$ by the predicate liftings λ^C, given by

$$\lambda^C_X(A) = \{(S_1, \ldots, S_n, f) \in TX \mid \\ \exists \sigma_C \in S_C. \forall \sigma_{\bar{C}} \in S_{\bar{C}}. f(\sigma_C, \sigma_{\bar{C}}) \in A\}.$$

Intuitively, $[C]\phi$ means that coalition C can force ϕ.

All the above examples can be canonically extended to systems that process inputs from a set I by passing from the signature functor T to the functor T^I. Similarly, output of elements of O is modelled by extending the functor T to the assignment $X \mapsto O \times TX$. We refer to [8] for a detailed account of the induced logics.

3. Proof Systems For Coalgebraic Modal Logic

Our decision procedure for rank-1 logics relies on a complete axiomatisation in a certain format. Deduction for modal logics with coalgebraic semantics has been considered in [21, 8, 15, 30]. It has been shown that every modal logic over coalgebras can be axiomatised in rank 1 using either rank-1 axioms or rules leading from rank 0 to rank 1 [30], essentially because functors, as opposed to comonads, only encode the one-step behaviour of systems. Here, we focus on rules. The crucial ingredients for the shallow model construction and the ensuing $PSPACE$ algorithm are novel notions of *resolution closure* and *strict one-step completeness* of rule sets.

For the remainder of the paper, we assume given a functor T and a set Λ of predicate liftings for T. We recall a few basic notions from propositional logic, as well as notation for coalgebraic modal logic introduced in [21, 8]:

Definition 3.1. We denote the set of propositional formulae over a set V by $\mathsf{Prop}(V)$. Here, we regard $\neg$ and $\wedge$ as the basic connectives, with all other connectives defined in the standard way. A *literal* over V is either an element of V or the negation of such an element. We use variables ϵ etc. to denote either nothing or $\neg$, so that a literal over V has the general form ϵa, $a \in V$. A *clause* is a finite, possibly empty, disjunction of literals. The set of all clauses over V is denoted by $\mathsf{Cl}(V)$. Although we regard clauses as formulae rather than sets of literals, we shall sometimes use terminology such as 'a literal is contained in a clause' or 'a clause contains another', with the obvious meaning. We denote by $\mathsf{Up}(V)$ the set $\{[\lambda]a \mid \lambda \in \Lambda, a \in V\}$.

If the elements of V are, or have an interpretation as, subsets of a given set X, then $\phi \in \mathsf{Prop}(V)$ can be interpreted as a subset $[\![\phi]\!]_X$ of X; we write $X \models \phi$ if $[\![\phi]\!]_X = X$. Similarly, if $a \in V$ is interpreted as a subset A of X, then we interpret $[\lambda]a \in \mathsf{Up}(V)$ as the subset $[\![[\lambda]a]\!] = \lambda_X(A)$ of TX. This can be iterated, leading to interpretations $[\![\phi]\!] \subset TX$ of $\phi \in \mathsf{Up}(\mathsf{Prop}(V))$ etc.

In case the elements of V are formulae in $\mathcal{L}(\Lambda)$, we also regard propositional formulae over V as formulae in $\mathcal{L}(\Lambda)$. We sometimes explicitly designate V as consisting of *propositional variables*; propositional variables retain their status across further applications of Up and Prop (e.g. if V is a set of propositional variables, then V and not $\mathsf{Prop}(V)$ is the set of propositional variables for $\mathsf{Up}(\mathsf{Prop}(V))$). Given a set L, an *L-substitution* is a substitution σ of the propositional variables by elements of L; for a formula ϕ over V, we call $\phi\sigma$ an *L-instance* of ϕ. If $L \subset \mathcal{P}(X)$ for some X, then we also refer to σ as an *L-valuation*.

Definition 3.2. A *(one-step) rule* R over a set V of propositional variables is a rule ϕ/ψ, where $\phi \in \mathsf{Prop}(V)$ and $\psi \in \mathsf{Cl}(\mathsf{Up}(V))$. We silently identify rules under α-equivalence. The rule R is *sound* if, whenever $\phi\sigma$ is valid for an $\mathcal{L}(\Lambda)$-substitution σ, then $\psi\sigma$ is valid. Moreover, R is *one-step sound* if for each set X and each $\mathcal{P}(X)$-valuation τ, $X \models \phi\tau$ implies $TX \models \psi\tau$.

Out hitherto informal use of the term *rank*-1 *logic* formally means *axiomatisable by one-step rules* (equivalently by rank-1-axioms [30]). The class of rank-1 logics includes many interesting cases (Example 3.17), but excludes logics whose axiomatisation needs nested modalities, e.g. $S4$.

Remark 3.3. We can always assume that every propositional variable a appearing in the premise ϕ of a one-step rule appears also in the conclusion: otherwise, we can eliminate a by passing from ϕ to $\phi[\top/a] \vee \phi[\bot/a]$.

Proposition 3.4. *Every one-step sound rule is sound.*

The converse holds under additional assumptions [29]; note however that the obviously sound rule $\bot/\bot$ is one-step sound iff $T\emptyset = \emptyset$ (as is the case e.g. for PML).

A given set $\mathcal{R}$ of one-step sound rules induces a proof system for $\mathcal{L}(\Lambda)$ as follows.

Definition 3.5. Let $\mathcal{R}_C$ denote the set of rules obtained by extending $\mathcal{R}$ with the *congruence rule*

$$(C)\quad \frac{a \leftrightarrow b}{[\lambda]a \rightarrow [\lambda]b}.$$

(This rule of course implies a rule where $\rightarrow$ is replaced by $\leftrightarrow$, which however does not fit the format for one-step rules.) The set of *derivable* formulae is the smallest set closed under propositional entailment and the rules in $\mathcal{R}_C$, with propositional variables instantiated to formulae in $\mathcal{L}(\Lambda)$.

It is easy to see that this proof system is sound. Completeness requires 'enough' rules in the following sense.

Definition 3.6. The set $\mathcal{R}$ is *(strictly) one-step complete* if, for each set X and each finite $\mathfrak{A} \subset \mathcal{P}(X)$, whenever $TX \models \chi$ for $\chi \in \mathsf{Cl}(\mathsf{Up}(\mathfrak{A}))$, then χ is *(strictly) derivable*; i.e. χ is propositionally entailed by clauses $\psi\tau$, where ϕ/ψ is in $\mathcal{R}$ and τ is a $\mathsf{Prop}(\mathfrak{A})$-valuation (an $\mathfrak{A}$-valuation) such that $X \models \phi\tau$.

The distinctive feature of *strict* one-step completeness is that strict derivation precludes intermediate reasoning over $\mathsf{Up}(\mathsf{Prop}(\mathfrak{A}))$. This plays a central role in the shallow model construction to be presented in Section 4.

Remark 3.7. It is easy to see that in the definition of one-step completeness, it does not matter whether elements of $\mathsf{Prop}(\mathfrak{A})$ are regarded as formulae or as subsets of X.

Lemma and Definition 3.8. *If $\mathcal{R}$ is strictly one-step complete, then for each set X, each $\phi \in \mathsf{Prop}(\mathsf{Up}(V))$, and each $\mathcal{P}(X)$-valuation τ such that $TX \models \phi\tau$, ϕ is* strictly congruence derivable*, i.e. propositionally entailed by clauses $\psi\sigma$, where ϕ/ψ is in $\mathcal{R}_C$ (Definition 3.5) and σ is a V-substitution such that $X \models \phi\sigma\tau$.*

Remark 3.9. It is implicitly shown in [30] that the set of all one-step sound rules is always strictly one-step complete and that the proof system induced by a one-step complete set of rules is *weakly complete*, i.e. proves all valid formulae.

Proposition 3.10. *A set $\mathcal{R}$ of one-step rules is (strictly) one-step complete iff for all* finite *sets X and all subsets $\mathfrak{A} \subset \mathcal{P}X$ that generate $\mathcal{P}(X)$ as a boolean algebra, χ is (strictly) derivable under $\mathcal{R}$ whenever $TX \models \chi$ for $\chi \in \mathsf{Cl}(\mathsf{Up}(\mathfrak{A}))$.*

Strictly one-step complete sets of rules are generally more complicated than one-step complete sets of rules or axioms [21, 30]. In our terminology, part of the effort of [34] and [23] is devoted to finding strictly one-step complete sets of rules. We now develop a systematic procedure for turning one-step complete rule sets into strictly one-step complete ones. For the following, recall that given clauses ϕ and ψ containing literals a and $\neg a$, respectively, a *resolvent* of ϕ and ψ (*at* a) is obtained by removing a and $\neg a$ from the clause $\phi \vee \psi$. A set Φ of clauses is called *resolution closed* if, for $\phi, \psi \in \Phi$, all resolvents of ϕ and ψ are in Φ.

Definition 3.11. A set $\mathcal{R}$ of one-step rules is *resolution closed* if it satisfies the following requirement. Let $R_1, R_2 \in \mathcal{R}$, where $R_1 = \phi_1/\psi_1$ and $R_2 = \phi_2/\psi_2$. We can assume that R_1 and R_2 have disjoint sets of propositional variables. Let $[\lambda]a$ be in ψ_1, and let $\neg[\lambda]b$ be in ψ_2 for some $\lambda \in \Lambda$, so that we have a resolvent ψ of ψ_1 and $\psi_2[a/b]$ at $[\lambda]a$. Then $\mathcal{R}$ is required to contain a rule $R = \phi/\psi$ such that ϕ is propositionally entailed by $\phi_1 \wedge \phi_2[a/b]$; in this case, R is called a *resolvent* of R_1 and R_2.

Remark 3.12. One can construct resolution closed sets by iterated addition of missing resolvents. Here, an obvious choice for a resolvent ϕ/ψ as above is to take ϕ as $\phi_1 \wedge \phi_2[a/b]$, with a eliminated according to Remark 3.3 if a is not contained in ψ; it is clear that $\phi_1 \wedge \phi_2[a/b]/\psi$ is one-step sound if R_1 and R_2 are one-step sound.

Remark 3.13. One should not confuse the terminology introduced above with existing resolution-based approaches to decision procedures for modal logic (e.g. [10]), which rely on translating modal logic into first-order logic.

Lemma 3.14. *Let V be a set of propositional variables, let $\psi \in \mathsf{Cl}(V)$, and let $\Phi \subset \mathsf{Cl}(V)$ be resolution closed. Then Φ propositionally entails ψ iff ψ contains a clause in Φ.*

Theorem 3.15. *If $\mathcal{R}$ is one-step complete and resolution closed, then $\mathcal{R}$ is strictly one-step complete.*

Proof. (Sketch) Let $\mathfrak{A} \subset \mathcal{P}X$, and let $\gamma \in \mathsf{Cl}(\mathsf{Up}(\mathfrak{A}))$ such that $TX \models \gamma$. By one-step completeness, γ is propositionally entailed by the set of clauses

$$\Psi = \{\psi\sigma \mid \phi/\psi \in \mathcal{R}, \sigma \text{ a } \mathsf{Prop}(\mathfrak{A})\text{-valuation}, X \models \phi\sigma\}.$$

Resolution closedness of $\mathcal{R}$ implies that Ψ is resolution closed. By Lemma 3.14, it follows that γ contains, and hence is propositionally entailed by, a clause $\psi\sigma$ in Ψ, where necessarily $\sigma(v) \in \mathfrak{A}$ for variables v of ψ. $\square$

In summary, strictly one-step complete rule sets can be constructed by resolving the rules of a one-step complete axiomatisation against each other. Below, we give examples of strictly one-step complete systems obtained in this way. In order to simplify the presentation for the case of graded modal logic and probabilistic modal logic, we use the following notation. If ϕ_i is a formula, $r_i \in \mathbb{Z}$ for all $i \in I$, and $k \in \mathbb{Z}$, we abbreviate

$$\sum_{i \in I} r_i \phi_i \geq k \equiv \bigwedge_{r(J) < k} \Big(\bigwedge_{j \in J} \phi_j \to \bigvee_{j \notin J} \phi_j \Big),$$

where $r(J) = \sum_{j \in J} r_j$. Moreover, if $r \in \mathbb{Z} - \{0\}$ and ϕ is a formula, then we put

$$sgn(r)\phi = \begin{cases} \phi & r > 0 \\ \neg\phi & r < 0. \end{cases}$$

The formula $\sum_{i \in I} r_i a_i \geq k$ translates into the arithmetic of characteristic functions as suggested by the notation:

Lemma 3.16. *An element $x \in X$ belongs to the interpretation of $\sum_{i \in I} r_i a_i \geq k$ under a $\mathcal{P}(X)$-valuation σ iff*

$$\sum_{i \in I} r_i \mathbb{1}_{\sigma(a_i)}(x) \geq k,$$

where $\mathbb{1}_A : X \to \{0, 1\}$ is the characteristic function of $A \subseteq X$.

In all examples, the resolution process, applied to known one-step complete rule sets, can be kept under control; by Theorem 3.15, the resulting rule sets are strictly one-step complete.

Example 3.17. 1. The empty set of rules is one-step complete for neighbourhood frame semantics (Example 2.5.2). This set is trivially resolution closed.

2. (Monotone modal logic) The one-step rule

$$(M) \quad \frac{a \to b}{\Box a \to \Box b}$$

is one-step complete for monotone neighbourhood frame semantics (Example 2.5.2), and clearly resolution closed.

3. (Standard modal logic K) The one-step rules

$$\frac{a}{\Box a} \qquad \frac{a \wedge b \to c}{\Box a \wedge \Box b \to \Box c}$$

are one-step complete for Kripke semantics (Example 2.5.1), i.e. for the modal logic K [21]. The resolution closure $\mathcal{R}$ of these rules consists of the rules

$$\frac{\bigwedge_{i=1}^n a_i \to b}{\bigwedge_{i=1}^n \Box a_i \to \Box b}$$

for all $n \in \mathbb{N}$ (here, strict one-step completeness is also easily seen directly).

4. (Coalition logic) In Lemma 6.1 of [23], the following set of one-step rules for coalition logic (Example 2.5.7), numbered as in loc. cit., is implicit:

$$(1) \;\; \frac{\bigvee_{i=1}^n \neg a_i}{\bigvee_{i=1}^n \neg [C_i] a_i} \quad (2) \;\; \frac{a}{[C]a} \quad (3) \;\; \frac{a \vee b}{[0]a \vee [N]b}$$

$$(4) \;\; \frac{\bigwedge_{i=1}^n a_i \to b}{\bigwedge_{i=1}^n [C_i] a_i \to [\bigcup C_i] b}$$

where $n \geq 0$, and rules (1) and (4) are subject to the side condition that the C_i are pairwise disjoint. This set of rules extends the axiomatisation of coalition logic, which one easily proves to be one-step complete given the results of [23]. The rules are moreover 'nearly' resolution closed (full resolution closure is not needed in [23] due to a slightly different notion of closed rule sets). Resolving rule (4) with rules (2) and (3), one obtains the rule schema

$$(4') \;\; \frac{\bigwedge_{i=1}^n a_i \to b \vee \bigvee_{j=1}^m c_j}{\bigwedge_{i=1}^n [C_i] a_i \to [D] b \vee \bigvee_{j=1}^m [N] c_j}$$

where $m, n \geq 0$, subject to the side condition that the C_i are pairwise disjoint subsets of D; this subsumes rules (2)–(4) above. The set consisting of the rules (1) and (4') is easily seen to be resolution closed.

5. (Graded modal logic) Using Proposition 3.10, one shows directly that the one-step rules

$$(W)\ \Diamond_{k+1} a \to \Diamond_k a \qquad (A_1)\ \frac{b \to \bigvee_{i=1}^m a_i}{\Diamond_{\sum_{i=1}^m k_i} b \to \bigvee_{i=1}^m \Diamond_{k_i} a_i}$$

$$(A_2)\ \frac{\bigwedge_{\substack{1\leq i,j\leq n\\ i\neq j}} (\neg b_i \vee \neg b_j) \quad \bigwedge_{j=1}^n (b_j \to a)}{\bigwedge_{j=1}^n \Diamond_{k_j} b_j \to \Diamond_k a}\ {\scriptstyle (\sum_{j=1}^n (k_j+1) = k+1)},$$

where $m \geq 0$, $n \geq 1$, are one-step complete for GML (Example 2.5.5). All these rules are subsumed by the rule schema

$$(G)\ \frac{\sum_{i=1}^m a_i - \sum_{j=1}^n b_j \geq 0}{\bigwedge_{j=1}^n \Diamond_{l_j} b_j \to \bigvee_{i=1}^m \Diamond_{k_i} a_i},$$

where $n, m \geq 0$ and $n + m \geq 1$, subject to the side condition $\sum_{j=1}^n (l_i + 1) \geq 1 + \sum_{i=1}^m k_i$. Soundness of this rule is seen analogously as for similar rules in probabilistic modal logic [13]. It is easy to see that the rule schema is closed under resolution.

6. (Probabilistic modal logic) By reformulating the one-step complete set of axioms for probabilistic modal logic given in [8] as one-step rules and subsequently applying resolution, one obtains the rules

$$(P_k)\ \frac{\sum_{i=1}^m a_i - \sum_{j=1}^n b_j \geq k}{\bigwedge_{j=1}^n L_{q_j} b_j \to \bigvee_{i=1}^m L_{p_i} a_i},$$

where $m, n \geq 0$, $m + n \geq 1$, and $k \in \mathbb{Z}$, subject to the side condition

$$\textstyle\sum_{i=1}^m p_i - \sum_{j=1}^n q_j \leq k, \text{ and}$$
$$\text{if } m = 0 \text{ then } -\textstyle\sum_{j=1}^n q_j < k.$$

This rule schema subsumes the axiomatisation in loc. cit. and hence is one-step complete. Using Lemma 3.16, one can show directly that (P_k) is one-step sound in the same way as for the axiomatisation in [8]. Moreover, it is easy to see that the rule schema is resolution closed: the required resolvent of an instance of (P_k) and an instance of (P_l) is obtained as an instance of (P_{k+l}).

4. The Shallow Model Construction

We now present the announced generic shallow model construction, which is based on strictly one-step complete axiomatisations. The construction is performed along with the proof of a recursive characterisation of satisfiable formulae which generalises results from [34] (where the use of axiomatisations is implicit in certain lemmas).

Definition 4.1. A set Σ of formulae is called *closed* if it is closed under subformulae and under *normalised negation* $\sim$, where $\sim\phi$ is defined to be ψ in case ϕ is of the form $\neg\psi$, and $\neg\phi$ otherwise. The smallest closed set containing a given formula ϕ is denoted $\Sigma(\phi)$. A subset H of Σ is called a Σ-*Hintikka set* if $\bot \notin H$ and, for $\phi \wedge \psi \in \Sigma$, $\phi \wedge \psi \in H$ iff $\phi, \psi \in H$, and, for $\neg\phi \in \Sigma$, $\neg\phi \in H$ iff $\phi \notin H$.

For a formula $\chi \in \mathsf{Prop}(V)$ and a Σ-substitution σ, we define satisfaction of $\chi\sigma$ in H ($H \models \chi\sigma$) inductively by

$$\begin{array}{lll} H \models (\chi_1 \wedge \chi_2)\sigma & :\iff & H \models \chi_1\sigma \text{ and } H \models \chi_2\sigma \\ H \models (\neg\chi)\sigma & :\iff & H \not\models \chi\sigma \\ H \models a\sigma & :\iff & \sigma(a) \in H \\ H \not\models \bot. & & \end{array}$$

This is well-defined because H is Hintikka.

Lemma 4.2. *Let Σ be closed, let H be a Σ-Hintikka set, and let $\phi, \psi \in \mathsf{Prop}(V)$. Then $H \models \phi \vee \psi$ iff $H \models \phi$ or $H \models \psi$.*

Lemma 4.3 (Soundness of propositional reasoning for Hintikka sets)**.** *Let Σ be closed, and let H be a Σ-Hintikka set. Let $\phi, \psi \in \mathsf{Prop}(V)$, and let σ be a Σ-substitution. If $H \models \phi\sigma$ and ϕ propositionally entails ψ, then $H \models \psi\sigma$.*

The following result generalises Propositions 3.2, 3.5, 3.8, 3.13, and 3.16 (but not 3.10 and 3.18, which concern logics outside rank 1) of [34] and Lemma 6.1 of [23].

Theorem 4.4. *Let $\mathcal{R}$ be strictly one-step complete. Then $\phi \in \mathcal{L}(\Lambda)$ is satisfiable iff $\phi \in H$ for some Hintikka set $H \subset \Sigma(\phi)$ such that, for every clause $\rho = \bigvee_{i=1}^n \epsilon_i[\lambda_i]\rho_i$ over $\Sigma(\phi)$ with $H \not\models \rho$ and for each rule $\psi / \bigvee_{i=1}^n (\epsilon_i[\lambda_i]a_i)$ in $\mathcal{R}_C$, the formula $\neg\psi[\rho_i/a_i]_{i=1,\dots,n}$ is satisfiable.*

Proof. 'Only if': Take H to be the intersection of $\Sigma(\phi)$ with the theory of a state satisfying ϕ.

'If': For each formula $\chi \equiv \neg\psi[\rho_i/a_i]_{i=1,\dots,n}$ as in the statement, there exists a coalgebra $C_\chi = (X_\chi, \xi_\chi)$ and a state x_χ in C_χ such that $x_\chi \models_{C_\chi} \chi$; we can assume that the X_χ are pairwise disjoint. Define the sets X and $\hat\rho$ by

$$X = \{x_0\} \cup \bigcup_\chi X_\chi \quad \text{and} \quad \hat\rho = A_\rho \cup \bigcup_\chi [\![\rho]\!]_{C_\chi},$$

where x_0 is a fresh element, $\rho \in \Sigma(\phi)$, and $A_\rho = \{x_0\}$ if $\rho \in H$, $A_\rho = \emptyset$ otherwise. We define a coalgebra structure ξ on X as follows. For $x \in X_\chi$, we put $\xi(x) = \xi_\chi(x) \in TX_\chi \subset TX$ (cf. Assumption 2.2). Then for $[\lambda]\rho \in \Sigma(\phi)$,

$$\xi(x) \in \lambda\hat\rho \iff x \models_{C_\chi} [\lambda]\rho, \tag{1}$$

because by naturality $(\lambda\hat\rho) \cap TX_\chi = \lambda(\hat\rho \cap X_\chi) = \lambda[\![\rho]\!]_{C_\chi}$. Moreover, we will show that there exists $\xi(x_0) \in TY \subset TX$, where Y is the set of all x_χ, such that for $[\lambda]\rho \in \Sigma(\phi)$,

$$\xi(x_0) \in \lambda\hat\rho \iff [\lambda]\rho \in H. \tag{2}$$

By structural induction, (1) and (2) then imply

$$x \models_C \rho \iff x \models_{C_\chi} \rho \text{ for } x \in X_\chi, \text{ and}$$
$$x_0 \models_C \rho \iff \rho \in H$$

for all $\rho \in \Sigma(\phi)$. In particular, $x_0 \models \phi$, and we are done.

It remains to prove that $\xi(x_0)$ satisfying (2) exists. Assume the contrary. Let V be the set of propositional variables b_ρ, where $[\lambda]\rho \in \Sigma(\phi)$ for some λ. Let $\theta \in \mathsf{Cl}(\mathsf{Up}(V))$ consist of the literals $\neg[\lambda]b_\rho$ for $[\lambda]\rho \in H$ and $[\lambda]b_\rho$ for $\neg[\lambda]\rho \in H$. By assumption, $TY \models \theta\tau^Y$, where τ^Y is the $\mathcal{P}(Y)$-valuation taking b_ρ to $\hat{\rho} \cap Y = \{x_\chi \mid x_\chi \models_{C_\chi} \rho\}$. By Lemma 3.8, it follows that θ is strictly congruence derivable from those $\zeta \in \mathsf{Prop}(V)$ such that $Y \models \zeta\tau^Y$.

From the derivation of θ, it now follows that $H \models \theta\sigma$, where σ is the $\Sigma(\phi)$-substitution taking b_ρ to ρ (note that $\theta\sigma$ is a propositional formula over atoms $[\lambda]\rho \in \Sigma(\phi)$), by Lemma 4.2 a contradiction to the construction of θ: by Lemma 4.3, the propositional steps are sound over H; it remains to be shown that if the derivation of θ uses a rule $R \equiv \psi/\bigvee_{i=1}^n(\epsilon_i[\lambda_i]a_i)$ in $\mathcal{R}_C$, instantiated for a V-substitution η, then the conclusion of $R\eta\sigma$ is satisfied over H. Assume the contrary. By Lemma 4.2, it follows that $\epsilon_i[\lambda_i]\sigma(\eta(a_i)) \notin H$ for all i. By construction, we have $x_\chi \models_{C_\chi} \chi$ for $\chi \equiv \neg\psi\eta\sigma$. But since $R\eta$ appears in the derivation of θ, $Y \models \psi\eta\tau^Y$ and hence $x_\chi \in [\![\psi\eta\tau^\chi]\!]$, where τ^χ is the $\mathcal{P}(X_\chi)$-valuation taking b_ρ to $[\![\rho]\!]_{C_\chi}$. Since $[\![\psi\eta\tau^\chi]\!] = [\![\psi\eta\sigma]\!]_{C_\chi}$, we have arrived at a contradiction. ☐

As a corollary to the above proof, we obtain that coalgebraic modal logic has the shallow model property. The formulation of this property requires the following notion.

Definition 4.5. A *supporting Kripke frame* of a T-coalgebra (X, ξ) is a Kripke frame (X, R) such that for each $x \in X$,

$$\xi(x) \in T\{y \mid xRy\} \subset TX.$$

As clauses suffice for satisfiability checking, we obtain

Corollary 4.6 (Shallow model property)**.** *Every satisfiable $\mathcal{L}(\Lambda)$-formula ϕ is satisfiable in a* shallow model*, i.e. in a T-coalgebra (X, ξ) that has a supporting Kripke frame which consists of a tree of depth at most the rank of ϕ and of branching degree at most 2^n, where n is the number of subformulae of ϕ, and possibly an additional* final *state $x_\top$, i.e. for all x, $xRx_\top$, and $x_\top Rx$ implies $x = x_\top$.*

(The state $x_\top$ may arise from the rule $\bot/\bot$, cf. Sect. 3.)

5. A Generic PSPACE Algorithm

The shallow model result (Theorem 4.4) will be exploited to design a decision procedure in the spirit of [34]. Since resolution closed rule sets are in general infinite, this requires ensuring that we never need to instantiate literals in the conclusions of rules with identical formulae: otherwise, an infinite number of rules could match a single given clause over a Hintikka set. This is formally captured as follows.

Definition 5.1. We call a clause over L *reduced* if all its literals are distinct. An L-instance $\phi\sigma/\psi\sigma$ of a rule $\phi/\psi \in \mathcal{R}$ is *reduced* if the clause $\psi\sigma$ is reduced. Finally, $\mathcal{R}$ is *closed under reduction* if for every V-instance $\phi\sigma/\psi\sigma$ of a rule ϕ/ψ over V in $\mathcal{R}$, there exists a reduced V-instance $\phi'\sigma'/\psi'\sigma'$ of a rule $\phi'/\psi' \in \mathcal{R}$ such that $\psi\sigma$ and $\psi'\sigma'$ are propositionally equivalent and and $\phi'\sigma'$ is propositionally entailed by $\phi\sigma$.

I.e. a rule set is reduced if every instance of a rule that duplicates literals in the conclusion can be replaced by a reduced instance of a different rule. Not all the rule sets discussed in Example 3.17 satisfy this property, but they can easily be extended to reduction closed sets: just add a rule ϕ'/ψ' for every rule ϕ/ψ over V in $\mathcal{R}$ and every V-substitution σ, where ϕ' is some suitably chosen propositional equivalent of $\phi\sigma$ and ψ' is obtained from $\psi\sigma$ by removing duplicate literals. It is clear that the new rules remain one-step sound. Note that there is no need to preserve closure under resolution when passing to a reduced rule set, as Theorem 4.4 requires only strict one-step completeness, which is preserved under extending the rule set.

Example 5.2. 1. The strictly one-step complete rule sets of Examples 3.17.1–4 (including monotone modal logic, K, and coalition logic) are easily seen to be closed under reduction, essentially because in all relevant rule schemas, the premise is a clause of the same general format as the conclusion.

2. (Graded modal logic) The rule schema (G) of Example 3.17.5 fails to be closed under reduction, as duplicating literals in the conclusion substantially affects both the premise and the side condition. We can close (G) under reduction as described above; this results in the rule schema

$$(G') \ \frac{\sum_{i=1}^n r_i a_i \geq 0}{\bigvee_{i=1}^n sgn(r_i)\Diamond_{k_i} a_i},$$

where $n \geq 1$ and $r_1, \dots, r_n \in \mathbb{Z} - \{0\}$, subject to the side condition $\sum_{r_i<0} r_i(k_i + 1) \geq 1 + \sum_{r_i>0} r_i k_i$.

3. (Probabilistic modal logic) The rule schema (P_k) of Example 3.17.6 fails to be closed under reduction. Closure under reduction as described above leads to the rule schema

$$(P_k') \ \frac{\sum_{i=1}^n r_i a_i \geq k}{\bigvee_{1\leq i\leq n} sgn(r_i) L_{p_i} a_i}$$

where $n \geq 1$ and $r_1, \dots, r_n \in \mathbb{Z} - \{0\}$, subject to the side condition

$$\sum_{i=1}^n r_i p_i \leq k, \text{ and}$$
$$\text{if } \forall i.\, r_i < 0 \text{ then } \sum_{i=1}^n r_i p_i < k.$$

As instances of the congruence rule never contain duplicate literals, we have the following trivial fact.

Lemma 5.3. *If $\mathcal{R}$ is closed under reduction, then so is $\mathcal{R}_C$.*

Thus the following is immediate from Theorem 4.4.

Corollary 5.4. *Let $\mathcal{R}$ be strictly one-step complete and closed under reduction. Then $\phi \in \mathcal{L}(\Lambda)$ is satisfiable iff $\phi \in H$ for some Hintikka set $H \subset \Sigma(\phi)$ such that, for every* reduced *clause $\rho = \bigvee_{i=1}^n \epsilon_i[\lambda_i]\rho_i$ over $\Sigma(\phi)$ with $H \not\models \rho$ and for each rule $\psi / \bigvee_{i=1}^n (\epsilon_i[\lambda_i]a_i)$ in $\mathcal{R}_C$, the formula $\neg\psi[\rho_i/a_i]_{i=1,\dots,n}$ is satisfiable.*

In the implementation of the algorithm suggested by Corollary 5.4, we need to pass around matches of rules with given clauses. Since rules, in particular their premises, are generally too large to pass around directly, we assume that every rule (i.e. every instance of a rule scheme) is given by a *code*, i.e. a string over some alphabet which identifies the rule; when rules appear as data, they are always represented by their code. Moreover, we assume that propositional variables a_i in rules are uniformly represented by indices that point to literals $\epsilon_i[\lambda_i]a_i$ of the conclusion.

Definition 5.5. We say that a rule $R \in \mathcal{R}$ *matches* a reduced clause $\rho \equiv \bigvee_{i=1}^n \epsilon_i[\lambda_i]\phi_i$ if the conclusion of R is of the form $\bigvee_{i=1}^n \epsilon_i[\lambda_i]a_i$. By the above variable convention, the instantiation $\psi[\phi_i/a_i]_{i=1,\dots,n}$ of a conjunct ψ of the premise of R can be computed in polynomial time from ψ and ρ; we denote the result by $\psi[\rho]$. Two matching rules are *equivalent* if their premises are propositionally equivalent; equivalence classes $[R]$ are called *$\mathcal{R}$-matchings*. The code of R is also a *code* for $[R]$.

We fix some size measures for complexity purposes:

Definition 5.6. The size $size(a)$ of an integer a is $\lceil \log_2(|a|+1) \rceil$, where $\lceil r \rceil = \min\{z \in \mathbb{Z} \mid z \geq r\}$ as usual. The size $size(p)$ of a rational number $p = a/b$, with a, b relatively prime, is $1 + size(a) + size(b)$. The *size* $|\phi|$ of a formula ϕ over V is defined by counting 1 for each propositional variable, boolean operator, or modal operator, and additionally the size of each index of a modal operator. (In the examples, indices are either numbers, with sizes as above, or subsets of $\{1, \dots, n\}$, assumed to be of size n.)

In particular, indices of graded or probabilistic modal operators are coded in binary.

Example 5.7. For the rules of Examples 3.17 and 5.2, we just take the parameters of a rule as its code in the obvious way. E.g. the code of an instance of (P'_k) as displayed in Example 5.2.3 consists of n, k, the r_i, and the p_i. The size of the code is determined by the sizes of these numbers plus separating letters, say, $\sum(1+size(a_i))+\sum(1+size(p_i))+size(n)+size(k)+1$. Note that not all such codes represent instances of (P'_k).

The following decision procedure on an alternating Turing machine generalises the algorithms in [34], given a strictly one-step complete and reduction closed set $\mathcal{R}$.

Algorithm 5.8. (Decide satisfiability of $\phi \in \mathcal{L}(\Lambda)$)

1. (Initialise) Construct the set $\Sigma(\phi)$.
2. (Existential) Guess a Hintikka set $H \subset \Sigma(\phi)$ with $\phi \in H$.
3. (Universal) Guess a reduced clause $\bot \neq \rho \in \Sigma(\phi)$ with $H \not\models \rho$ and an $\mathcal{R}_C$-matching $[R]$ of ρ.
4. (Existential) Guess a clause γ from the conjunctive normal form (CNF) of the premise of R and recursively check that $\neg\gamma[\rho]$ is satisfiable.

The algorithm succeeds if all possible choices at steps marked *universal* lead to successful termination, and for all steps marked *existential*, there exists a choice leading to successful termination.

Correctness of the algorithm is guaranteed by Corollary 5.4. Note that the algorithm terminates successfully in Step 3 if there are no rules matching clauses over H. In particular, the algorithm terminates either in Step 2 or in Step 3 if ϕ has rank 0. We emphasise that in Step 3, it suffices to guess one code for each matching.

The crucial requirement for the effectivity of Algorithm 5.8 is that Steps 3 and 4 can be performed in polynomial time, i.e. by suitable nondeterministic polynomial-time multivalued functions (NPMV) [5]. We recall that a function $f : \Sigma^* \to \mathcal{P}(\Delta^*)$, where Σ and Δ are alphabets, is NPMV iff

1. there exists a polynomial p such that $|y| \leq p(|x|)$ for all $y \in f(x)$, where $|\cdot|$ denotes size, and
2. the graph $\{(x, y) \mid y \in f(x)\}$ of f is in NP.

Thus, the following conditions guarantee that Algorithm 5.8 has polynomial running time:

Definition 5.9. A set $\mathcal{R}$ of rules is called *$PSPACE$-tractable* if there exists a polynomial p such that all $\mathcal{R}$-matchings of a reduced clause ρ over $\mathcal{L}(\Lambda)$ have a code of size at most $p(|\rho|)$, and it can be decided in NP

1. whether a given code is the code of some rule in $\mathcal{R}$;
2. whether a rule matches a given reduced clause; and

3. whether a clause belongs to the CNF of the premise of a given rule.

Theorem 5.10 (Space Complexity)**.** *Let $\mathcal{R}$ be strictly one-step complete, closed under reduction, and $PSPACE$-tractable. Then the satisfiability problem for $\mathcal{L}(\Lambda)$ is in $PSPACE$.*

Remark 5.11. A more careful analysis of Algorithm 5.8 reveals that it suffices for the decision problems in Definition 5.9 to be in PH, the polynomial time hierarchy. In our examples, however, the complexity is in fact P rather than NP. We expect that this situation is typical, with the crucial condition for $PSPACE$-tractability being the polynomial bound on $\mathcal{R}$-matchings. We are not aware of any natural examples of intractable rule sets (contrived examples are easy to construct, e.g. by using hard side conditions).

The next lemma, which follows directly from size estimates in linear integer programming [28], is crucial for establishing $PSPACE$-tractability in the examples. Following usual practice, we take the *size* $|W|$ of a rational inequality $W \equiv (\sum_{i=1}^{n} u_i x_i \text{ op } u_0)$, $\text{op} \in \{<, \leq, >, \geq\}$, to be $1 + n + \sum_{i=0}^{n} size(u_i)$.

Lemma 5.12. *There exists a polynomial p such that for every rational linear inequality W and every solution $r_0, \ldots, r_n \in \mathbb{Z}$ of W, there exists a solution $r'_0, \ldots, r'_n \in \mathbb{Z}$ of W such that $size(r'_i) \leq p(|W|)$ for all i, and the formulae $\sum_{i=1}^{n} r_i a_i \geq r_0$ and $\sum_{i=1}^{n} r'_i a_i \geq r'_0$ are propositionally equivalent.*

We now illustrate how Theorem 5.10 allows us to establish $PSPACE$ bounds for many modal logics in a uniform way. In particular, we obtain a new (tight) $PSPACE$ bound for probabilistic modal logic.

Example 5.13. Conditions 1 and 2 of Definition 5.9 are immediate for all the rule sets of Example 3.17 — the decision problems in question involve no more than checking computationally harmless side conditions in the case of Condition 1 (disjointness and containment of finite sets, linear inequalities), and comparing clauses of polynomial (in fact, linear) size in the case of Condition 2. Moreover, Condition 3 is immediate in those cases where the premises of rules are just single clauses. This leaves only GML and PML; but the definition of $\sum_{i \in I} r_i a_i \geq k$ is already in CNF, and checking whether a given clause belongs to this CNF is clearly in P.

It remains to establish the polynomial bound on the matchings. For GML and PML, this is guaranteed by Lemma 5.12. In all other cases, every reduced clause ρ matches at most one rule, whose code has size linear in the size of ρ.

We thus have obtained $PSPACE$-tractability and hence decidability in $PSPACE$ for all logics in Example 3.17. The logic of neighbourhood frames and monotone modal logic are of lesser interest here, as the corresponding modal logics are in NP [34]. We briefly comment on the algorithms and bounds for the other cases.

1. For the modal logic K (Example 3.17.3), Algorithm 5.8 is essentially the witness algorithm [34, 4], with reduced clauses violated by H corresponding to *demands*.
2. For coalition logic (Example 3.17.4), we arrive, due to minor differences of the rule sets, at a slight variant of Pauly's $PSPACE$-algorithm [23].
3. For graded modal logic, we obtain a new algorithm which confirms the known $PSPACE$ bound [31]. One might claim that the new algorithm is not only nicely embedded into a unified framework, but also conceptually simpler than the constraint-based algorithm of [31] (which corrects an erroneous algorithm previously given elsewhere).
4. For probabilistic modal logic, we obtain a new algorithm which yields a previously unknown $PSPACE$-bound (to our knowledge, the best previously published bound for PML is $EXPTIME$ [30]). The bound is tight, as PML contains the $PSPACE$-complete logic KD as a fragment (embedded by mapping $\Box$ to L_1).

6. Conclusion

Generalising results of [34], we have shown that coalgebraic modal logic has the shallow model property, and we have presented a generic $PSPACE$ algorithm for satisfiability based on depth-first exploration of shallow models. We have thus

- reproduced the *witness algorithm* for K [4]
- obtained a slight variant of the known $PSPACE$ algorithm for coalition logic [23]
- obtained a new $PSPACE$ algorithm for graded modal logic, recovering the known $PSPACE$ bound [31]
- obtained a novel $PSPACE$ bound for probabilistic modal logic [17, 13].

In all these cases, the bound obtained is tight.

The crucial prerequisite for the generic algorithm is an axiomatisation by so-called one-step rules (going from rank 0 to rank 1) obeying two closedness conditions: closedness under resolution and under removal of duplicate literals. In the examples, it has not only turned out that is it feasible to keep this closure process under control, but also that the axiomatisations obtained have pleasingly compact presentations — often, one ends up with a single rule schema. Nevertheless, it remains desirable to prove a $PSPACE$ bound relying on purely semantic conditions such as the ones appearing in [30]; this is the subject of further research, as is the extension of the theory beyond rank 1 by means of comonads and the treatment of iteration, possibly using automata theoretic methods [35, 36].

Ongoing work indicates that every modal logic can be equipped with a coalgebraic semantics, provided it is axiomatisable in rank 1 and satisfies the congruence rule. This means in particular that the method employed here applies to every such modal logic, i.e. one obtains a purely syntactic criterion (tractability of a certain closure of the axiomatisation) for rank-1 modal logics to be in $PSPACE$.

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