Sequent Systems for Lewis' Conditional Logics*

Björn Lellmann and Dirk Pattinson

Department of Computing, Imperial College London, UK

Abstract. We present unlabelled cut-free sequent calculi for Lewis' conditional logic $\mathbb{V}$ and extensions, in both the languages with the entrenchment connective and the strong conditional. The calculi give rise to PSPACE-decision procedures, also in the language with the weak conditional. Furthermore, they are used to prove the Craig interpolation property for all the logics under consideration, and yield a PSPACE-decision procedure for a recently considered hybrid version of $\mathbb{V}$.

1 Introduction

Although the use of conditional logics in artificial intelligence and automated reasoning has a long tradition (e.g. [6]), there has been slow progress in the development of proof systems for these logics. Even today, we still see conditional logics for which no proof systems of optimal complexity have been found. In general, the development of proof systems follows two main approaches: one can derive labelled tableau systems from the semantics [14, 8] or convert a Hilbert-style axiomatisation to an unlabelled sequent system which is then saturated to guarantee cut-elimination [15, 17, 10].

Although proof systems for some of the more prominent logics have been developed quite early on [2, 9, 4, 1], the systematic exploration of systems with optimal complexity has attracted interest only recently. In particular, there are no complexity-optimal proof systems for an important class of logics, those that are interpreted over *sphere models*, originally proposed by Lewis [11]. These logics can be characterised using different connectives: the *entrenchment connective* $\preccurlyeq$, the *strong conditional* $\Box\!\!\Rightarrow$, and the *weak conditional* $\Box\!\!\rightarrow$. While these connectives are interdefinable, the translations in general yield an exponential blow-up, and thus complexity results do not necessarily carry over.

Although the logics in the weak conditional language have long been known to be decidable in polynomial space [3], the best proof systems for this language so far only yield a coNEXPTIME upper bound [8]. For the entrenchment connective, there are systems for the logics $\mathbb{VC}$ and $\mathbb{VCS}$, which implicitly yield a PSPACE upper bound [2, 4], but no systematic treatment has been given yet, a gap that this paper now closes.

Our main results are the following: we present complexity-optimal unlabelled sequent calculi for the logics $\mathbb{V}$, $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$ and $\mathbb{VC}$ in the entrenchment and strong conditional language. With the exception of the calculus for $\mathbb{VC}$ in the entrenchment language these seem to be new. Cut elimination for our calculi follows from the more general approach of *cut elimination by saturation*, and yields purely syntactical decision procedures of optimal PSPACE complexity. A PSPACE decision procedure for the logics in the weak conditional language is established by means of translation. As an application,

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we establish the Craig interpolation property for all logics considered (in any connective), which we believe is also a new result. Our second application yields a previously unknown PSPACE result for a hybrid version of $\mathbb{V}_{\Box\!\Rightarrow}$ recently considered in [16].

2 Preliminaries

For $n \in \{0, 1, 2, \dots\}$ we write $[n]$ for $\{1, \dots, n\}$. Throughout, $\mathcal{V}$ denotes a denumerable set of propositional variables, written $p, q, \dots$ and we use bold face $\boldsymbol{p}, \boldsymbol{q}, \dots$ to denote finite sequences of propositional variables. We fix a set Λ of modal operators with associated arities (later, we will specialise Λ to consist of just one binary conditional operator). The set of Λ-*formulae* is defined by $\mathcal{F}(\Lambda) \ni A, B, A_1, \dots, A_n ::= \bot \mid p \mid A \wedge B \mid A \vee B \mid A \to B \mid \heartsuit(A_1, \dots, A_n)$ for $p \in \mathcal{V}$ and $\heartsuit \in \Lambda$ with arity n, with remaining connectives defined as usual. We write $\Lambda(S) = \{\heartsuit(A_1, \dots, A_n) \mid \heartsuit \in \Lambda \text{ } n\text{-ary}, A_1, \dots, A_n \in S\}$ for the set of formulae constructed from S using a single connective in Λ and $\mathsf{var}(A)$ for the set of propositional variables occurring in the formula A. Uniform substitution of all propositional variables in a formula A using a substitution $\sigma : \mathcal{V} \to \mathcal{F}(\Lambda)$ is denoted by $A\sigma$. A Λ-*logic*, or just *logic* is a set $\mathcal{L} \subseteq \mathcal{F}(\Lambda)$ that contains all propositional tautologies and is closed under uniform substitution, modus ponens and the congruence rule: from $A_i \leftrightarrow B_i$ for $i = 1, \dots, n$ infer $\heartsuit(A_1, \dots, A_n) \leftrightarrow \heartsuit(B_1, \dots, B_n)$ for every n-ary modality $\heartsuit \in \Lambda$. We think of logics as given semantically as the set of universally valid formulae on some class of models and write $\models_{\mathcal{L}} A$ for $A \in \mathcal{L}$. The set $\mathcal{S}(F)$ of *sequents* over F consists of tuples of multisets Γ, Δ of formulae in F, written $\Gamma \Rightarrow \Delta$. The multiset union of two multisets Γ and Δ is written Γ, Δ and we identify formulae with singleton multisets. Substitution extends to both multisets of formulae and sequents in the obvious way (perserving multiplicity), e.g. $(A_1, A_2 \Rightarrow B)\sigma = A_1\sigma, A_2\sigma \Rightarrow B\sigma$. We use the system $\mathsf{G3}cp$ of [18] with axioms $\Gamma, A \Rightarrow \Delta, A$ (where A ranges over the set of formulae) as basis for all systems that extend classical propositional logic and denote its proof rules by G. We adopt the standard structural rules

$$\frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta, \Pi}\ \mathsf{W}, \quad \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathsf{ConL}, \quad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \mathsf{ConR}, \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Sigma, A \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{Cut}$$

and write $\models_{\mathcal{L}} \Gamma \Rightarrow \Delta$ if $\mathcal{L}$ is a logic and $\models_{\mathcal{L}} \bigvee \Gamma \to \bigwedge \Delta$.

3 Conditional Logics: Calculi and Main Results

We consider the conditional logics $\mathbb{V}, \mathbb{VN}, \mathbb{VT}, \mathbb{VW}$ and $\mathbb{VC}$ [11, 13] in the languages of (binary) *entrenchment* $\preccurlyeq$ and (binary) *weak* and *strong conditionals* $\Box\!\!\rightarrow$ and $\Box\!\!\Rightarrow$. We read entrenchment $A \preccurlyeq B$ as 'A is at least as plausible as B' and adopt Lewis' *sphere semantics*: a *sphere model* is a triple $\mathcal{I} = (I, (\$_i)_{i \in I}, \pi)$ where I is a set (of worlds), each $\$_i \subseteq \mathcal{P}(I)$ is a *system of spheres*, i.e. a family of nested subsets of I closed under unions and nonempty intersections, and $\pi : \mathcal{V} \to \mathcal{P}(I)$ is a valuation. We think of $\$_i$ as providing a measure of comparative similarity, which provides the truth condition

$$\mathcal{I}, i \models A \preccurlyeq B \iff \text{for all spheres } S \in \$_i \ (\, S \cap [\![B]\!] \neq \emptyset \text{ only if } S \cap [\![A]\!] \neq \emptyset\,)$$

(CP)	$\dfrac{\vdash B \to (A_1 \vee \cdots \vee A_n)}{\vdash (A_1 \preccurlyeq B) \vee \cdots \vee (A_n \preccurlyeq B)} \quad (n \geq 1)$	(N)	$\neg(\bot \preccurlyeq \top)$
(TR)	$((A \preccurlyeq B) \wedge (B \preccurlyeq C)) \to (A \preccurlyeq C)$	(T)	$(\bot \preccurlyeq \neg A) \to A$
(CN)	$(A \preccurlyeq B) \vee (B \preccurlyeq A)$	(W)	$((\bot \preccurlyeq \neg A) \vee \neg(\neg A \preccurlyeq \top)) \to A$
		(C)	$((A \preccurlyeq \top) \wedge (\top \preccurlyeq A)) \to A$

$\mathcal{HV}_{\preccurlyeq}$: CP, TR, CN $\quad \mathcal{H}\mathbb{VN}_{\preccurlyeq} : \mathcal{HV}, N \quad \mathcal{H}\mathbb{VT}_{\preccurlyeq} : \mathcal{HV}, T \quad \mathcal{H}\mathbb{VW}_{\preccurlyeq} : \mathcal{HV}, W \quad \mathcal{H}\mathbb{VC}_{\preccurlyeq} : \mathcal{HV}, C$

Fig. 1. Hilbert axiomatisation of the V-logics as smallest logics closed under rules/axioms

where $[\![A]\!] = \{i \in I \mid i \models A\}$ is the truth set of a formula A, together with the standard clauses for propositional variables and boolean connectives. The *strong conditional operator* $\Box\!\!\Rightarrow$ can be defined in terms of entrenchment by $(A \Box\!\!\Rightarrow B) \leftrightarrow \neg((A \wedge \neg B) \preccurlyeq (A \wedge B))$. Over a sphere model, $A \Box\!\!\Rightarrow B$ asserts that $A \wedge B$ is more possible or plausible than $A \wedge \neg B$. This leads to the interpretation

$$\mathcal{I}, i \models A \Box\!\!\Rightarrow B \iff \text{for some sphere } S \in \$_i \; (\, S \cap [\![A]\!] \neq \emptyset \text{ but } S \cap [\![A \wedge \neg B]\!] = \emptyset \,)\;.$$

Similarly, the weak conditional $\Box\!\!\rightarrow$ can be expressed in terms of entrenchment by $(A \Box\!\!\rightarrow B) \leftrightarrow ((\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \wedge B)))$ where the only difference is that a weak conditional $A \Box\!\!\rightarrow B$ is also accepted if the conditional antecedent A is considered impossible, i.e. false in every sphere for the current world.

If A is a formula and C is a class of sphere models, then A is *universally valid* on C if $\mathcal{I}, i \models A$ for all $\mathcal{I} = (I, (\$_i)_{i \in I}, \pi) \in C$ and all $i \in I$. We write $\mathbb{V}_*$ for the logic of all sphere models, i.e. the set of all formulae that are universally valid in all sphere models in the language of the binary connective $* \in \{\preccurlyeq, \Box\!\!\Rightarrow, \Box\!\!\rightarrow\}$. We consider the following extensions [11, page 120] of $\mathbb{V}_*$ determined by the following additional conditions on sphere models $\mathcal{I} = (I, (\$_i)_{i \in I}, \pi)$, understood as universally quantified over all $i \in I$:

- The logic $\mathbb{VN}_*$ is determined by all *normal* sphere models, i.e. those with $\bigcup \$_i \neq \emptyset$
- The logic $\mathbb{VT}_*$ is determined by all *totally reflexive* sphere models, i.e. $i \in \bigcup \$_i$
- The logic $\mathbb{VW}_*$ is the logic of all *weakly centered* sphere models, i.e. those for which there is $S \in \$_i$ with $S \neq \emptyset$ and $i \in S'$ whenever $\emptyset \neq S' \in \$_i$
- The logic $\mathbb{VC}_*$ is the logic of all *centered* sphere models, i.e. those with $\{i\} \in \$_i$.

Those logics are known [11, pages 124–130] to enjoy a sound and complete axiomatisation in a Hilbert calculus with rules and axioms summarised in Figure 1. By reducing the decision problem for standard modal logics K, D, T to the decision problems for the corresponding conditional logics [11, p.137] using the translations $\Diamond A \leftrightarrow \neg(\bot \preccurlyeq A)$ and $\Diamond A \leftrightarrow (A \Box\!\!\Rightarrow \top)$ and $\Diamond A \leftrightarrow \neg(A \Box\!\!\rightarrow \bot)$ all the logics are easily seen to be Pspace-hard. Our main contribution are new, cut-free sequent calculi for the logics above that enable backwards proof search in polynomial space. Our calculi contain the standard rules for the propositional connectives together with the rules summarised in Figure 2. Intuitively, rules $R_{1,2}$ and $R_{2,0}$ guarantee derivability of the axioms (TR) and (CN), while the rules $R_{n,0}$ cover the rules of (CP). The remaining rules of $\mathcal{R}_{\mathbb{V}_{\preccurlyeq}}$ are needed to guarantee saturation (see Section 5), and additional rules for the other logics correspond to additional axioms. The rule sets for $\Box\!\!\Rightarrow$ are constructed by translation.

$$\frac{\{\, B_k \Rightarrow A_1, \dots, A_n, D_1, \dots, D_m \mid k \leq n \,\} \;\cup\; \{\, C_k \Rightarrow A_1, \dots, A_n, D_1, \dots, D_{k-1} \mid k \leq m \,\}}{\Gamma, (C_1 \preccurlyeq D_1), \dots, (C_m \preccurlyeq D_m) \Rightarrow \Delta, (A_1 \preccurlyeq B_1), \dots, (A_n \preccurlyeq B_n)}\; R_{n,m}$$

$$\frac{A \Rightarrow \qquad \Rightarrow B}{\Gamma, (A \preccurlyeq B) \Rightarrow \Delta}\; R_N \qquad\qquad \frac{A \Rightarrow \qquad \Gamma \Rightarrow \Delta, B}{\Gamma, (A \preccurlyeq B) \Rightarrow \Delta}\; R_T$$

$$\frac{\{\, C_k \Rightarrow A_1, \dots, A_n, D_1, \dots, D_{k-1} \mid k \leq m \,\} \qquad \Gamma \Rightarrow \Delta, A_1, \dots, A_n, D_1, \dots, D_m}{\Gamma, (C_1 \preccurlyeq D_1), \dots, (C_m \preccurlyeq D_m) \Rightarrow \Delta, (A_1 \preccurlyeq B_1), \dots, (A_n \preccurlyeq B_n)}\; W_{n,m}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, (A \preccurlyeq B)}\; R_{C1} \qquad\qquad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, B}{\Gamma, (A \preccurlyeq B) \Rightarrow \Delta}\; R_{C2}$$

$$\frac{\begin{array}{c}\{\, C_k, \{B_i \mid i \in I\} \Rightarrow \{A_i \mid i \notin I\}, \{C_j \mid j \in J\}, \{D_j \mid k > j \notin J\} \mid k \leq m,\ I \subseteq [n],\ J \subseteq [k-1] \,\} \\ \cup\, \{\, A_k, B_k, \{\, B_i \mid i \in I \,\} \Rightarrow \{\, A_i \mid i \notin I \,\}, \{C_j \mid j \in J\}, \{D_j \mid j \notin J\} \mid k \leq n, I \subseteq [n], J \subseteq [m] \,\}\end{array}}{\Gamma, (A_1 \mathrel{\Box\!\!\Rightarrow} B_1), \dots, (A_n \mathrel{\Box\!\!\Rightarrow} B_n) \Rightarrow \Delta, (C_1 \mathrel{\Box\!\!\Rightarrow} D_1), \dots, (C_m \mathrel{\Box\!\!\Rightarrow} D_m)}\; R'_{n,m}$$

$$\frac{\Rightarrow A \qquad \Rightarrow B}{\Gamma \Rightarrow \Delta, (A \mathrel{\Box\!\!\Rightarrow} B)}\; R'_N \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A \qquad A \Rightarrow B}{\Gamma \Rightarrow \Delta, (A \mathrel{\Box\!\!\Rightarrow} B)}\; R'_T$$

$$\frac{\begin{array}{c}\{\, C_k, \{B_i \mid i \in I\} \Rightarrow \{A_i \mid i \notin I\}, \{C_j \mid j \in J\}, \{D_j \mid k > j \notin J\} \mid k \leq m,\ I \subseteq [n],\ J \subseteq [k-1] \,\} \\ \cup\, \{\, \Gamma, \{B_i \mid i \in I\} \Rightarrow \{A_i \mid i \notin I\}, \{C_j \mid j \in J\}, \{D_j \mid j \notin J\} \mid I \subseteq [n], J \subseteq [m] \,\}\end{array}}{\Gamma, (A_1 \mathrel{\Box\!\!\Rightarrow} B_1), \dots, (A_n \mathrel{\Box\!\!\Rightarrow} B_n) \Rightarrow \Delta, (C_1 \mathrel{\Box\!\!\Rightarrow} D_1), \dots, (C_m \mathrel{\Box\!\!\Rightarrow} D_m)}\; W'_{n,m}$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, (A \mathrel{\Box\!\!\Rightarrow} B) \Rightarrow \Delta}\; R'_{C1} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, (A \mathrel{\Box\!\!\Rightarrow} B)}\; R'_{C2}$$

$$\begin{array}{ll}
\mathcal{R}_{\mathbb{V}_{\preccurlyeq}} = \{R_{n,m} \mid n \geq 1, m \geq 0\} & \mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} = \{R'_{n,m} \mid n \geq 1, m \geq 0\} \\
\mathcal{R}_{\mathbb{VN}_{\preccurlyeq}} = \mathcal{R}_{\mathbb{V}} \cup \{R_N\} & \mathcal{R}_{\mathbb{VN}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} \cup \{R'_N\} \\
\mathcal{R}_{\mathbb{VT}_{\preccurlyeq}} = \mathcal{R}_{\mathbb{V}} \cup \{R_T\} & \mathcal{R}_{\mathbb{VT}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} \cup \{R'_T\} \\
\mathcal{R}_{\mathbb{VW}_{\preccurlyeq}} = \mathcal{R}_{\mathbb{VT}} \cup \{W_{n,m} \mid n \geq 1, m \geq 0\} & \mathcal{R}_{\mathbb{VW}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{VT}_{\Box\Rightarrow}} \cup \{W'_{n,m} \mid n \geq 1, m \geq 0\} \\
\mathcal{R}_{\mathbb{VC}_{\preccurlyeq}} = \mathcal{R}_{\mathbb{V}} \cup \{R_{C1}, R_{C2}\} & \mathcal{R}_{\mathbb{VC}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} \cup \{R'_{C1}, R'_{C2}\}
\end{array}$$

Fig. 2. The rules and rule sets.

As usual, we call a formula *principal* in a rule if it appears in the conclusion of the rule but not in any premiss. A premiss of a rule is *contextual* if it inherits the context (written $\Gamma \Rightarrow \Delta$ in Figure 2) from the conclusion. That is, the right hand premiss of R_T and the premisses of both R_{C_1} and R_{C_2} are contextual premisses of the respective rules. If $\mathcal{R}$ is one of the rule sets of Figure 2, we write $\mathcal{R}^*$ for the rule set that arises by adding the principal formulae of each rule to each of its contextual premisses and refer to $\mathcal{R}^*$ as the *modification* of $\mathcal{R}$. For example, the right (contextual) premiss of the rule R_T then becomes $\Gamma, (A \preccurlyeq B) \Rightarrow \Delta, B$ whereas the left (non-contextual) premiss of R_T remains unchanged. We write $\vdash_{\mathcal{R}} \Gamma \Rightarrow \Delta$ in case $\Gamma \Rightarrow \Delta$ is derivable using rules in $\mathcal{R}$, and $\vdash_{\mathcal{R}^*}$ for derivability using the modification of $\mathcal{R}$. We denote use of additional rules by juxtaposition, e.g. $\mathsf{G}\mathcal{R}\mathsf{ConCut}$ denotes derivability where Cut and Contraction (both on the left and on the right) may be used in addition to rules in G and $\mathcal{R}$. The remainder of the paper establishes our main contributions, the first being soundness and completeness of the corresponding rules in presence of contraction (see Sections 4,5,6).

Theorem 1 (Soundness and Completeness). *If* $* \in \{\preccurlyeq, \Box\!\!\Rightarrow\}$ *and* $\mathcal{L}$ *is one of the logics* $\mathbb{V}_*, \mathbb{VN}_*, \mathbb{VT}_*, \mathbb{VW}_*, \mathbb{VC}_*$ *then* $\vdash_{\mathsf{G}\mathcal{R}_{\mathcal{L}}\mathsf{Con}} \Gamma \Rightarrow \Delta$ *if and only if* $\models_{\mathcal{L}} \Gamma \Rightarrow \Delta$.

The primary purpose of the modifications of the rules in Figure 2 is to achieve admissibility of contraction between principal formulae and those in the context. It is easy to see that this does not change the set of derivable sequents (Sections 5,6).

Proposition 2 (Elimination of Contraction). *If $* \in \{\preccurlyeq, \Box\!\!\Rightarrow\}$ and $\mathcal{L}$ is one of the logics $\mathbb{V}_*, \mathbb{VN}_*, \mathbb{VT}_*, \mathbb{VW}_*, \mathbb{VC}_*$ then $\vdash_{(\mathsf{G}\mathcal{R}_\mathcal{L})^*} \Gamma \Rightarrow \Delta$ if and only if $\vdash_{\mathsf{G}\mathcal{R}_\mathcal{L}\mathsf{Con}} \Gamma \Rightarrow \Delta$.*

This already implies that cut elimination holds for all logics formulated in terms of entrenchment and strong conditional. The calculi are complexity optimal (Sections 5,6):

Theorem 3 (Complexity). *If $* \in \{\preccurlyeq, \Box\!\!\Rightarrow\}$ and $\mathcal{L}$ is one of the logics $\mathbb{V}_*, \mathbb{VN}_*, \mathbb{VT}_*, \mathbb{VW}_*, \mathbb{VC}_*$, then derivability in $(\mathsf{G}\mathcal{R}_\mathcal{L})^*$ is decidable in* Pspace *using backwards proof search. If $* = \Box\!\!\rightarrow$, then $\mathcal{L}$ is decidable in* Pspace *by translating to $\Box\!\!\Rightarrow$.*

As an immediate application, the calculi above allow us to establish, for the first time, that all logics considered here have the Craig interpolation property (Section 7).

Theorem 4 (Craig Interpolation). *If $* \in \{\preccurlyeq, \Box\!\!\Rightarrow, \Box\!\!\rightarrow\}$ and $\mathcal{L}$ is one of the logics $\mathbb{V}_*, \mathbb{VN}_*, \mathbb{VT}_*, \mathbb{VW}_*, \mathbb{VC}_*$, then $\mathcal{L}$ has the Craig interpolation property.*

We prove the above theorems and give precise definitions in the following sections.

4 Soundness and Completeness of The Entrenchment Rules

We first consider the rules in the entrenchment language. The corresponding results for the rules for the strong implication will be established in Section 6.

Theorem 5. *For $\mathcal{L} \in \{\mathbb{V}_\preccurlyeq, \mathbb{VN}_\preccurlyeq, \mathbb{VT}_\preccurlyeq, \mathbb{VW}_\preccurlyeq, \mathbb{VC}_\preccurlyeq\}$ the rules in $\mathcal{R}_\mathcal{L}$ are sound for $\mathcal{L}$.*

Proof. We proceed by induction on the derivations and refer to [4] for $\mathcal{R}_{\mathbb{VC}_\preccurlyeq}$.

For $\mathcal{R}_{\mathbb{V}_\preccurlyeq}$: Suppose the last applied rule was $R_{n,m}$, with conclusion $(C_1 \preccurlyeq D_1), \dots, (C_m \preccurlyeq D_m) \Rightarrow (A_1 \preccurlyeq B_1), \dots, (A_n \preccurlyeq B_n)$ and premisses as given in Figure 2, and suppose all the premisses are valid. Let $\mathcal{I} = (I, (\$_i)_{i\in I}, \pi)$ be a sphere model and $i \in I$. Suppose $i \in [\![C_k \preccurlyeq D_k]\!]$ for all $k \in [m]$ and that for a $k \in [n]$ we have $i \notin [\![A_\ell \preccurlyeq B_\ell]\!]$ for all $\ell \in [n]$, $\ell \neq k$. Choose $S \in \$_i$ and $j \in S \cap [\![B_k]\!]$. Since $\models_{\mathbb{V}_\preccurlyeq} B_k \to \bigvee_{\ell\in[m]} D_\ell \vee \bigvee_{\ell\in[n]} A_\ell$ we have $j \in \bigcup_{\ell\in[n]}[\![A_\ell]\!] \cup \bigcup_{\ell\in[m]}[\![D_\ell]\!]$. Thus either $j \in \bigcup_{\ell\in[n]}[\![A_\ell]\!]$ or $j \in [\![D_\ell]\!]$ for a $\ell \in [m]$. In the latter case, since $i \in [\![C_\ell \preccurlyeq D_\ell]\!]$ we find a $j_2 \in S \cap [\![C_\ell]\!]$, and since $\models_{\mathbb{V}_\preccurlyeq} C_\ell \to \bigvee_{\ell'<\ell} D_{\ell'} \vee \bigvee_{\ell'\in[n]} A_{\ell'}$ we have $j_2 \in \bigcup_{\ell'<\ell}[\![D_{\ell'}]\!] \cup \bigcup_{\ell'\in[n]}[\![A_{\ell'}]\!]$. Continuing like this we find a $j' \in I$ with $j' \in S \cap \bigcup_{\ell\in[n]}[\![A_\ell]\!]$. Now if $j' \notin [\![A_k]\!]$ there is a $\ell \neq k$ with $j' \in [\![A_\ell]\!]$. But since $i \notin [\![A_\ell \preccurlyeq B_\ell]\!]$ there is an $S' \in \$_i$ with $S' \subsetneq S$ and $S' \cap [\![B_\ell]\!] \neq \emptyset = S' \cap [\![A_\ell]\!]$. As above we get a $j'' \in S' \cap \bigcup_{t\in[n]}[\![A_t]\!] = S' \cap \bigcup_{t\in[n],t\neq\ell}[\![A_t]\!]$. Repeating the argument we finally get an $S'' \in \$_i$ with $\emptyset \neq S'' \cap \bigcup_{\ell\in[n]}[\![A_\ell]\!] = S'' \cap [\![A_k]\!]$, and since by construction $S'' \subseteq S$ we have $i \in [\![A_k \preccurlyeq B_k]\!]$.

For $\mathcal{R}_{\mathbb{VN}_\preccurlyeq}$: Assume $\models_{\mathbb{VN}_\preccurlyeq} \neg A$ and $\models_{\mathbb{VN}_\preccurlyeq} B$, let $\mathcal{I}$ be a normal sphere model, i.e., for all $i \in I$ we have $\bigcup \$_i \neq \emptyset$, and let $i \in I$. Since $\bigcup \$_i \neq \emptyset$ there is a $j \in S \in \$_i$. But then $j \in [\![B]\!]$ and for all $t \in S$ we have $t \notin [\![A]\!]$. Thus $i \notin [\![A \preccurlyeq B]\!]$.

For $\mathcal{R}_{\mathbb{VT}_\preccurlyeq}$: Suppose $\models_{\mathbb{VT}_\preccurlyeq} \neg A$ and $\models_{\mathbb{VT}_\preccurlyeq} \bigwedge \Gamma \to \bigvee \Delta \vee B$, and let $\mathcal{I}$ be totally reflexive, i.e., for all $i \in I$ we have $i \in \bigcup \$_i$. Then for any $i \in I$ we have either $i \in [\![B]\!]$ and are done, or we can choose a $S \in \$_i$ with $i \in S$. But we know that $j \notin [\![A]\!]$ for all $j \in S$, and thus we get $i \notin [\![A \preccurlyeq B]\!]$.

For $\mathcal{R}_{\mathbb{VW}_\preccurlyeq}$: Similar to $\mathbb{V}_\preccurlyeq$. Let $\models_{\mathbb{VW}_\preccurlyeq} \Gamma \Rightarrow D_1, \dots, D_m, A_1, \dots, A_n, \Delta$ and $\models_{\mathbb{VW}_\preccurlyeq} C_k \Rightarrow D_1, \dots, D_{k-1}, A_1, \dots, A_n$ for all $k \in [m]$, and suppose that $\mathcal{I}$ is weakly centered, i.e., for all $i \in I$ there is an $S \in \$_i$ with $S \neq \emptyset$ and for all $S \in \$_i$ with $S \neq \emptyset$ we have $i \in S$. Then for $i \in I$ we have either $i \notin \bigcup_{\ell \in [m]} [\![D_\ell]\!] \cup \bigcup_{\ell \in [n]} [\![A_\ell]\!]$ and are done; or we have $i \in [\![A_\ell]\!]$ for a $\ell \in [n]$ and are done; or we have $i \in [\![D_k]\!]$ for a $k \in [m]$. In the latter case we take $S \in \$_i$ with $S \neq \emptyset$. Then $i \in S$. If $i \notin [\![C_k \preccurlyeq D_k]\!]$ we are done; otherwise there is a $i_1 \in S \cap C_k$. Since $\models_{\mathbb{VW}_\preccurlyeq} C_k \to \bigvee_{\ell<k} D_\ell \vee \bigvee_{\ell \in [n]} A_\ell$ we have $i_1 \in \bigcup_{\ell<k} [\![D_\ell]\!] \cup \bigcup_{\ell \in [n]} [\![A_\ell]\!]$. Repeating the argument yields a $j \in S \cap \bigcup_{\ell \in [n]} [\![A_\ell]\!]$. Choose k_1 with $j \in [\![A_{k_1}]\!]$. If $i \notin [\![A_{k_1} \preccurlyeq B_{k_1}]\!]$, then there is a $S' \subsetneq S$ with $S' \cap [\![A_{k_1}]\!] = \emptyset$ and $S' \cap [\![B_{k_1}]\!] \neq \emptyset$. As above we get a $j_2 \in S' \cap \bigcup_{\ell \in [n]} [\![A_\ell]\!] = S' \cap \bigcup_{\ell \neq k_1} [\![A_\ell]\!]$. Repeating the argument again we successively eliminate the A_ℓ's and get a $k' \in [n]$ such that for all $S \in \$_i$ with $S \cap [\![B_{k'}]\!] \neq \emptyset$ we have $S \cap [\![A_{k'}]\!] \neq \emptyset$. But this means $i \in [\![A_{k'} \preccurlyeq B_{k'}]\!]$. □

Next we establish completeness of the sequent systems with the cut rule. Cut-free completeness follows from the generic cut elimination result of the next section. Since all our systems include the congruence rule and thus are closed under uniform substitution, it suffices to show that all the rules and axioms of the Hilbert-style characterisation $\mathcal{HL}$ of a given logic $\mathcal{L}$ from Figure 1 are derivable in the corresponding sequent system with cut. Since the Hilbert-systems are complete [11], this establishes the result.

Theorem 6 (Completeness). *For* $\mathcal{L} \in \{\mathbb{V}_\preccurlyeq, \mathbb{VN}_\preccurlyeq, \mathbb{VT}_\preccurlyeq, \mathbb{VW}_\preccurlyeq, \mathbb{VC}_\preccurlyeq\}$ *the sequent system* $\mathsf{G}\mathcal{R}_\mathcal{L}\mathsf{ConWCut}$ *is complete with respect to* $\mathcal{L}$.

Proof. Showing that the rules and axioms of $\mathcal{H}\mathbb{V}_\preccurlyeq$, $\mathcal{H}\mathbb{VN}_\preccurlyeq$ and $\mathcal{H}\mathbb{VT}_\preccurlyeq$ can be derived in the corresponding sequent system is easy. For $\mathcal{H}\mathbb{VW}_\preccurlyeq$ note that adding the axiom (W) is equivalent to adding the axioms (T) and $(\neg A \preccurlyeq \top) \vee A$, where the latter is easily derived using $W_{1,0}$. For $\mathcal{H}\mathbb{VC}_\preccurlyeq$, using R_{C2} we get $(A \preccurlyeq \top) \to A$ and thus (C). □

5 Cut Elimination for the Entrenchment Rules

Our approach towards proving cut elimination for the sequent systems of the previous section is based on a general method for the construction of cut-free calculi: *cut elimination by saturation*. We call a set of (sequent) rules *saturated* if it is closed under the operations of cut and contraction, introduced below. Cut elimination by saturation elevates both cut and contraction from the level of *proof rules* to the level of *operations* on proof rules, i.e. constructions that allow us to derive new proof rules while preserving soundness. Cut closure holds if for any two given rules, performing a cut on the conclusions and collecting the premisses of both rules results in a (cut-free) derivable rule (after eliminating variables that no longer occur in the conclusion) and contraction closure stipulates that the result of identifying literals in the conclusion of a rule gives a rule already present in the rule set. Assuming saturation cut elimination holds, every cut can be replaced by a derivable rule, reducing level or rank of the cuts. The key ingredient in sequent systems for non-iterative logics is the concept of a *shallow rule*, introduced in previous work [10]. Intuitively, a shallow rule adds one layer of modalities in the conclusion, while its premisses may or may not propagate the context.

Definition 7. A *shallow rule* is a triple $R = (P_n; P_c; \Sigma \Rightarrow \Pi)$ where $P_n \subseteq \mathcal{S}(\mathcal{V})$ and $P_c \subseteq \mathcal{S}(\mathcal{V})$ are finite sets of sequents (the *non-contextual* and *contextual* premisses, respectively) and $\Sigma \Rightarrow \Pi \in \mathcal{S}(\Lambda(\mathcal{V}))$ are the *principal formulae* subject to the following variable restriction: every variable $p \in \mathcal{V}$ may occur at most once in $\Sigma \Rightarrow \Pi$ and occurs in the premisses iff it occurs in the principal formulae. An *instance* of a shallow rule

$$\frac{\{\Upsilon\sigma \Rightarrow \Omega\sigma \mid \Upsilon \Rightarrow \Omega \in P_n\} \quad \cup \quad \{\Gamma, \Theta\sigma \Rightarrow \Delta, \Xi\sigma \mid \Theta \Rightarrow \Xi \in P_c\}}{\Gamma, \Sigma\sigma \Rightarrow \Delta, \Pi\sigma}$$

is given by a context $\Gamma \Rightarrow \Delta$ and a substitution $\sigma : \mathcal{V} \to \mathcal{F}(\Lambda)$. We often annotate the contextual premisses with the context (usually $\Gamma \Rightarrow \Delta$) if no confusion can arise.

Remark 8. The variable restriction on the principal formulae is for technical convenience and not restrictive, as a duplicate occurrence of a variable p is avoided by replacing it by a fresh variable q and adding non-contextual premisses $p \Rightarrow q$ and $q \Rightarrow p$.

Example 9. The rules of classical propositional logic such as $\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$ ($\wedge$R) and all of our rules for conditional logics in Figure 2 are shallow. All premisses in $\wedge$R and R_{C2} are contextual, while all premisses in $R_{n,m}$ and R_N are non-contextual. Rule R_T has both a contextual and a non-contextual premiss.

A set $\mathcal{R}$ of shallow rules induces a sequent system in the standard way.

Definition and Convention 10. Whenever we speak about a *set of shallow rules* $\mathcal{R}$ we assume that $\mathcal{R}$ is closed under injective renaming of propositional variables. Let $\mathcal{R}$ be a set of shallow rules and $S \subseteq \mathcal{S}(\mathcal{F}(\Lambda))$ a set of sequents. A sequent $\Gamma \Rightarrow \Delta$ is *$\mathcal{R}$-derivable from S*, in symbols $S \vdash_{\mathcal{R}} \Gamma \Rightarrow \Delta$, if it is an element of the least set $S \vdash_{\mathcal{R}}$ containing S and closed under the axiom rules $\frac{}{\Gamma, A \Rightarrow \Delta, A}$ and the *congruence rules* $\frac{A_1 \Rightarrow B_1 \quad B_1 \Rightarrow A_1 \quad \dots \quad A_n \Rightarrow B_n \quad B_n \Rightarrow A_n}{\Gamma, \heartsuit(A_1, \dots, A_n) \Rightarrow \Delta, \heartsuit(B_1, \dots, B_n)}$ and all instances of rules in $\mathcal{R}$. We write $S \vdash_{\mathcal{R}\mathcal{R}'}$ for $S \vdash_{\mathcal{R} \cup \mathcal{R}'}$ and simply $\vdash_{\mathcal{R}}$ for $\emptyset \vdash_{\mathcal{R}}$. The rule set $\mathcal{R}'$ is *$\mathcal{R}$-admissible* if $\vdash_{\mathcal{R}\mathcal{R}'} \subseteq \vdash_{\mathcal{R}}$. *Derivations* are defined as usual [18] and a (not necessarily shallow) rule $R = P_1 \dots P_n / C$ with premisses $P_1, \dots, P_n$ and conclusion C is *$\mathcal{R}$-derivable* if $\{P_1, \dots, P_n\} \vdash_{\mathcal{R}} C$.

Lemma 11 (Admissibility of Weakening). *$\vdash_{\mathcal{R}} \Gamma \Rightarrow \Delta$ whenever $\vdash_{\mathcal{R}\mathsf{W}} \Gamma \Rightarrow \Delta$.*

The proof is standard. For admissibility of Contraction and Cut, the rule set needs to be closed under the operations of *rule contraction* and *cut between rules* described next.

Definition 12 (Cut as an Operation on Proof Rules). If (O_n, O_c) are sets of sequents (that we think of as non-contextual and contextual premisses, respectively) and p is a variable, then the *p-elimination on O_n and O_c* is the pair $(O_n, O_c) \ominus p := (O'_n, O'_c)$ where

$$\begin{aligned} O'_n &= \{\Gamma, \Sigma \Rightarrow \Delta, \Pi \mid \langle \Gamma, p \Rightarrow \Delta; \Sigma \Rightarrow \Pi, p \rangle \in O_n \times O_n\} \cup \{\Gamma \Rightarrow \Delta \in O_n \mid p \notin \Gamma, \Delta\} \\ O'_c &= \{\Gamma, \Sigma \Rightarrow \Delta, \Pi \mid \langle \Gamma, p \Rightarrow \Delta; \Sigma \Rightarrow \Pi, p \rangle \in (O_n \cup O_c)^2 \setminus (O_n \times O_n)\} \\ &\quad \cup \{\Gamma \Rightarrow \Delta \in O_c \mid p \notin \Gamma \cup \Delta\} \end{aligned}$$

and we write $(O_n, O_c) \ominus p_1, \dots, p_n$ for the repeated application of variable elimination. If $R = (P_n; P_c; \Sigma \Rightarrow \Pi, \heartsuit\boldsymbol{p})$ and $R' = (P'_n; P'_c; \heartsuit\boldsymbol{p}, \Sigma' \Rightarrow \Pi')$ are shallow rules, the *cut* *of R and R' on $\heartsuit \boldsymbol{p}$* is the shallow rule $\mathsf{cut}(R, R', \heartsuit \boldsymbol{p}) = (Q_n; Q_c; \Sigma, \Sigma' \Rightarrow \Pi, \Pi')$ where $(Q_n, Q_c) = (P_n \cup P'_n, P_c \cup P'_c) \ominus \boldsymbol{p}$. A set $\mathcal{R}$ of shallow rules is *cut closed* if for any $R, R' \in \mathcal{R}$ with principal formulae $\Gamma \Rightarrow \Delta, \heartsuit \boldsymbol{p}$ and $\heartsuit \boldsymbol{p}, \Gamma' \Rightarrow \Delta'$ the rule $\mathsf{cut}(R, R', \heartsuit \boldsymbol{p})$ is $\mathcal{R}\mathsf{WCon}$-derivable.

That is, the cut between R and R' is a (shallow) rule, whose principal formulae arise by applying cut to the principal formulae of R and R' and whose premisses are the premisses of both R and R' with superfluous variables eliminated by variable elimination, i.e. cuts on the variables that no longer appear in the conclusion. Note that a premiss is contextual in the cut between two rules if at least one step in the variable elimination process did involve a contextual premiss. Cut closed rule sets are simply closed under performing cuts between rules. Also note that in presence of the rules for classical propositional logic the constructed rules are derivable using the old rules and Cut, since we can reconstruct the cut formulae for the premisses using the rules from G:

Lemma 13 ([10]). *For shallow rules R_1, R_2 with principal formulae $\Sigma \Rightarrow \Pi, \heartsuit \boldsymbol{p}$ and $\heartsuit \boldsymbol{p}, \Sigma' \Rightarrow \Pi'$ the rule $\mathsf{cut}(R_1, R_2, \heartsuit \boldsymbol{p})$ is $\mathsf{G}R_1R_2\mathsf{Cut}$-derivable.*

A similar construction applies to contraction:

Definition 14 (Contraction as Operation on Proof Rules). If S is a set of sequents and $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ are n-tuples of variables, then $S[\boldsymbol{q} \leftarrow \boldsymbol{p}]$ is the result of replacing every occurrence of q_i in a sequent in S by p_i for all $i = 1, \ldots, n$ and contracting duplicate instances of $p_1, \ldots, p_n$. Let $R = (P_n; P_c; \Gamma, \heartsuit \boldsymbol{p}, \heartsuit \boldsymbol{q} \Rightarrow \Delta)$ be a shallow rule. The *left contraction of R on $\heartsuit \boldsymbol{p}$ and $\heartsuit \boldsymbol{q}$* is the shallow rule $\mathsf{ConL}(R, \heartsuit \boldsymbol{p}, \heartsuit \boldsymbol{q}) = (P_n[\boldsymbol{q} \leftarrow \boldsymbol{p}]; P_c[\boldsymbol{q} \leftarrow \boldsymbol{p}]); \Gamma, \heartsuit \boldsymbol{p} \Rightarrow \Delta)$. The *right contraction* $\mathsf{ConR}(R, \heartsuit \boldsymbol{p}, \heartsuit \boldsymbol{q})$ is defined dually. A rule set $\mathcal{R}$ is *contraction closed* if for every rule $R \in \mathcal{R}$ the rules $\mathsf{ConL}(R, \heartsuit \boldsymbol{p}, \heartsuit \boldsymbol{q})$ and $\mathsf{ConR}(R, \heartsuit \boldsymbol{p}, \heartsuit \boldsymbol{q})$ can be simulated by applications of Weakening and Contraction, followed by at most one application of a rule $R' \in \mathcal{R}$ and Weakening.

Saturated rule sets combine both properties.

Definition 15. A set of shallow rules is *saturated* if it is both cut and contraction closed.

Theorem 16. *For $\mathcal{L} \in \{\mathbb{V}_{\preccurlyeq}, \mathbb{VN}_{\preccurlyeq}, \mathbb{VT}_{\preccurlyeq}, \mathbb{VW}_{\preccurlyeq}, \mathbb{VC}_{\preccurlyeq}\}$ the rule set $\mathsf{G}\mathcal{R}_{\mathcal{L}}$ is saturated.*

Proof (Sketch). It is easy to see that the rules of G are saturated. Since cuts between propositional and conditional rules on principal formulae of both rules are impossible we thus only need to consider the rule sets $\mathcal{R}_{\mathcal{L}}$. For cut closure of $\mathcal{R}_{\mathbb{V}_{\preccurlyeq}}$ it can be seen that cuts between two rules $R_{n,m}$ and $R_{k,\ell}$ are subsumed by the rule $R_{n+k-1,m+\ell-1}$. Contraction closure is evident. The sets $\mathcal{R}_{\mathbb{VN}_{\preccurlyeq}}$ and $\mathcal{R}_{\mathbb{VT}_{\preccurlyeq}}$ are cut- and contraction closed, since cuts between a rule $R_{n,m}$ and R_N or R_T are subsumed by the rule $R_{n-1,m}$. Cut- and contraction closure of $\mathcal{R}_{\mathbb{VW}_{\preccurlyeq}}$ follows since $\mathcal{R}_{\mathbb{V}_{\preccurlyeq}}$ is cut closed and since cuts between $R_{n,m}$ or $W_{n,m}$ and $W_{k,\ell}$ are subsumed by $W_{n+k-1,m+\ell-1}$. For $\mathcal{R}_{\mathbb{VC}_{\preccurlyeq}}$ note that cuts between $R_{n,m}$ and R_{C1} or R_{C2} can be replaced by a number of applications of R_{C2} and R_{C1}. □

Saturation enables a general cut elimination theorem following [5]: (multi-)cuts on context formulae are propagated upwards in the proof trees, and (multi-)cuts on principal formulae can be eliminated using cut and contraction closure.

Theorem 17 (Generic Cut Elimination). *Let $\mathcal{R}$ be a saturated set of shallow rules. Then* Cut *is admissible in* $\mathcal{R}$Con, *i.e.* $\vdash_{\mathcal{R}\mathsf{Con}} \Gamma \Rightarrow \Delta$ *whenever* $\vdash_{\mathcal{R}\mathsf{ConCut}} \Gamma \Rightarrow \Delta$.

Proof. Similar to [10, Prop. 21].

Corollary 18. *For* $\mathcal{L} \in \{\mathbb{V}_{\preccurlyeq}, \mathbb{VN}_{\preccurlyeq}, \mathbb{VT}_{\preccurlyeq}, \mathbb{VW}_{\preccurlyeq}, \mathbb{VC}_{\preccurlyeq}\}$ $\mathsf{G}\mathcal{R}_{\mathcal{L}}\mathsf{Con}$ *has cut elimination.*

Note that contraction closure only allows to eliminate Contraction on *principal formulae*, but not on a principal formula and a context formula. Nevertheless, after establishing cut elimination, admissibility of Contraction and a generic PSPACE complexity result are obtained in the modification of the rule set, where in a standard move the principal formulae are copied into the contextual premisses.

Definition 19 (Modified Instances). A *modified instance*

$$\frac{\{\Upsilon\sigma \Rightarrow \Omega\sigma \mid \Upsilon \Rightarrow \Omega \in P_n\} \quad \cup \quad \{\Gamma, \Sigma\sigma, \Theta\sigma \Rightarrow \Delta, \Xi\sigma, \Pi\sigma \mid \Theta \Rightarrow \Xi \in P_c\}}{\Gamma, \Sigma\sigma \Rightarrow \Delta, \Pi\sigma}$$

of a shallow rule $(P_n; P_c; \Sigma \Rightarrow \Delta)$ is given by a substitution σ and a (context) sequent $\Gamma \Rightarrow \Delta$. For the *modification* $\mathcal{R}^*$ of $\mathcal{R}$ the notion of $\mathcal{R}^*$-admissibility and $\mathcal{R}^*$-derivability are as for $\mathcal{R}$ using modified instances of rules in $\mathcal{R}$ instead of instances.

The purpose of modified instances is the elimination of Contraction, where Contraction between context and principal formulae is absorbed by moving principal formulae upwards in the context. Moving to modified instances, e.g. the (standard) instance $\frac{\Gamma,\Theta\Rightarrow\Delta,\Xi \quad \Upsilon\Rightarrow\Omega}{\Gamma,\heartsuit A\Rightarrow\Delta,\clubsuit B}$ is replaced by the modified instance $\frac{\Gamma,\heartsuit A,\Theta\Rightarrow\Delta,\clubsuit B,\Xi, \quad \Upsilon\Rightarrow\Omega}{\Gamma,\heartsuit A\Rightarrow\Delta,\clubsuit B}$. We can now apply the following result from [10] for *tractable* rule sets, i.e., sets where codes of the rules can be computed in space polynomial in the length of the conclusion and where the premisses can be computed in space polynomial in the code of the rule. It can easily be checked that all of the rule sets in Figure 2 as well as the rules of G are tractable.

Theorem 20. *If* $\mathcal{R}$ *is saturated, then* $\vdash_{\mathcal{R}\mathsf{ConCut}} = \vdash_{\mathcal{R}\mathsf{Con}} = \vdash_{\mathcal{R}^*\mathsf{Con}} = \vdash_{\mathcal{R}^*}$. *In particular,* Con *is* $\mathcal{R}^*$*-admissible. If* $\mathcal{R}$ *is also tractable, then backwards proof search in* $\mathcal{R}^*$ *is in* PSPACE.

Corollary 21. *For* $\mathcal{L} \in \{\mathbb{V}_{\preccurlyeq}, \mathbb{VN}_{\preccurlyeq}, \mathbb{VT}_{\preccurlyeq}, \mathbb{VW}_{\preccurlyeq}, \mathbb{VC}_{\preccurlyeq}\}$ *we have* $\models_{\mathcal{L}} = \vdash_{(\mathsf{G}\mathcal{R}_{\mathcal{L}})^*}$ *and backwards proof search in* $(\mathsf{G}\mathcal{R}_{\mathcal{L}})^*$ *is in* PSPACE.

Remark 22. Theorems 17 and 20 remain valid for languages that do not contain all Boolean connectives. As the propositional rules are shallow, they can be absorbed into the general treatment and it is easy to see that, for every Boolean connective, adding the corresponding left and right rules preserves saturation.

6 Strong And Weak Conditional Implication

For the systems in the language with the strong conditional our strategy for proving soundness and completeness is slightly different.

Theorem 23. *For* $\mathcal{L} \in \{\mathbb{V}_{\boxRight}, \mathbb{VN}_{\boxRight}, \mathbb{VT}_{\boxRight}, \mathbb{VW}_{\boxRight}, \mathbb{VC}_{\boxRight}\}$ *the sequent system* $\mathsf{G}\mathcal{R}_{\mathcal{L}}\mathsf{ConCut}$ *is sound and complete for* $\mathcal{L}$.

Proof. Since the strong conditional is defined in terms of entrenchment by the translation $(A \boxRightarrow B) \leftrightarrow \neg((A \wedge \neg B) \preccurlyeq (A \wedge B))$ from [11], we get the translation rules

$$\frac{A \Rightarrow C \quad A, D \Rightarrow \quad C \Rightarrow A, D \quad B \Rightarrow C \quad B \Rightarrow D \quad C, D \Rightarrow B}{\Gamma, (A \preccurlyeq B), (C \boxRightarrow D) \Rightarrow \Delta} \; R_{t1}$$

$$\frac{A \Rightarrow C \quad A, D \Rightarrow \quad C \Rightarrow A, D \quad B \Rightarrow C \quad B \Rightarrow D \quad C, D \Rightarrow B}{\Gamma \Rightarrow \Delta, (A \preccurlyeq B), (C \boxRightarrow D)} \; R_{t2}$$

which together are equivalent to the translation axiom. The rule sets $\mathcal{R}_{\mathbb{V}_{\boxRightarrow}}, \mathcal{R}_{\mathbb{VN}_{\boxRightarrow}}, \mathcal{R}_{\mathbb{VT}_{\boxRightarrow}}, \mathcal{R}_{\mathbb{VW}_{\boxRightarrow}}$ and $\mathcal{R}_{\mathbb{VC}_{\boxRightarrow}}$ arise from the rule sets for the entrenchment connective by cutting every literal of every rule with the appropriate translation rule. The resulting rules have the translation built in which gives completeness and soundness (using Lemma 13). ◻

Since cuts between the translation rules are subsumed by congruence and since the entrenchment rules are saturated, saturation for these rule sets is not unexpected.

Theorem 24. *For $\mathcal{L} \in \{\mathbb{V}_{\boxRightarrow}, \mathbb{VN}_{\boxRightarrow}, \mathbb{VT}_{\boxRightarrow}, \mathbb{VW}_{\boxRightarrow}, \mathbb{VC}_{\boxRightarrow}\}$ the rule set $\mathsf{G}\mathcal{R}_{\mathcal{L}}$ is saturated and thus $\mathsf{G}\mathcal{R}_{\mathcal{L}}\mathsf{Con}$ has cut elimination. Hence $\mathcal{L}$ is decidable in* Pspace.

Proof. Cut closure is seen analogous to the entrenchment case. E.g. for $\mathcal{R}_{\mathbb{V}_{\boxRightarrow}}$ a cut between rules $R'_{n,m}$ and $R'_{k,\ell}$ is subsumed by the rule $R'_{n+k-1,\,m+\ell-1}$. Note that for some of the premisses of the latter rule we need to cut three of the original premisses and apply Contraction. Contraction closure is straightforward. ◻

Unfortunately, this technique does not work not work for Lewis' weak conditional $\boxright$, since the translations of $\boxright$ into $\preccurlyeq$ or $\boxRightarrow$ are more subtle. Nevertheless, since the translation $(A \boxright B) \leftrightarrow ((\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \wedge B)))$ of $\boxright$ into $\preccurlyeq$ from [11, p.26,53] increases the number of subformulae only by a constant factor, we may represent formulae as directed acyclic graphs instead of trees, to obtain a purely syntactical Pspace decision procedure of optimal complexity for these logics in the language with $\boxright$.

Theorem 25. *The logics $\mathbb{V}_{\boxright}, \mathbb{VN}_{\boxright}, \mathbb{VT}_{\boxright}, \mathbb{VW}_{\boxright}, \mathbb{VC}_{\boxright}$ are decidable in* Pspace.

Proof. Since the important measure for the backwards proof search procedure from [10] is the nesting depth of connectives and not the size of the formulae, careful inspection of the proofs together with the fact that the translation is linear for the representation of formulae by directed acyclic graphs yields the result. ◻

7 Applications

Interpolation. The sequent systems presented above enable us to establish Theorem 4 (Craig interpolation) for all logics considered in this paper in a standard way. A logic $\mathcal{L}$ has the *(Craig) interpolation property* (CIP) if whenever $\models_{\mathcal{L}} A \to B$, then there is an *interpolant* I satisfying the *variable condition* $\mathsf{var}(I) \subseteq \mathsf{var}(A) \cap \mathsf{var}(B)$ such that $\models_{\mathcal{L}} A \to I$ and $\models_{\mathcal{L}} I \to B$. We use split sequents [18] to establish the CIP, the intuition being that whenever we split a provable sequent into two, we can find an interpolant:

Definition 26 (split sequent). An expression $\Gamma_1 \mid \Gamma_2 \Rightarrow \Delta_1 \mid \Delta_2$ is a *split sequent*, if $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ is a sequent, and we say that $\Gamma_1 \mid \Gamma_2 \Rightarrow \Delta_1 \mid \Delta_2$ is a *splitting* of $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. A formula I is an *interpolant* in $\mathcal{R}_\mathcal{L}$ for the split sequent $\Gamma_1 \mid \Gamma_2 \Rightarrow \Delta_1 \mid \Delta_2$ if it satisfies the *variable condition* $\mathsf{var}(I) \subseteq \mathsf{var}(\Gamma_1 \Rightarrow \Delta_1) \cap \mathsf{var}(\Gamma_2 \Rightarrow \Delta_2)$ and $\vdash_{\mathcal{R}_\mathcal{L}} \Gamma_1 \Rightarrow \Delta_1, I$ and $\vdash_{\mathcal{R}_\mathcal{L}} I, \Gamma_2 \Rightarrow \Delta_2$. A sequent $\Gamma \Rightarrow \Delta$ *admits interpolation* in $\mathcal{R}_\mathcal{L}$ if all its splittings have an interpolant in $\mathcal{R}_\mathcal{L}$. A shallow rule R *supports interpolation* in $\mathcal{R}_\mathcal{L}$ if whenever all its premisses admit interpolation in $\mathcal{R}_\mathcal{L}$, then so does its conclusion.

It is routine to prove the following lemma by induction.

Lemma 27. *If $\mathsf{G}\mathcal{R}_\mathcal{L}$ is a sound and complete set of shallow rules for a logic $\mathcal{L}$ and all the rules in $\mathsf{G}\mathcal{R}_\mathcal{L}$ support interpolation in $\mathsf{G}\mathcal{R}_\mathcal{L}$, then $\mathcal{L}$ has the interpolation property.*

Theorem 28. *$\mathbb{V}_\preccurlyeq$ has the Craig interpolation property.*

Proof. We need to show that the rules in $\mathsf{G}\mathcal{R}_{\mathbb{V}_\preccurlyeq}$ support interpolation. For the propositional rules this is standard [18]. For $R_{n,m}$ we construct an interpolant for a splitting of the conclusion from interpolants of the corresponding splittings of the premisses. First, consider the rule $R_{2,m}$ and the splitting $\Gamma_1 \mid \Gamma_2 \Rightarrow \Delta_1 \mid \Delta_2$ of its conclusion given by

$$\{(C_i \preccurlyeq D_i) \mid i \in [m], i \text{ odd}\} \mid \{(C_i \preccurlyeq D_i) \mid i \in [n], i \text{ even}\} \Rightarrow (A_1 \preccurlyeq B_1) \mid (A_2 \preccurlyeq B_2)\ .$$

For $k \in [m]$ let I_k be the interpolant for the corresponding splitting of the premiss $C_k \Rightarrow \{D_\ell \mid \ell < k\}, A_1, A_2$ and for $k \in \{1,2\}$ let J_k be the one for the corresponding splitting of the premiss $B_k \Rightarrow \{D_\ell \mid \ell \in [m]\}, A_1, A_2$. For every odd $k \in [m]$ we introduce

$$X_k = \bigvee_{\ell \le k, \ell \text{ odd}} I_\ell \qquad Y_k = \begin{cases} \neg I_{k+1} \vee \neg J_2 & k = \max\{\ell \in [m] \mid \ell \text{odd}\} \\ \neg I_{k+1} & \text{otherwise} \end{cases} \qquad Z_k = J_1 \vee \bigvee_{\ell \in [m],\, \ell > k,\, \ell \text{ odd}} I_\ell$$

$$V_k = (X_k \preccurlyeq Y_k) \qquad W_k = (Y_k \preccurlyeq Z_k) \qquad I = \bigwedge_{k \in [m],\, k \text{ odd}} (\neg W_k \vee V_k)\ .$$

Claim 1: For every odd $k \in [m]$ we have $\vdash_{\mathcal{R}_{\mathbb{V}_\preccurlyeq}} \Gamma_1, W_k \Rightarrow \Delta_1, V_k$. The idea is to insert W_k instead of $(C_{k+1} \preccurlyeq D_{k+1})$ and V_k instead of $(A_2 \preccurlyeq B_2)$ into the rule pattern. Then using the definitions of W_k and V_k together with applications of Weakening it is straightforward to check that $R_{2, |\{\ell \in [m] | \ell \text{ odd}\}|+1}$ can be applied.

Claim 2: For every partition (F, S) of $\{k \in [m] \mid k \text{ odd}\}$ we have $\vdash_{\mathcal{R}_{\mathbb{V}_\preccurlyeq}} \Gamma_2, \{V_k \mid k \in F\} \Rightarrow \Delta_2, \{W_k \mid k \in S\}$. The idea is to insert the V_k instead of the $(C_k \preccurlyeq D_k)$, and the W_k as positive literals instead of $(A_1 \preccurlyeq B_1)$. Then again it is straightforward to check that we have all the necessary premisses for an application of $R_{|S|+1, |F|+|\{\ell \in [m] | \ell \text{ even}\}|}$.

By propositional reasoning, both claims give $\vdash_{\mathcal{R}_{\mathbb{V}_\preccurlyeq}} \Gamma_1 \Rightarrow \Delta_1, I$ and $\vdash_{\mathcal{R}_{\mathbb{V}_\preccurlyeq}} I, \Gamma_2 \Rightarrow \Delta_2$ and the interpolant I satisfies the variable condition, since all its constituents satisfy it.

For the general case consider the splitting $\Gamma_1 \mid \Gamma_2 \Rightarrow \Delta_1 \mid \Delta_2$ of the conclusion, and write I'_k for the interpolant for the corresponding splitting of the premiss $C_k \Rightarrow \{D_\ell \mid \ell < k\}, \{A_\ell \mid \ell \in [n]\}$ and J'_k for the one for the premiss $B_k \Rightarrow \{A_\ell \mid \ell \in [n]\}, \{D_\ell \mid \ell \in [m]\}$. In the construction of the interpolant above we replace J_1 by $\bigvee_{(A_\ell \preccurlyeq B_\ell) \in \Delta_1} J'_\ell$ and $\neg J_2$ by $\bigvee_{(A_\ell \preccurlyeq B_\ell) \in \Delta_2} \neg J'_\ell$. The formulae I_ℓ in X_k and Z_k are replaced by $\bigvee_{j \in T_\ell} I'_j$ where T_ℓ is the ℓ-th block of consecutive indices j with $(C_j \preccurlyeq D_j) \in \Gamma_1$. The formulae $\neg I_{k+1}$ in

Y_k are replaced by $\bigvee_{j \in S_k} \neg I'_j$ where S_k is the k-th block of consecutive indices j with $(C_j \preccurlyeq D_j) \in \Gamma_2$. Then in the proofs of the claims the formulae W_k and V_k are inserted instead of the blocks $\{(C_\ell \preccurlyeq D_\ell) \mid \ell \in T_k\}$ and $\{(C_\ell \preccurlyeq D_\ell) \mid \ell \in S_k\}$. □

Corollary 29. $\mathbb{VT}_\preccurlyeq, \mathbb{VN}_\preccurlyeq, \mathbb{VW}_\preccurlyeq, \mathbb{VC}_\preccurlyeq$ *have the CIP.*

Proof. For $\mathbb{VT}_\preccurlyeq, \mathbb{VN}_\preccurlyeq$ and $\mathbb{VC}_\preccurlyeq$ this is immediate since the additional axioms trivially support interpolation. For the rules $W_{n,m}$ of $\mathcal{R}_{\mathbb{VW}_\preccurlyeq}$ we only need to modify the proof for the rules $R_{n,m}$ by replacing the interpolants J_1, J_2 in the construction of I by the interpolant J of the contextual premiss and its negation. □

Corollary 30. *For* $* \in \{\Box\!\!\Rightarrow, \Box\!\!\rightarrow\}$ *the logics* $\mathbb{V}_*, \mathbb{VT}_*, \mathbb{VN}_*, \mathbb{VW}_*, \mathbb{VC}_*$ *have the CIP.*

Proof. By translating the formula $A \to B$ into the entrenchment language, and translating the interpolant back into the original language. Since both translations are identity on propositional variables the variable condition holds, and we obtain an interpolant since translating back and forth yields logically equivalent formulae. □

Hybrid conditional logic. In [16] a hybridisation of conditional logic $\mathbb{V}_{\Box\Rightarrow}$ is proposed to extend Lewis' interpretation of $\Box\!\!\Rightarrow$ in terms of *contextually definite descriptions*. Worlds in a sphere model represent things or individuals, the sphere systems give degrees of salience, and a formula like $\mathsf{Pig} \;\Box\!\!\Rightarrow\; \mathsf{Grunting}$ is interpreted as "The (most salient) pig is grunting". *Nominals i* are introduced as names for specific individuals together with the satisfaction operators $@_i\, A$ stating that A is true for individual i.

Following [12] the sequent system for $\mathbb{V}_{\Box\Rightarrow}$ can easily be turned into a sequent system for the hybrid logic $\mathbb{V}_{\mathcal{HC}(@)}$ in the language of the strong conditional. Sphere models are captured coalgebraically as coalgebras for the functor Sp with $\mathsf{Sp}(X) = \{\$ \in \mathcal{PP}(X) \mid \$ \text{ a system of spheres}\}$ and $\mathsf{Sp}(f)$ the double direct image of f. The correct semantics for $\Box\!\!\Rightarrow$ is then given by the predicate lifting $[\![\Box\!\!\Rightarrow]\!]_X(A, B) = \{\$ \in \mathsf{Sp}(X) \mid \exists S \in \$ \text{ s.t. } (S \cap A \neq \emptyset \text{ and } S \cap A \cap B^c = \emptyset)\}$. Our proof of soundness and completeness for $\mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}}$ over $\mathbb{V}_{\Box\Rightarrow}$ can be adapted to show that the rules are indeed one-step sound and cut-free complete with respect to the coalgebraic semantics. By [12] this induces a sequent system which is sound and complete with respect to $\mathbb{V}_{\mathcal{HC}(@)}$. In particular, backwards proof search in this system can be implemented in polynomial space.

Theorem 31. *Hybrid conditional logic* $\mathbb{V}_{\mathcal{HC}(@)}$ *is decidable in* PSPACE.

8 Conclusion

We presented the first unlabelled sequent systems for the conditional logics $\mathbb{V}, \mathbb{VN}, \mathbb{VT}$ and $\mathbb{VW}$ in the entrenchment and strong conditional languages and for $\mathbb{VC}$ in the strong conditional language. Since these systems have cut elimination and (after a slight modification) admissibility of contraction, backwards proof search can be implemented in polynomial space, giving the first purely syntactical PSPACE decision procedures for these logics. Furthermore, translating the weak conditional into our systems gives to our knowledge the first purely syntactical PSPACE decision procedures for the logics in the weak conditional language. All the algorithms are of optimal complexity. Moreover, we used our calculi to show that all the logics mentioned have the Craig interpolation property, and to give a PSPACE decision procedure for the hybrid version of $\mathbb{V}_{\Box\!\Rightarrow}$.

Related Work. Our calculus for $\mathbb{VC}_{\preccurlyeq}$ is the sequent version of the tableau calculus in [4, 2], but we also systematically cover weaker logics and different languages. The calculi in [8] for the weak conditional language are labelled and thus conceptually more involved, and not complexity optimal. In [1] a system for $\mathbb{V}_{\Box\!\rightarrow}$ involving second degree sequents is given, but it is not used for deciding the logic. The complexity results in [3] are obtained via small model theorems which complements our purely syntactical treatment. Calculi for the flat fragments of conditional logics corresponding to logics of the KLM framework are given in [7].

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