

# Coalgebraic Weak Bisimulation from Recursive Equations over Monads

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**Abstract.** Strong bisimulation for labelled transition systems is one of the most fundamental equivalences in process algebra, and has been generalised to numerous classes of systems that exhibit richer transition behaviour. Nearly all of the ensuing notions are instances of the more general notion of *coalgebraic bisimulation*. Weak bisimulation, however, has so far been much less amenable to a coalgebraic treatment. Here we attempt to close this gap by giving a coalgebraic treatment of (parametrized) weak equivalences, including weak bisimulation. Our analysis requires that the functor defining the transition type of the system is based on a suitable order-enriched monad, which allows us to capture weak equivalences by least fixpoints of recursive equations. Our notion is in agreement with existing notions of weak bisimulations for labelled transition systems, probabilistic and weighted systems, and simple Segala systems.

## 1 Introduction

Both strong and weak bisimulations are fundamental equivalences in process algebra [13]. Both have been adapted to systems with richer behaviour such as probabilistic and weighted transition systems. For each class of systems, strong bisimulation is defined in a similar way which is explained by universal coalgebra where strong bisimulation is recovered as a canonical equivalence that parametrically depends on the type of system [16]. Weak bisimulations are much more difficult to analyse even for labelled transition systems (LTS), and much less canonical in status (e.g. branching and delay bisimulations [21]).

We present a unified, coalgebraic treatment of various types of weak bisimulation. An important special (and motivating) case of our definition is *probabilistic weak bisimulation* of Baier and Herrmanns [2]. Unlike labelled transition systems, probabilistic weak bisimulation needs to account for *point-to-set* transitions, while *point-to-point* transitions, as for labelled transition systems, do not suffice: Every LTS with a transition relation  $\rightarrow$  induces an LTS with a *weak transition relation*  $\Rightarrow$  and weak bisimulation for the original system is strong bisimulation of the transformed one. This approach fails in the probabilistic case, as weak point-to-point transitions no longer form a probability distribution: in a system where  $x \xrightarrow{a(0.5)} y$  and  $x \xrightarrow{\tau(0.5)} x$ , we obtain  $x \xrightarrow{a(1)} y$  as the probability that  $x$  evolves to  $y$  along a trace of the form  $\tau^* \cdot a \cdot \tau^*$  is clearly one, but also  $x \xrightarrow{\tau(1)} x$  as the system will also evolve from  $x$  to  $x$  along  $\tau^*$  also with probability one

(by simply doing nothing). Crucially, both events are not independent. This is resolved by relating states to state sets along transition sequences, and the probability  $P(x, A, S)$  of  $x$  evolving to a state in  $S$  along a trace in  $A$  is the probability of the event that contains all execution sequences leading from  $x$  to  $S$  via  $A$ , called *total probability* in *op.cit.* By re-formulating this idea axiomatically, we show that it is applicable to a large class of systems, specifically coalgebras of the form  $X \rightarrow T(X \times A)$  where  $T$  is enriched over directed complete partial orders with least element (pointed dcpos) and non-strict maps. Not surprisingly, similar (but stronger) assumptions also play a prominent role in coalgebraic trace semantics [8], and have two ramifications: the fact that the functor  $T$  that describes the branching behaviour extends to a monad allows us to consider *transition sequences*, and order-enrichment permits us to compute the cumulative effect of (sets of) transition sequences recursively using Kleene’s fixpoint theorem. Our construction is parametric in an *observation pattern* that can be varied to obtain e.g. weak and delay bisimulation. We demonstrate by example that our definition generalises concrete definitions of probabilistic and weak weighted and probabilistic bisimulation found in the literature [2, 5, 18, 17].

A special role in our model is played by the operation of binary join, which is a continuous operation of the monad. We show that if it is also *algebraic* in the sense of Plotkin and Power [15], which holds in the case of LTS, then weak bisimulation can be recovered as a strong bisimulation for a system of the same type, thus reestablishing Milner’s weak transition construction. In the probabilistic case, for which join is unsurprisingly nonalgebraic, we show that weak bisimulation arises as strong bisimulation of a system based on the continuation monad.

## 2 Preliminaries

We use basic notions of category theory and coalgebra, see e.g. [16] for an overview. For a functor  $F : \mathbf{Set} \rightarrow \mathbf{Set}$ , an  $F$ -coalgebra is a pair  $(X, f)$  with  $f : X \rightarrow TX$ . Coalgebras form a category where the morphisms between  $(X, f)$  and  $(Y, g)$  are functions  $\phi : X \rightarrow Y$  with  $g \circ \phi = F\phi \circ f$ . A relation  $E \subseteq X \times X$  is a *kernel bisimulation* on  $(X, f)$  if there is an  $F$ -coalgebra  $(Z, h)$  and two morphisms  $\phi : (X, f) \rightarrow (Z, h)$  and  $\psi : (X, g) \rightarrow (Z, h)$  such that  $E = \text{Ker}(\phi) = \{(x, y) \in X \times X \mid \phi(x) = \psi(y)\}$  is the kernel of  $\phi$ . Clearly, kernel bisimulations are equivalence relations, and we only consider kernel bisimulations in what follows. Kernel bisimulation agrees with Aczel-Mendler bisimulation (and its variants) in case  $F$  preserves pullbacks weakly but is mathematically better behaved in case  $F$  does not. It also agrees (in all cases) with the notion of behavioural equivalence: a thorough comparison is provided in [20].

We take *monads* (on sets) as given by their extension form, i.e. as Kleisli triples  $\mathbb{T} = (T, \eta, -^\dagger)$  where  $T : \mathbf{Set} \rightarrow \mathbf{Set}$  is a functor,  $\eta_X : X \rightarrow TX$  is a map for all sets  $X$  and  $f^\dagger : TX \rightarrow TY$  is a map for all  $f : X \rightarrow TY$  subject to the equations  $f^\dagger \eta_X = f$ ,  $\eta_X^\dagger = \text{id}_{TX}$  and  $(f^\dagger g)^\dagger = f^\dagger g^\dagger$  for all sets  $X$  and all  $f, g$  of appropriate type. Throughout, we write  $T$  for the underlying functor of a monad  $\mathbb{T}$ . The *Kleisli category* induced by a monad  $\mathbb{T}$  has sets as objects, but Kleisli-morphisms between  $X$  and  $Y$  are functions  $f : X \rightarrow TY$  with Kleisli composition  $g \circ f = g^\dagger \circ f$  where  $g^\dagger \circ f$  is function composition in  $\mathbf{Set}$  and  $\eta_X$  is

the identity at  $X$ . We use Haskell-style **do**-notation to manipulate monad terms: for any  $p \in TX$  and  $q : X \rightarrow TY$  we write  $\mathbf{do} \ x \leftarrow p; q(x)$  to denote  $q^\dagger(p) \in TY$ ; if  $p \in T(X \times Y)$  we write  $\mathbf{do} \langle x, y \rangle \leftarrow p; q(x, y)$ .

In the sequel, we consider (among other examples) monads induced by semirings: A *semiring* is a structure  $(R, +, \cdot, 0, 1)$  such that  $(R, +, 0)$  is a commutative monoid,  $(R, \cdot, 1)$  is a monoid and multiplication distributes over addition, i.e.  $x \cdot (y + z) = x \cdot y + x \cdot z$  and  $(y + z) \cdot x = y \cdot x + z \cdot x$ . A *positively ordered semiring* is a semiring  $(R, +, \cdot, 0, 1, \leq)$  equipped with a partial order  $\leq$  that is positive ( $0 \leq r$  for all  $r \in R$ ) and compatible with the ring structure ( $x \leq y$  implies that  $x \square z \leq y \square z$  and  $z \square x \leq z \square y$  for all  $x, y, z \in R$  and  $\square \in \{+, \cdot\}$ ). A *continuous semiring* is a positively ordered semiring where every directed set  $D \subseteq R$  has a least upper bound  $\sup D \in R$  that is compatible with the ring structure ( $r \square \sup D = \sup\{r \square d \mid d \in D\}$  and  $\sup D \square r = \sup\{d \square r \mid d \in D\}$  for all directed sets  $D \subseteq R$ , all  $r \in R$ ) and  $\square \in \{+, \cdot\}$ . Every continuous semiring  $R$  is a *complete semiring*, i.e. has infinite sums given by  $\sum_{i \in I} r_i = \sup\{\sum_{i \in J} r_i \mid J \subseteq I \text{ finite}\}$ . We refer to [7] for details. If  $R$  is complete, the functor  $T_R X = X \rightarrow R$  extends to a monad  $\mathbb{T}_R$ , called the *complete semimodule monad* (c.f. [9]) with  $\eta_X(x)(y) = 1$  if  $x = y$  and  $\eta_X(x)(y) = 0$ , otherwise, and  $f^\dagger(\phi)(y) = \sum_{x \in X} \phi(x) \cdot f(x)(y)$  for  $f : X \rightarrow T_R Y$ . Note if  $R$  is continuous then all  $T_R X$  are pointed dcpos under the pointwise ordering of  $R$  and the same applies to Kleisli homsets, i.e. the set of Kleisli-maps of type  $X \rightarrow TY$ .

### 3 Examples

We illustrate our generic approach to weak bisimulation by means of the following examples. For all examples, strong bisimulation is well understood and known to coincide with kernel bisimulation. As we will see later, the same is true for weak bisimulation, introduced in the next section.

**Labelled Transition Systems.** We consider the monad  $\mathbb{T}_Q$  where  $Q = \{0, 1\}$  is the boolean semiring. Clearly  $T_Q \cong \mathcal{P}$  where  $\mathcal{P}$  is the covariant powerset functor. A *labelled transition system* can now be described as a coalgebra  $(X, f : X \rightarrow T_Q(X \times A))$ . It is well known that bisimulation equivalences on labelled transition systems coincide with kernel bisimulations as introduced in the previous section.

**Probabilistic Systems.** Consider the monad  $\mathbb{T}_{[0, \infty]}$  induced by the complete semiring of non-negative real numbers, extended with infinity. Various types of probabilistic systems arise as sub-classes of systems of type  $(X, f : X \rightarrow T_{[0, \infty]}(X \times A))$ . For *reactive systems*, one postulates  $\sum_{y \in X} f(x)(y, a) \in \{0, 1\}$  for all  $x \in X$  and all  $a \in A$ . *Generative systems* satisfy  $\sum_{(y, a) \in Y \times A} f(x)(y, a) \in \{0, 1\}$  for all  $x \in X$ , and *fully probabilistic systems* satisfy  $\sum_{(y, a) \in X \times A} f(x)(y, a) = 1$  for all  $x \in X$ . We refer to [3] for a detailed analysis of various types of probabilistic systems in coalgebraic terms. It is known that probabilistic bisimulation equivalence [10] and kernel bisimulations agree [6]. Our justification of viewing these various types of probabilistic systems as  $[0, \infty]$  weighted transition systems comes from the fact that kernel bisimulations are reflected by embeddings:

**Lemma 1.** *Let  $\kappa : F \rightarrow G$  be a monic natural transformation between two set-functors  $F$  and  $G$  and  $(X, f)$  be an  $F$ -coalgebra. Then kernel bisimulations on the  $F$ -coalgebra  $(X, f)$  agree with kernel bisimulations on the  $G$ -coalgebra  $(X, \kappa_X \circ f)$ .*

**Integer Weighted Transition Systems.** Weighted transition systems, much like probabilistic systems, arise as coalgebras for the functor  $FX = T_{\mathbb{N} \cup \{\infty\}}(X \times A)$  where  $\mathbb{N} \cup \{\infty\}$  is the (complete) semiring of natural numbers extended with  $\infty$  and the usual arithmetic operations. In an (integer) weighted transition system, every labelled transition comes with a weight, and we can write  $x \xrightarrow{a(n)} y$  if  $f(x)(y, a) = n$ . In process algebra, weights represent different ways in which the same transition can be derived syntactically, e.g.  $a.0 + a.0 \xrightarrow{a(2)} 0$ , according to the reduction of the term on the left and right, respectively. The ensuing (strong) notion of equivalence has been studied in [1] and shown to be coalgebraic.

The three examples above are a special instance of semiring-weighted transition systems, studied for instance in [11]. This is not the case for systems that combine probability and non-determinism.

**Non-Deterministic Probabilistic Systems.** As we have motivated in the introduction, a coalgebraic analysis of weak bisimulation hinges on the ability to sequence transitions, i.e. the fact that the functor  $F$  defining the concrete shape of a transition system  $(X, f : X \rightarrow F(X \times A))$  extends to a monad. The naive combination of probability and non-determinism, i.e. considering the functor  $F = \mathcal{P} \circ \mathcal{D}$  where  $\mathcal{D}(X)$  is the set of finitely distributed probability distributions does not extend to a monad [22]. One solution, discussed in *op.cit.* and elaborated in [9] is to restrict to *convex* sets of valuations. Informally, we use monad  $\mathcal{C}_0\mathcal{M}$  (a variant of the  $\mathcal{CM}$  monad from [9]), encompassing two semiring structures, for probability and non-determinism, and the former distributes over the latter, i.e.  $a +_p (b + c) = (a +_p b) + (a +_p c)$  where  $+$  is nondeterministic choice and  $+_p$  is probabilistic choice (choose ‘left’ with probability  $p$  and ‘right’ with probability  $1 - p$ ). Concretely, for the underlying functor  $C_0M$  of the monad  $\mathcal{C}_0\mathcal{M}$ ,  $C_0MX$  is the set of nonempty *convex* sets of finite valuations over  $[0, \infty)$ , i.e. finitely supported maps to  $[0, \infty)$ , containing the trivial valuation identically equal to 0. A set  $S$  is convex if  $\sum_i r_i \cdot \xi_i \in S$  whenever all  $\xi_i \in S$  and  $\sum_i r_i = 1$ . Our definition deviates slightly from [9] in that we require that  $C_0MX$  contains the zero valuation, whereas in *op.cit.* (and also in [4]) this condition is used to restrict the class of systems to which the theory is applied.

## 4 Weak Bisimulation, Coalgebraically

Capturing weak bisimulation for transition systems  $(X, f : X \rightarrow T(X \times A))$  coalgebraically, where  $A$  is a set of labels that we keep fixed throughout, amounts to two requirements: first,  $T$  needs to extend to a monad which enables us to sequence transitions. Second, we need to be able to compute the cumulative effect of transitions which requires the monad to be enriched over the category of directed complete partial orders (and non-strict morphisms).

**Definition 2 (Completely ordered monads).** A monad  $\mathbb{T}$  is *completely ordered* if its Kleisli category is enriched over the category  $\mathbf{DCPO}_\perp$  of directed-complete partial orders with least element (pointed dcpos) and continuous maps: every hom-set  $\mathbf{Set}(X, TY)$  is a pointed dcpo and Kleisli composition is continuous, i.e. the joins  $f^\dagger \circ (\bigsqcup_i g_i) = \bigsqcup_i f^\dagger \circ g_i$  and  $(\bigsqcup_i f_i)^\dagger \circ g = \bigsqcup_i f_i^\dagger \circ g$  exist and are equal whenever the join on the left hand side is taken over a directed set. A *continuous operation* of arity  $n$  on a completely ordered monad is a natural transformation  $\alpha : T^n \rightarrow T$  for which every component  $\alpha_X$  is Scott-continuous.

The diligent reader will have noticed that the same type of enrichment is also required in the coalgebraic treatment of trace semantics [8]. This is by no means a surprise, as the observable effect of weak transitions are precisely given in terms of (sets of) traces.

Often, these sets are defined in terms of weak transitions of the form  $\xrightarrow{a} \cdot \xrightarrow{\tau^*}$ . We think of weak transitions as transitions along trace sets closed under Brzozowski derivatives which enables us to recursively decompose a weak transition into a (standard) transition, followed by a weak transition.

**Definition 3 (Observation pattern).** An *observation pattern* over a set  $A$  of labels is a subset  $B \subseteq \mathcal{P}(A^*)$  that is closed under Brzozowski derivatives, i.e.  $b/a = \{w \in A^* \mid aw \in b\} \in B$  for all  $b \in B$  and all  $a \in A$ .

Different observation patterns capture different notions of weak bisimulation, and  $B$  stands for both bisimulation and closure under Brzozowski derivatives.

**Example 4 (Observation patterns).** Let  $A$  contain a silent action  $\tau$ .

- (i) the *strong pattern* over  $A$  is given by  $B = \{\{a\} \mid a \in A\} \cup \{\emptyset, \{\epsilon\}\}$ .
- (ii) the *weak pattern* over  $A$  is given by  $B = \{\hat{a} \mid a \in A\}$  where  $\hat{\tau} = \tau^*$  and  $\hat{a} = \tau^* \cdot a \cdot \tau^*$  for  $a \neq \tau$ .
- (iii) The *delay pattern* is  $B = \{\tau^* a \mid a \in A \setminus \{\tau\}\} \cup \{\tau^*\}$ .

It is immediate that all are closed under Brzozowski derivatives.

Given an observation pattern that determines the notion of traces, our definition of weak bisimulation relies on the fact that the cumulative effect of transitions can be computed recursively. This is ensured by enrichment, and we have the following (see Section 2 for the **do**-notation):

**Lemma 5.** *Suppose  $B$  is an observation pattern over  $A$ ,  $\mathbb{T}$  is a completely ordered monad and  $\oplus : T^2 \rightarrow T$  is continuous. Then the equation*

$$f_h^B(x)(b) = \left\{ \begin{array}{ll} \eta(h(x)) & \text{if } \epsilon \in b \\ \perp & \text{otherwise} \end{array} \right\} \oplus \mathbf{do} \langle y, a \rangle \leftarrow f(x); f_h^B(y)(b/a) \quad (\star)$$

has a unique least solution  $f_h^B : X \rightarrow (TY)^B$  for all  $f : X \rightarrow T(X \times A)$  and all  $h : X \rightarrow Y$ .

Lemma (5) follows from Kleene's fixpoint theorem [23] using order-enrichment. The central notion of our paper can now be given as follows:

**Definition 6.** Suppose that  $\mathbb{T}$  is a completely ordered monad with a continuous operation  $\oplus$ ,  $B$  is an observation pattern over  $A$  and let  $f : X \rightarrow T(X \times A)$ . An equivalence relation  $E \subseteq X \times X$  is a  $B$ - $\oplus$ -*bisimulation* if  $E \subseteq \text{Ker}(f_\pi^B)$  where  $\pi : X \rightarrow X/E$  is the canonical projection (and  $f_\pi^B$  is the unique least solution of  $(\star)$ ). We often elide the continuous operation, and say that  $x, x' \in X$  are  $B$ -bisimilar, if they are related by a  $B$ -bisimulation.

Some remarks are in order before we show that the above definition agrees with various notions of weak bisimulation studied in the literature.

**Remark 7.** (i) Intuitively, the requirement  $E \subseteq \text{Ker}(f_\pi^B)$  expresses that any two  $E$ -related states  $x$  and  $x'$  have the same cumulative behaviour under all trace sets in  $B$ , *provided* that  $E$ -related states are not distinguished. In other words, a state  $[x]_E$  of the quotient of the original system exhibits the same behaviour with respect to all trace sets in  $B$ , as the representative  $x$  of  $[x]_E$ . This intuition is made precise in Section 6 where we show how  $B$ -bisimulation can be recovered as strong bisimulation (and hence quotients can be constructed).

(ii) The definition of weak bisimulation above caters for systems of the form  $(X, f : X \rightarrow T(X \times A))$ , i.e. we implicitly consider the labels as part of the observable behaviour, or as 'output'. The role of labels appears to be reversed when computing the cumulative effect of transitions via the function  $f_\pi^B : X \rightarrow T(X/E)^B$ . This apparent reversal of roles is due to the fact that every element of  $B$  is a set of traces. Accordingly, the function application  $f_h^B(x)(b)$  represents the totality of behaviour that can be observed along traces in  $b$ , starting from  $x$ , and trace sets are now 'input'.

As a slogan,  $B$ -bisimilarity is a  $B$ -bisimulation:

**Lemma 8.** *Let  $(E_i)_{i \in I}$  be a family of  $B$ -bisimulation equivalences on  $(X, f : X \rightarrow T(X \times A))$ . Then so is the transitive closure of  $\bigcup_{i \in I} E_i$ .*

## 5 Examples, Revisited

We demonstrate that  $B$ -bisimulation agrees with the known (and expected) notion of weak bisimulation for the examples in Section 3. To instantiate the general definition to coalgebras of the form  $X \rightarrow T(X \times A)$ , we need to verify that the monad  $\mathbb{T}$  is completely ordered. This is the case for complete semimodule monads over continuous semirings.

**Lemma 9.** *Let  $R$  be a continuous semiring. Then the monad  $\mathbb{T}_R$  is completely ordered, and both join  $\sqcup$  and semiring sum  $+$  are continuous operations on  $T$ .*

This lemma in particular ensures that  $B$ -bisimulation is meaningful for transitions systems weighted in a complete semiring, and in particular for labelled, probabilistic and integer-weighted systems.

**Labelled transitions systems.** As in Section 3, labelled transition systems are coalgebras for the functor  $FX = \mathcal{P}(X \times A)$ . For an  $F$ -coalgebra  $(X, f)$ , Equation  $(\star)$  stipulates that

$$f_h(x)(b) = \{h(x) \mid \epsilon \in b\} \cup \bigcup_{x \xrightarrow{a} y} f_h(y)(b/a)$$

where  $x \xrightarrow{a} y$  iff  $\langle y, a \rangle \in f(x)$ . By Kleene's fixpoint theorem, the least solution is

$$f_h(x)(b) = \left\{ h(x_k) \mid x \xrightarrow{a_1} x_1 \xrightarrow{a_2} \dots \xrightarrow{a_k} x_k, \quad a_1 \cdots a_k \in b \right\}.$$

If  $\tau \in A$ ,  $B$  is the weak pattern and  $E \subseteq X \times X$  is an equivalence, this gives

$$[x']_E \in f_\pi^B(x)(\hat{a}) \quad \text{iff} \quad x \xrightarrow{\hat{a}} x'$$

where  $x \xrightarrow{\hat{a}} x'$  if there are  $(y_1, a_1), \dots, (y_n, a_n)$  such that  $x \xrightarrow{a_1} y_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} y_n = x'$  and  $a_1 \cdots a_n \in \hat{a}$ . By Definition 6,  $E$  is a  $B$ -bisimulation if for any  $\langle x, y \rangle \in E$ ,  $\{[x']_E \mid x \xrightarrow{\hat{a}} x'\} = \{[y']_E \mid y \xrightarrow{\hat{a}} y'\}$  for any  $a \in A$  (including  $\tau$ ). The latter is easily shown to be equivalent to the standard notion of weak bisimulation equivalence. By analogous reasoning one readily recovers delay bisimulation equivalences from the delay pattern.

**Probabilistic systems.** Fully probabilistic system (Section 3) are coalgebras of type  $(X, f : X \rightarrow T_{[0, \infty]}(X \times A))$ , where  $T_{[0, \infty]}$  is the complete semimodule monad induced by  $[0, \infty]$  and additionally satisfy  $\sum_{(y, a) \in X \times A} f(x)(y, a) = 1$  for all  $x \in X$ . In [2], an equivalence relation  $E \subseteq X \times X$  is a weak bisimulation, if

$$P(x, \hat{a}, [y]_E) = P(x', \hat{a}, [y]_E)$$

for all  $a \in A, y \in X$  and  $(x, x') \in E$ . Here  $\hat{a}$  is given as in Example 4 and  $P(x, \Lambda, C)$  is the *total probability* of the system evolving from state  $x$  to a state in  $C$  via a trace in  $\Lambda \subseteq A^*$ . *Op.cit.* states that total probabilities satisfy the recursive equations:  $P(x, \Lambda, C) = 1$  if  $\epsilon \in C$  and  $x \in \Lambda$ , and

$$P(x, \Lambda, C) = \sum_{(y, a) \in X \times A} f(x)(y, a) \cdot P(y, \Lambda/a, C)$$

otherwise. In fact, total probabilities are the *least* solution (with respect to the pointwise order on  $[0, \infty]$ ) of the recursive equations above.

**Lemma 10.** *Let  $(X, f : X \rightarrow T_{[0, \infty]}(X \times A))$  be a fully probabilistic system,  $B$  an observation pattern over  $A$  and  $E \subseteq X \times X$  an equivalence relation. If  $\pi : X \rightarrow X/E$  is the canonical projection, then  $P(x, b, [y]_E) = f_\pi^B(x)(b)([y]_E)$  for all  $x, y \in X$  and all  $b \in B$ , using  $\sqcup$  as continuous operation.*

As a corollary, we obtain that weak bisimulation of fully probabilistic systems is a special case of  $B$ -bisimulation for the weak pattern.

**Weighted transition systems.** Weighted transition systems are technically similar to probabilistic systems as they also appear as coalgebras for a (complete) semimodule monad, but without any restriction on the sum of weights. The associated notion of *weak resource bisimulation* is described syntactically in [5]. Abstracting from the concrete syntax and taking weighted transition systems as primitive, we are faced with a situation that is reminiscent of the probabilistic case: a weak resource bisimulation equivalence on a weighted transition system  $(X, f : X \rightarrow T_{\mathbb{N} \cup \infty}(X \times A))$  is an equivalence relation  $E \subseteq X \times X$  such that  $xEy$  and  $a \in A$  implies that  $W(x, \Lambda, C) = W(y, \Lambda, C)$  for all equivalence classes

$C \in X/E$  and all  $\Lambda$  that are of the form  $\tau^*a\tau^*$  for  $a \neq \tau$  and  $\tau^*$ . Here  $W(x, \Lambda, C)$  is the *total weight*, i.e. the maximal number of possibilities in which  $x$  can evolve into a state in  $C$  via a path from  $\Lambda$ . Total weights can be understood as (weighted) sums over all *independent* paths that lead from  $x$  into  $C$  via a trace in  $\Lambda$ , where two paths are independent if neither is a prefix of the other. Analogously to the probabilistic case, these weights are given by the least solution of the recursive equations

$$W(x, \Lambda, C) = \begin{cases} 1 & \epsilon \in \Lambda, x \in C \\ 0 & \text{otherwise} \end{cases} \sqcup \sum_{(y,a) \in X \times A} f(x)(y,a) \cdot W(y, \Lambda/a, C)$$

and represent the total number of possibilities in which a process  $x$  can evolve into a process in  $C$  along a trace in  $\Lambda$ . For example, we have that  $W(0 + \tau.0 + \tau.\tau.0, \tau^*, \{0\}) = 3$  representing the three different possibilities in which the given process can become inert along a  $\tau$ -trace, and  $W(x, \tau^*, z) = 6$  for the triangle-shaped system  $x \xrightarrow{\tau^{(2)}} y$ ,  $x \xrightarrow{\tau^{(2)}} z$  and  $y \xrightarrow{\tau^{(2)}} z$ . It is routine to check that  $W(x, b, [x']_E) = f_\pi^B(x)(b)([x']_E)$ . Unlike the probabilistic case, the number of different ways in which processes may evolve is strictly additive. For the weak pattern,  $B$ -bisimulation is therefore the semantic manifestation of weak resource bisimulation advocated in [5].

**Probability and nondeterminism.** Systems that combine probabilistic and nondeterministic behaviour arise as coalgebras of type  $(X, f : X \rightarrow C_0M(X \times A))$  where  $C_0M$  is the monad from Section 3. Systems of this type capture so-called Segala systems. Here we stick to simple Segala systems, which are colagebras of type  $\mathcal{P}(\mathcal{D} \times A)$  and for which the ensuing notion of weak probabilistic bisimulation was introduced in [18]. These systems extend probabilistic systems by additionally allowing non-deterministic transitions. As was essentially elaborated in [4], every simple Segala system embeds into a coalgebra  $(X, f : X \rightarrow C_0M(X \times A))$ .

Completing a simple Segala system to a coalgebra over  $C_0M$  amounts to forming *convex* sets of valuations; convexity arises from probabilistic choice as follows: given non-deterministic transitions  $x \rightarrow \xi$  and  $x \rightarrow \zeta$ , where  $\xi$  and  $\zeta$  are valuations over  $X \times A$  induces a transition  $x \rightarrow \xi +_p \zeta$  where  $+_p$  is probabilistic choice. Following [22], one way to understand this is to also consider non-deterministic choice  $+$  and to observe that

$$\xi + \zeta = (\xi + \zeta + \xi) +_p (\xi + \zeta + \zeta) = (\xi + \zeta) +_p (\xi + \zeta) + (\xi +_p \zeta) = \xi + \zeta + (\xi +_p \zeta)$$

by the axioms  $\xi + \xi = \xi +_p \xi = \xi$ ,  $(\xi + \zeta) +_p \theta = (\xi +_p \theta) + (\zeta +_p \theta)$ , the last one describing the interaction between probabilistic and non-deterministic choice.

We argue that  $B$ -bisimulation where  $B$  is the weak observation pattern agrees with the notion from [18, 17]. We make a forward reference to Theorem 16 which shows that  $B$ -bisimulation for  $(X, f)$  amounts to strong bisimulation for  $(X, f_{\text{id}}^B)$ . In other words, weak bisimilarity can be recovered from strong bisimilarity for the system whose transitions are weak transitions of the original system. Solving the recursive equation for  $f_{\text{id}}^B$  (where  $B$  is the weak pattern and we use the notation of Example 4) we can write  $x \xrightarrow{\hat{a}} \xi$  if  $\xi \in f_{\text{id}}^B(x)(a)$ . Intuitively, this

represents that  $x$  can evolve along a trace in  $\hat{a}$  to the valuation  $\xi$ , interleaving probabilistic and nondeterministic steps. We then obtain that an equivalence relation  $E \subseteq X \times X$  is a  $B$ -bisimulation if, whenever  $(x, y) \in E$  and  $x \xrightarrow{\hat{a}} \xi$ , there exists  $\zeta$  such that  $y \xrightarrow{\hat{a}} \zeta$  and  $\xi$  and  $\zeta$  are 'equivalent up to  $E$ ', that is,  $(F\pi)\xi = (F\pi)\zeta$  where  $\pi : X \rightarrow X/E$  is the projection and  $FX = [0, \infty)^X$ . More concretely, the weak relation  $\Rightarrow \in X \times B \times [0, \infty)^X$  is obtained by  $(\star)$  and is the least solution of the following system:

$$\begin{aligned} x &\xrightarrow{\hat{\tau}} \delta_x \\ x &\xrightarrow{\hat{a}} \zeta \quad \text{iff} \quad \exists \xi \in f(x). \zeta \in \left\{ \sum_{y \in X} \xi(y, a) \cdot \theta_y^\tau + \xi(y, \tau) \cdot \theta_y^a \mid \forall y. y \xrightarrow{\hat{b}} \theta_y^b \right\} \end{aligned}$$

where  $x \xrightarrow{\hat{b}} \zeta$  ( $b \in \{a, \tau\}$ ) abbreviates  $\langle x, b, \zeta \rangle \in \Rightarrow$ ;  $\delta_y(y') = 1$  if  $y = y'$  and  $\delta_y(y') = 0$  otherwise; and scalar multiplication and summation act on valuations pointwise. Kleene's fixpoint theorem underlying Lemma 5 ensures that the relation  $\Rightarrow$  can be calculated iteratively, i.e.  $\Rightarrow = \bigcup_i \Rightarrow_i$  where the  $\Rightarrow_i$  replace  $\Rightarrow$  in the above recursive equations in the obvious way, hence making them recurrent. Then  $x \xrightarrow{\hat{b}} \zeta$  iff there is  $i$  such that  $x \xrightarrow{\hat{b}}_i \zeta$ . The resulting definition in terms of weak transitions  $\Rightarrow_i$  matches weak probabilistic bisimulation from [18, 17]. Note that convexity of the monad precisely ensures that  $\xi$  in the recursive clause above for  $x \xrightarrow{\hat{a}} \zeta$  represents a *combined step* of the underlying Segala system, which by definition, is exactly a convex combination of ordinary probabilistic transitions.

## 6 Weak Bisimulation as Strong Bisimulation

Milner's weak transition construction characterises weak bisimilarity as bisimilarity for a (modified) system whose transitions are the weak transitions of the original system. This construction does not transfer to the general case, witnessed by the case of (fully) probabilistic systems. The pivotal role is played by the continuous operation  $\oplus$  that determines  $B$ -bisimulation. We show that Milner's construction generalises if  $\oplus$  is *algebraic* and present a variation of the construction if algebraicity fails. An *algebraic operation* of arity  $n$  on a monad  $\mathbb{T}$  (e.g. [15]) is a natural transformation  $\alpha : T^n \rightarrow T$  such that  $\alpha_Y \circ (f^\dagger)^n = f^\dagger \circ \alpha_X$  for all  $f : X \rightarrow TY$ . Algebraic operations are automatically continuous:

**Lemma 11.** *Algebraic operations of completely ordered monads are continuous.*

**Example 12 (Algebraic operations).** Semiring summation  $+$  is algebraic on continuous semimodule monads. If the underlying semiring is idempotent, e.g. the boolean semiring, summation coincides with the join operation  $\sqcup$  which is therefore also algebraic. The bottom element  $\perp$  is a nullary algebraic operation (constant). The join operation is algebraic on the monad  $\mathcal{C}_0\mathcal{M}$  from Section 3. The join operation  $\sqcup$  is generally not algebraic for free (complete) semimodule monads unless the semiring is idempotent.

Algebraicity of  $\oplus$  allows us to lift Milner’s construction to the coalgebraic case:  $B$ -bisimulations coincide with kernel bisimulations for a modified system of the *same transition type*. This instantiates to labelled transition systems, as  $\sqcup$  is algebraic on the semimodule monad induced by the boolean semiring. We show this using a sequence of lemmas, the first asserting that algebraic operations commute over fixpoints.

**Lemma 13.** *Suppose  $h : X \rightarrow Y$  and  $u : Y \rightarrow Z$ . Given a coalgebra  $(X, f : X \rightarrow T(X \times A))$  we have that  $f_{uh}^B = T^B u \circ f_h^B$  if  $\oplus$  is algebraic.*

Similarly, sans algebraicity,  $B$ -bisimulations commute with morphisms.

**Lemma 14.** *Let  $h : X \rightarrow Y$  be a morphism from  $(X, f : X \rightarrow T(X \times A))$  to  $(Y, g : Y \rightarrow T(Y \times A))$ . Then  $g_u^B \circ h = f_{uh}^B$  for all  $u : Y \rightarrow Z$ .*

Consequently, kernel bisimulations are  $B$ -bisimulations:

**Corollary 15.** *Let  $h : X \rightarrow Y$  be a morphism of coalgebras  $(X, f : X \rightarrow T(X \times A))$  and  $(Y, g : Y \rightarrow T(Y \times A))$ . Then  $\text{Ker } h \subseteq \text{Ker } f_h^B$ .*

Lemma 13 shows that for monads equipped with an algebraic operation  $\oplus$  (such as the monad defining) labelled transition systems, we can recover  $B$ -bisimilarity as strong bisimilarity of a transformed system.

**Theorem 16.** *Provided  $\oplus$  is algebraic,  $E$  is a  $B$ -bisimulation on a monad-type coalgebra  $(X, f)$  iff  $E$  is a kernel bisimulation equivalence on  $(X, f_{\text{id}}^B)$ .*

If  $\oplus$  is not algebraic it can still be possible to recover  $B$ -bisimulation as a kernel bisimulation for a system of a different type. For probabilistic systems this was done in [19]. Here, we obtain a similar result in a more conceptual way using the *continuous continuation monad*  $\mathbb{T}$ , which is obtained from the standard continuation monad [14] by restricting to continuous functions: the functorial part of  $\mathbb{T}$  is  $\mathbb{T}X = (X \rightarrow D) \rightarrow_c D$  where  $\rightarrow_c$  is the continuous function space,  $D$  is a directed-complete partial order, and  $(X \rightarrow D)$  is ordered pointwise.

**Lemma 17.** *For a pointed dcpo  $D$ ,  $\mathbb{T}X = (X \rightarrow D) \rightarrow_c D$  extends to a submonad  $\mathbb{T}$  of the corresponding continuation monad,  $\mathbb{T}$  is completely ordered, and every  $\oplus : T^2 \rightarrow T$ , given pointwise, i.e.  $(p \oplus q)(c) = p(c) \oplus q(c)$ , is algebraic.*

The following lemma is the  $B$ -bisimulation analogue of Lemma 1 and is the main technical tool for reducing  $B$ -bisimulation to kernel bisimulation.

**Lemma 18.** *Let  $(X, f : X \rightarrow T(X \times A))$  be a coalgebra and  $\kappa : T \rightarrow \widehat{T}$  an injective monad morphism. If  $\widehat{\oplus}$  is an algebraic operation on  $\widehat{T}$  such that  $\widehat{\oplus} \circ \kappa^2 = \kappa \circ \oplus$  then  $B$ - $\oplus$ -bisimulation equivalences on  $(X, f)$  and  $B$ - $\widehat{\oplus}$ -bisimulation equivalences on  $(X, \kappa f)$  agree.*

We use Lemma 18 as follows. Given a complete semimodule monad  $\mathbb{T}$  over a (complete) semiring  $R$ , we embed  $\mathbb{T}X$  into  $\widehat{\mathbb{T}}X = (X \rightarrow T1) \rightarrow_c T1$  (where  $T1 = R$ ) by mapping  $p \in \mathbb{T}X$  to the function  $\lambda c : X \rightarrow T1. c^\dagger(p)$ . This embedding is injective, and the conditions of Lemma 18 are fulfilled with  $\oplus = \sqcup$  and  $\widehat{\oplus}$  the pointwise extension of  $\oplus$  (which is algebraic by Lemma 17). This gives:

**Theorem 19.** *Let  $\mathbb{T}$  be a continuous semimodule monad over a continuous semiring  $R$ . Let  $(X, f : X \rightarrow (X \times A))$  be a coalgebra and let  $\oplus$  be the join on  $R$ . Then  $E$  is a  $B$ -bisimulation equivalence on  $(X, f)$  iff it is a bisimulation equivalence on  $(X, (\kappa_X \circ f)_{\text{id}}^B : X \rightarrow (X \times A \rightarrow_c R) \rightarrow R)$ .*

In summary, Milner’s weak transition construction generalises to the coalgebraic case if  $\sqcup$  is algebraic, and lifts to a different transition type for semirings.

## 7 Conclusions and Related Work

We have presented a generic definition, and basic structural properties, of weak bisimulation in a general, coalgebraic framework. We use coalgebraic methods and enriched monads, similar to the coalgebraic treatment of trace semantics [8]. Our definition applies uniformly to labelled transition systems, probabilistic and weighted systems, and to Segala systems from [18]. Most of our results, including the notions of  $B$ -bisimulation as a solution of the recursive equation ( $\star$ ), easily transfer to categories other than **Set**. An important conceptual contribution is the fact that algebraicity allows to generalise Milner’s weak transition construction to the coalgebraic setting (Theorem 16), recovering  $B$ -bisimulation as kernel bisimulation for a (modified) system of the same transition type. We also provide an alternative for cases where this fails (Theorem 19).

**Related work.** Results similar to ours are presented both in [4] and [12]. Brengos [4] uses a remarkably similar tool set (order-enriched monads) but in a substantially different way: Given a system of type  $T(F + -)$  with  $\mathbb{T}$  order-enriched, the monad structure on  $\mathbb{T}$  extends to  $T(F + -)$ , and saturation w.r.t. internal transitions is achieved by iterating the obtained monad in a way resembling the weak transition construction for LTS. Examples include labelled transition systems and (simple) Segala systems. For both underlying monads, join is algebraic, so that both examples are covered by our lifting Theorem 16. Fully probabilistic systems, for which algebraicity fails, are not treated in [4]. Miculan and Peresotti [12] also approach weak bisimulation by solving recurrence relations, but only treat (continuous) semimodule monads and do not account for (simple) Segala systems. Our treatment covers all examples considered in both [4] and [12], and additionally identifies the pivotal role of algebraicity in the generalisation of Milner’s construction. Sokolova et.al. [19] are concerned with probabilistic systems only and reduce probabilistic weak bisimulation to strong (kernel) bisimulation for a system of type  $(- \times A \rightarrow 2) \rightarrow [0, 1]$ . This is similar to our Theorem 19, which establishes an analogous transformation (to a system of type  $(- \times A \rightarrow [0, \infty]) \rightarrow [0, \infty])$  by a rather more high-level argument.

**Future work.** We plan to investigate to what extent our treatment extends to coalgebras  $X \rightarrow T(X + FX)$  for a monad  $\mathbb{T}$  (the branching type) and a functor  $F$  (the transition type) and are interested in both a logical and an equational characterisation of  $B$ -bisimulation, and in algorithms to compute  $B$ -bisimilarity.

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