Asymptotic Enumeration of Tournaments Containing a Specified Digraph

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Abstract

This paper studies the probability that a random tournament with specified score sequence contains a specified subgraph. The exact asymptotic value is found in the case that the scores are not too far from regular and the subgraph is not too large. An n-dimensional saddle-point method is used. As a sample application, we

prove that almost all tournaments with a given score sequence (not too far from regular) have a trivial automorphism group.

1 Introduction

A *tournament* is a digraph such that between every pair of vertices there is exactly one arc. Throughout this paper, we fix the vertex set to be $V = \{1, 2, \ldots, n\}$. Let d_j^-, d_j^+ be the in-degree and out-degree of vertex j in a tournament. Define $\delta_j = d_j^+ - d_j^-$ and call $\delta_1, \delta_2, \ldots, \delta_n$ the *excess sequence* of the tournament. Let $\delta = \max\{|\delta_1|, \ldots, |\delta_n|\}$.

Let H be a digraph with vertex set V and arc set $A(H)$ such that between every pair of distinct vertices there is at most one arc. We use $d_j^+(H)$ and $d_j^-(H)$ to denote the out-degree and in-degree, respectively, of vertex j in H. Define $\delta_j(H) = d_j^+(H) - d_j^-(H)$, $d_j(H) = d_j^+(H) + d_j^-(H)$, $\delta(H) = \max\{|\delta_1(H)|, \ldots, |\delta_n(H)|\}$, and $d(H) = \max\{d_1(H), \ldots, d_n(H)\}$.

Let $T(H; \delta_1, \ldots, \delta_n)$ be the number of tournaments that contain a specified digraph H and have excess sequence $\delta_1, \ldots, \delta_n$. As special cases, we have $T(\delta_1, \delta_2, \ldots, \delta_n)$ to denote the number of all tournaments that have excess sequence $\delta_1, \ldots, \delta_n$, and $T(n) = T(0, 0, \ldots, 0)$ to denote the number of labelled regular tournaments with n vertices.

Spencer [1] evaluated $T(n)$ to within a factor of $(1 + o(1))^n$ and obtained the estimate

$$T(\delta_1, \ldots, \delta_n) = T(n) \exp\Big(\big(-\tfrac{1}{2} + o(1)\big) \sum_{j=1}^{n} \delta_j^2 / n \Big)$$

for tournaments close to regular. The asymptotic value of $T(n)$ was obtained much later by McKay [2], who showed that

$$T(n) \sim \left(\frac{2^{n+1}}{n\pi} \right)^{(n-1)/2} \left(\frac{n}{e} \right)^{1/2} \qquad (n \text{ odd}).$$

Recently, McKay and Wang [3] obtained the asymptotic value of $T(\delta_1, \ldots, \delta_n)$ for $\delta = o(n^{3/4})$. The following is an immediate consequence of [3, Theorem 4.4].

Theorem 1 *Suppose $\delta = o(n^{2/3})$. Then*

$$T(\delta_1, \ldots, \delta_n) \sim n^{1/2} \left(\frac{2^{n+1}}{n\pi} \right)^{(n-1)/2} \exp\left(-\frac{1}{2} - \left(\frac{1}{2n} - \frac{1}{n^2} \right) \sum_{j=1}^{n} \delta_j^2 - \frac{1}{12n^3} \sum_{j=1}^{n} \delta_j^4 - \frac{1}{4n^4} \left(\sum_{j=1}^{n} \delta_j^2 \right)^2 \right).$$

McKay suggested that a similar argument can be used to obtain asymptotics for $T(H; \delta_1, \ldots, \delta_n)$. We carry out this task in this paper. To simplify the analysis, we shall restrict ourselves in the range $\delta = o(n^{2/3})$ and $d(H) = O(n^{1/2-\epsilon'})$, where ϵ' is any positive constant.

For a given digraph H and a given excess sequence $\delta_1, \delta_2, \ldots, \delta_n$, define

$$\begin{aligned}\beta_1 &= \frac{1}{2n} \sum_{1\le j\le n} \big(2\delta_j \delta_j(H) - \delta_j^2(H)\big) + \frac{1}{3n^3} \sum_{1\le j\le n} \delta_j^3 \delta_j(H) \\ &\quad + \frac{1}{n^4} \sum_{1\le j\le n} \delta_j^2 \sum_{1\le j\le n} \delta_j \delta_j(H), \end{aligned} \tag{1}$$

and

$$\beta_2 = -\frac{1}{2n^2} \sum_{(j,k)\in A(H)} \big(\delta_j - \delta_k - \delta_j(H) + \delta_k(H)\big)^2.$$

We shall prove

Theorem 2 *Suppose $\delta = o(n^{2/3})$, $d(H) = O(n^{1/2-\epsilon})$ and $d(H)\,\delta = o(n)$, where ϵ is any positive constant. Then*

$$T(H; \delta_1, \delta_2, \ldots, \delta_n)/T(\delta_1, \delta_2, \ldots, \delta_n) \sim 2^{-m} \exp(m/n + \beta_1 + \beta_2)$$

uniformly as $n \to \infty$.

The rest of the paper is organized as follows. In Section 2 we derive the asymptotic value of an integral. In Section 3, we use Cauchy's Theorem to represent $T(H; \delta_1, \delta_2, \ldots, \delta_n)$ as an integral and then apply the saddle point method and the results from Section 2 to obtain Theorem 2. In Section 4, we discuss several consequences of Theorem 2.

2 An Integral

In this section, we approximate the value of an n-dimensional integral we will need later. Define

$$U_n(t) = \big\{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \;\big|\; |x_i| \le t, i = 1, 2, \ldots, n\big\}$$

Lemma 1 *Let E, F and $0 < \epsilon < 1/20$ be constants and let $A_{jk}(n)$, $B_k(n)$, $C_{jk}(n)$, $D_{jk}(n)$, $\alpha_j(n)$ be real-valued functions.*
Suppose

(i)

$$\sum_{j=1}^{n-1} \big(|A_{jk} + A_{kj}|\big) = O(n^{1/2-3\epsilon})$$

uniformly for all $1 \le k \le n-1$, and

(ii) $B_k(n) = O(n^{-4\epsilon})$, $C_{j,k}(n) = O(n^{-4\epsilon})$, $D_{jk}(n) = O(n^{1/2-5\epsilon})$, $\alpha_j(n) = O(n^{-1/2-6\epsilon})$ *uniformly for* $1 \le j,k \le n-1$.

Define

$$\begin{aligned} f(\boldsymbol{x}) &= \exp\biggl(-\frac{1}{2}(n-1)\sum_{1\le j\le n-1}\bigl(1-\alpha_j(n)\bigr)x_j^2 + \frac{1}{2}\sum_{j\ne k} x_j x_k + nE\sum_{1\le k\le n-1} x_k^4 \\ &\qquad + F\Bigl(\sum_{1\le k\le n-1} x_k^2\Bigr)^2 + \sum_{j\ne k} A_{jk}(n)x_jx_k + i\,n\sum_{1\le k\le n-1} B_k(n)x_k^3 \\ &\qquad + i\sum_{j\ne k} C_{jk}(n)x_k^2x_j + \sum_{j\ne k} D_{jk}(n)x_k^3x_j + o(1)\biggr), \end{aligned}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_{n-1})$. *Then*

$$\int_{U_{n-1}(n^{-1/2+\epsilon})} f(\boldsymbol{x})\,d\boldsymbol{x} = n^{1/2}\Bigl(\frac{2\pi}{n}\Bigr)^{(n-1)/2}\exp\biggl(\sum_{1\le j\le n-1}\alpha_j(n)/2 + 3E + F + o(1)\biggr).$$

Proof. Let I be the above integral and define the linear transformation T_1 :

$$z_j = \bigl(1-\alpha_j(n)\bigr)^{1/2}x_j, \quad 1\le j\le n-1.$$

Let V_1 be the image of $U_{n-1}(n^{-1/2+\epsilon})$ under T_1. It is clear that V_1 is between $U_{n-1}(n^{-1/2+\epsilon/2})$ and $U_{n-1}(n^{-1/2+2\epsilon})$, and

$$\begin{aligned} I &= \prod_{1\le j\le n-1}(1-\alpha_j(n))^{-1/2} \times \int_{V_1}\exp\Bigl(-\tfrac{1}{2}(n-1)\sum z_j^2 + \tfrac{1}{2}\sum_{j\ne k} z_jz_k \\ &\qquad + nE\sum z_j^4 + F\Bigl(\sum z_j^2\Bigr)^2 + \sum_{j\ne k} A'_{jk}(n)z_jz_k + i\,n\sum_{1\le k\le n-1} B_k(n)z_k^3 \\ &\qquad + i\sum_{j\ne k} C_{jk}(n)z_k^2z_j + \sum_{j\ne k} D_{jk}(n)z_k^3z_j + o(1)\Bigr)d\boldsymbol{z}, \end{aligned}$$

where A'_{jk} satisfies the same conditions as A_{jk}.

Next we perform a second linear transformation T_2 to diagonalise the major quadratic terms of the integrand:

$$z_j = y_j - \beta\mu_1, \quad 1\le j\le n-1,$$

where $\beta = 1/(\sqrt{n}+1)$ and $\mu_m = \sum_{j=1}^{n-1} y_j^m$ for any m. Let V_2 be the image of V_1 under T_2. It is easily determined that the determinant of T_2 is $\sqrt{n}$, and so

$$I = \sqrt{n}\prod_{1\le j\le n-1}(1-\alpha_j(n))^{-1/2} \times \int_{V_2}\exp\Bigl(-\tfrac{1}{2}n\mu_2 + nE\mu_4 + F\mu_2^2$$

$$
\begin{aligned}
&+\sum_{j\neq k} A'_{jk}(n) y_j y_k + i\,n \sum_{1\le k\le n-1} B_k(n) y_k^3 \\
&+ i \sum_{j\neq k} C_{jk}(n) y_k^2 y_j + \sum_{j\neq k} D_{jk}(n) y_k^3 y_j + o(1)\Bigg) d\boldsymbol{y},
\end{aligned}
$$

The region of integration V_2 is somewhat irregular, but by the same method as used in [2, Theorem 2.1], we can see that it can be replaced by $U_{n-1}(n^{-1/2+\epsilon})$ with negligible change of value.

Finally we use an average technique introduced in [4, Lemma 3] to show that some unsymmetrical terms are negligible. Let $f_0(\boldsymbol{y}) = -\frac{1}{2}n\mu_2 + nE\mu_4 + F\mu_2^2$ and let $f(\boldsymbol{y})$ be the integrand of the previous integral.

For $1 \le m \le n$, define

$$
\begin{aligned}
\psi_m(\boldsymbol{y}) \;=\; \exp\Bigg(& f_0(\boldsymbol{y}) + \sum_{k=m}^{n-1}\sum_{j=m}^{n-1} A'_{jk}(n) y_j y_k + i\,n \sum_{k=m}^{n-1} B_k(n) y_k^3 \\
& + i \sum_{k=1}^{n-1}\sum_{j=m}^{n-1} C_{jk}(n) y_k^2 y_j + \sum_{k=m}^{n-1}\sum_{j=m}^{n-1} D_{jk}(n) y_k^3 y_j \Bigg),
\end{aligned}
$$

where $A'_{jj}(n)$, $C_{jj}(n)$ and $D_{jj}(n)$ are interpreted as zero for $1 \le j \le n-1$. Then

$$
\psi_1(\boldsymbol{y}) = f(\boldsymbol{y}) \exp(o(1)), \quad \psi_n(\boldsymbol{y}) = \exp(f_0(\boldsymbol{y})),
$$

and

$$
\psi_m(\boldsymbol{y}) = \psi_{m+1}(\boldsymbol{y}) \exp(Z),
$$

with

$$
\begin{aligned}
Z \;=\; & \sum_{j=m}^{n-1} \big(A'_{jm}(n) + A'_{mj}\big) y_j y_m + i\,n B_m(n) y_m^3 \\
& + i \sum_{k=1}^{n-1} C_{mk} y_k^2 y_m + \sum_{j=m}^{n-1} D_{jm} y_m^3 y_j + \sum_{k=m}^{n-1} D_{mk} y_m y_j^3.
\end{aligned}
$$

Define

$$
\bar{\psi}_m(\boldsymbol{y}) = \tfrac{1}{2}\big(\psi_m(\boldsymbol{y}) + \psi_m(y_1, \ldots, y_{m-1}, -y_m, y_{m+1}, \ldots, y_{n-1})\big).
$$

Since $U_{n-1}(n^{-1/2+\epsilon})$ is symmetric about the origin, we have

$$
\int_{U_{n-1}(n^{-1/2+\epsilon})} \bar{\psi}_m(\boldsymbol{y})\, d\boldsymbol{y} = \int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_m(\boldsymbol{y})\, d\boldsymbol{y}.
$$

Using

$$(e^Z + e^{-Z})/2 = \exp\bigl(O(Z^2)\bigr)$$

we obtain, for $\boldsymbol{y} \in U_{n-1}(n^{-1/2+\epsilon})$, that

$$\bar{\psi}_m(\boldsymbol{y}) = \psi_{m+1}(\boldsymbol{y}) \exp\bigl(O(n^{-1-2\epsilon})\bigr)$$

uniformly over m, and hence

$$\begin{aligned} \biggl| \int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_m(\boldsymbol{y})\, d\boldsymbol{y} - \int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_{m+1}(\boldsymbol{y})\, d\boldsymbol{y} \biggr| \\ = \exp\bigl(O(n^{-1-2\epsilon})\bigr) \int_{U_{n-1}(n^{-1/2+\epsilon})} |\psi_{m+1}(\boldsymbol{y})|\, d\boldsymbol{y}. \end{aligned}$$

Applying the same argument to $|\psi_m(\boldsymbol{y})|$, we obtain

$$\int_{U_{n-1}(n^{-1/2+\epsilon})} |\psi_m(\boldsymbol{y})|\, d\boldsymbol{y} = \exp\bigl(O(n^{-1-2\epsilon})\bigr) \int_{U_{n-1}(n^{-1/2+\epsilon})} |\psi_{m+1}(\boldsymbol{y})|\, d\boldsymbol{y}.$$

Therefore

$$\int_{U_{n-1}(n^{-1/2+\epsilon})} |\psi_m(\boldsymbol{y})|\, d\boldsymbol{y} = \exp\bigl(O(n^{-2\epsilon})\bigr) \int_{U_{n-1}(n^{-1/2+\epsilon})} |\psi_n(\boldsymbol{y})|\, d\boldsymbol{y},$$

and finally

$$\begin{aligned} \biggl| \int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_1(\boldsymbol{y})\, d\boldsymbol{y} - \int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_n(\boldsymbol{y})\, d\boldsymbol{y} \biggr| \\ = \exp\bigl(O(n^{-2\epsilon})\bigr) \int_{U_{n-1}(n^{-1/2+\epsilon})} |\psi_n(\boldsymbol{y})|\, d\boldsymbol{y}. \end{aligned}$$

Putting these results together, we find that

$$I \sim \sqrt{n} \prod_{1 \le j \le n-1} \bigl(1 - \alpha_j(n)\bigr)^{-1/2} \times \int_{U_{n-1}(n^{-1/2+\epsilon})} f_0(\boldsymbol{y})\, d\boldsymbol{y},$$

which is covered by [3, Theorem 2.1]. This gives the desired result on noting that

$$\prod_{1 \le j \le n-1} \bigl(1 - \alpha_j(n)\bigr)^{-1/2} \sim \exp\Biggl(\tfrac{1}{2} \sum_{1 \le j \le n-1} \alpha_j \Biggr). \qquad \blacksquare$$

3 Proof of Theorem 2

Throughout this section, we assume $\delta = o(n^{2/3})$, $d(H) = O(n^{1/2-\epsilon'})$, $d(H)\,\delta = o(n)$, and that $\epsilon' < 1/100$ is a positive constant.

For a given digraph H, define $\chi_{jk} = 1$ if $(j,k) \in A(H)$, and $\chi_{jk} = 0$ otherwise. Also define $\gamma_j = \delta_j - \delta_j(H)$. Let

$$G(\boldsymbol{x}) = G(x_1, x_2, \ldots, x_n) = \prod_{1\le j<k\le n} (x_j^{-1}x_k + x_j x_k^{-1}) \prod_{(j,k)\in A(H)} \frac{x_j x_k^{-1}}{x_j x_k^{-1} + x_j^{-1}x_k}.$$

Then $T(H;\delta_1,\ldots,\delta_n)$ is the coefficient of $x_1^{\delta_1}\cdots x_n^{\delta_n}$ in $G(\boldsymbol{x})$. Setting $x_j = r_j \exp(i\theta_j)$, we have by Cauchy's Theorem that

$$T(H;\delta_1,\ldots,\delta_n) = (2\pi)^{-n} \prod_{1\le j\le n} r_j^{-\delta_j} \int_{U_n(\pi)} G(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) \exp\Bigl(-i \sum_{1\le j\le n} \delta_j\theta_j\Bigr)\, d\boldsymbol{\theta}.$$

Define

$$\begin{aligned} T_{jk}(\boldsymbol{\theta}) &= \frac{r_j^2 \exp\bigl(i(\theta_j-\theta_k)\bigr) + r_k^2 \exp\bigl(i(\theta_k-\theta_j)\bigr)}{r_j^2 + r_k^2}, \\ g(\boldsymbol{\theta}) &= \exp\Bigl(-i \sum_{1\le j\le n} (\delta_j\theta_j)\Bigr) \prod_{1\le j<k\le n} T_{jk}(\boldsymbol{\theta}) \prod_{(j,k)\in A(H)} e^{i(\theta_j-\theta_k)}/T_{jk}(\boldsymbol{\theta}), \end{aligned}$$

and

$$I = \int_{U_n(\pi/2)} g(\boldsymbol{\theta})\, d\boldsymbol{\theta}. \tag{2}$$

Since $g(\boldsymbol{\theta})$ is invariant under the translation of any θ_j by π, we obtain

$$\begin{aligned} T(H;\delta_1,\ldots,\delta_n) &= \pi^{-n} I \prod_{1\le j\le n} r_j^{-\delta_j} \prod_{1\le j<k\le n} (r_j/r_k + r_k/r_j) \\ &\qquad \times \prod_{(j,k)\in A(H)} r_j^2/(r_j^2 + r_k^2). \end{aligned} \tag{3}$$

Since the integrand is invariant under a uniform translation of θ_j by θ_n, and $\sum_{j=1}^n \delta_j = 0$, we have

$$I = \pi \int_{U_{n-1}(\pi/2)} g(\theta_1, \theta_2, \ldots, \theta_{n-1}, 0)\, d\boldsymbol{\theta}',$$

where $\boldsymbol{\theta}' = (\theta_1, \ldots, \theta_{n-1})$. For a positive constant ϵ satisfying $\epsilon < \epsilon'/6$, let I_1 be the contribution to I from $\boldsymbol{\theta} \in U_{n-1}(n^{-1/2+\epsilon})$. As in [3], we first estimate I_1 and then show that $I_1 \sim I$. In the following analysis, we shall assume $\boldsymbol{\theta}' \in U_{n-1}(n^{-1/2+\epsilon})$ and $\theta_n = 0$. To apply the saddle point method, it is convenient to choose $r_j = \sqrt{(1+b_j)/(1-b_j)}$, where

$$b_j = \gamma_j/n + d_j(H)\gamma_j/n^2 - \sum_{k=1}^{n}(\chi_{jk}+\chi_{kj})\gamma_k/n^2 + \gamma_j \sum_{k=1}^{n} \gamma_k^2/n^4. \tag{4}$$

It is important to note that $b_j = \gamma_j/n + o(1/n) = o(n^{-1/3})$ and $\sum_{1\le j\le n} b_j = 0$. Let

$$a_{jk} = (r_j^2 - r_k^2)/(r_j^2 + r_k^2) = (b_j - b_k)/(1 - b_j b_k). \tag{5}$$

Using Taylor expansion, we have, for $\boldsymbol{\theta}' \in U_{n-1}(n^{-1/2+\epsilon})$, that

$$\begin{aligned} T_{jk}(\boldsymbol{\theta}) &= \exp\Big(ia_{jk}(\theta_j-\theta_k) + (-\tfrac{1}{2}+\tfrac{1}{2}a_{jk}^2)(\theta_j-\theta_k)^2 \\ &\qquad + \tfrac{1}{3}a_{jk}i(\theta_j-\theta_k)^3 - \tfrac{1}{12}(\theta_j-\theta_k)^4 + O(n^{-2-\epsilon})\Big). \end{aligned} \tag{6}$$

Noting that $\sum_{k\ge 1}(\chi_{jk}+\chi_{kj}) = d_j(H) = O(n^{1/2-\epsilon'})$ and expanding the powers of $\theta_j - \theta_k$, we obtain

$$\begin{aligned} g(\boldsymbol{\theta}) &= \exp\Bigg(i\sum_{1\le j\le n}\Bigg(\sum_{1\le k\le n} a_{jk} - \delta_j + \delta_j(H) - \sum_{1\le k\le n}(\chi_{jk}+\chi_{kj})a_{jk}\Bigg)\theta_j \\ &\qquad + \sum_{1\le j\le n-1}\Bigg(-\tfrac{1}{2}(n-1) + \sum_{1\le k\le n} a_{jk}^2/2 + d_j(H)/2\Bigg)\theta_j^2 \\ &\qquad + \tfrac{1}{2}\sum_{j\ne k}\theta_j\theta_k + \sum_{j\ne k}\big(-a_{jk}^2/2 - \chi_{jk}(1-a_{jk}^2)\big)\theta_j\theta_k \\ &\qquad + i\,n\sum_{1\le j\le n-1} O(n^{-1/3})\theta_j^3 + i\sum_{j\ne k} O(n^{-1/3})\theta_j\theta_k^2 \\ &\qquad - \tfrac{1}{12}n\sum_{1\le j\le n-1}\theta_j^4 - \tfrac{1}{4}\Bigg(\sum_{1\le j\le n-1}\theta_j^2\Bigg)^2 + \sum_{j\ne k}\theta_j\theta_k^3 + o(1)\Bigg). \end{aligned}$$

Using (4), (5) and the comment after (4), we have

$$\sum_{1\le k\le n} a_{jk} = \gamma_j + d_j(H)\gamma_j/n - \sum_{1\le k\le n}(\chi_{jk}+\chi_{kj})\gamma_k/n + o(n^{-2/3})$$

and

$$\sum_{1\le k\le n}(\chi_{jk}+\chi_{kj})a_{jk} = d_j(H)\gamma_j/n - \sum_{1\le k\le n}(\chi_{jk}+\chi_{kj})\gamma_k/n + o(n^{-2/3}),$$

and hence

$$\begin{aligned} g(\boldsymbol{\theta}) \;=\; & \exp\biggl(\sum_{1\le j\le n-1} \Bigl(-\tfrac{1}{2}(n-1) + \sum_{1\le k\le n} a_{jk}^2/2 + d_j(H)/2 \Bigr)\theta_j^2 \\ & + \tfrac{1}{2}\sum_{j\ne k}\theta_j\theta_k + \sum_{j\ne k}\bigl(-a_{jk}^2/2 - \chi_{jk}(1-a_{jk}^2)\bigr)\theta_j\theta_k \\ & + i\,n \sum_{1\le j\le n-1} O(n^{-1/3})\theta_j^3 + i\sum_{j\ne k} O(n^{-1/3})\theta_j\theta_k^2 \\ & - \tfrac{1}{12}n \sum_{1\le j\le n-1}\theta_j^4 - \tfrac{1}{4}\Bigl(\sum_{1\le j\le n-1}\theta_j^2\Bigr)^2 + \sum_{j\ne k}\theta_j\theta_k^3 + o(1)\biggr). \end{aligned} \qquad (7)$$

Applying Lemma 1 and using

$$\frac{1}{2(n-1)} \sum_{1\le j\le n-1}\ \sum_{1\le k\le n} a_{jk}^2 = \frac{1}{n^2}\sum_{1\le j\le n}\delta_j^2 + o(1)$$

and

$$\frac{1}{2(n-1)} \sum_{1\le j\le n-1} d_j(H) = \frac{m}{n} + o(1),$$

we obtain

$$I_1 \sim \pi n^{1/2}\Bigl(\frac{2\pi}{n}\Bigr)^{(n-1)/2} \exp\Bigl(m/n + \sum_{1\le j\le n}\delta_j^2/n^2 - 1/2\Bigr). \qquad (8)$$

The proof of the fact that the contribution to I from the region other than that of I_1 is negligible is essentially the same as that of [3] and will be omitted.

Using (4) and (5) with some calculation, we obtain

$$\begin{aligned} & \prod_{1\le j<k\le n}(r_j/r_k + r_k/r_j) \prod_{1\le j\le n} r_j^{-\delta_j} \\ & = 2^{n(n-1)/2}\exp\biggl(-\frac{1}{2n}\sum_{1\le j\le n}\delta_j^2 - \frac{1}{12n^3}\sum_{1\le j\le n}\delta_j^4 - \frac{1}{4n^4}\Bigl(\sum_{1\le j\le n}\delta_j^2\Bigr)^2 \\ & \qquad + \frac{1}{n^2}\sum_{(j,k)\in A(H)}\bigl(\delta_j(H)\gamma_k + \delta_k(H)\gamma_j\bigr) \\ & \qquad + \frac{1}{2n}\sum_{1\le j\le n}\delta_j^2(H) - \frac{1}{n^2}\sum_{1\le j\le n} d_j(H)\delta_j(H)\gamma_j + o(1)\biggr) \end{aligned} \qquad (9)$$

and

$$
\begin{aligned}
&\prod_{(j,k)\in A(H)} r_j^2/(r_j^2+r_k^2) \\
&\quad = 2^{-m}\exp\Bigg(\frac{1}{n}\sum_{1\le j\le n}\delta_j(H)\gamma_j + \frac{1}{n^2}\sum_{1\le j\le n} d_j(H)\delta_j(H)\gamma_j \\
&\qquad + \frac{1}{3n^3}\sum_{1\le j\le n}\delta_j(H)\delta_j^3 + \frac{1}{n^4}\sum_{1\le j\le n}\delta_j(H)\delta_j\sum_{1\le j\le n}\delta_j^2 \\
&\qquad - \frac{1}{n^2}\sum_{(j,k)\in A(H)}\big((\gamma_j-\gamma_k)^2/2 + \delta_j(H)\gamma_k + \delta_k(H)\gamma_j\big) + o(1)\Bigg). \qquad (10)
\end{aligned}
$$

Now Theorem 2 follows from (3) and (8)–(10).

4 Consequences

From Theorem 2, we see that $T(H,\delta_1,\delta_2,\dots,\delta_n)$ usually depends on the structure of the digraph H. However, it can have much simpler form in some special cases. Noting that

$$\beta_1 = \frac{1}{2n}\sum_{1\le j\le n}\big(2\delta_j(H)\delta_j - \delta_j^2(H)\big) + o(1), \quad \beta_2 = -\frac{1}{2n^2}\sum_{(j,k)\in A(H)}(\delta_j-\delta_k)^2 + o(1)$$

when

$$\sum_{1\le j\le n}|\delta_j(H)| = O(n),$$

we obtain the following two corollaries.

Corollary 1 *Suppose $\delta = o(n^{2/3})$, $d(H) = O(n^{1/2-\epsilon})$, $d(H)\delta = o(n)$ and $\sum_{1\le j\le n}|\delta_j(H)| = O(n)$. Then*

$$
\begin{aligned}
T(H;\delta_1,\delta_2,\dots,\delta_n)/T(\delta_1,\delta_2,\dots,\delta_n) \;\sim\; & 2^{-m}\exp\Bigg(\frac{m}{n} + \frac{1}{2n}\sum_{1\le j\le n}\big(2\delta_j(H)\delta_j - \delta_j^2(H)\big) \\
& - \frac{1}{2n^2}\sum_{(j,k)\in A(H)}(\delta_j-\delta_k)^2\Bigg).
\end{aligned}
$$

Corollary 2 *Suppose $\delta = o(n^{2/3})$, $m = O(n^{1/2-\epsilon})$, and $\delta(H)\delta = o(n)$. Then*

$$T(H;\delta_1,\delta_2,\dots,\delta_n)/T(\delta_1,\delta_2,\dots,\delta_n) \sim 2^{-m}\exp\Bigg(\frac{1}{n}\sum_{1\le j\le n}\delta_j(H)\delta_j\Bigg).$$

In particular,

$$T(H;\delta_1,\delta_2,\ldots,\delta_n)/T(\delta_1,\delta_2,\ldots,\delta_n) \sim 2^{-m}$$

uniformly for all $\delta = o(n^{2/3})$ *and* $m = O(n^{1/3})$.

For regular tournaments, we have

Corollary 3 *Let* $T_n(H)$ *be the number of regular tournaments with* n *vertices containing the digraph* H. *Suppose* $d(H) = O(n^{1/2-\epsilon})$. *Then, for odd* n,

$$\begin{aligned} T_n(H) \;\sim\; & \left(\frac{2^{n+1}}{n\pi}\right)^{(n-1)/2}\left(\frac{n}{e}\right)^{1/2}\left(\frac{1}{2}\right)^{m} \\ & \times \exp\biggl(\frac{m}{n} - \frac{1}{2n}\sum_{1\le j\le n}\delta_j^2(H) - \frac{1}{2n^2}\sum_{(j,k)\in A(H)}\bigl(\delta_j(H)-\delta_k(H)\bigr)^2\biggr). \end{aligned}$$

A simple application of Theorem 2 is the unsurprising fact that very few tournaments with $\delta = o(n^{2/3})$ have nontrivial automorphisms. This allows us to estimate the number of isomorphism types.

Corollary 4 *Suppose* $\delta = o(n^{2/3})$. *Then the number of unlabelled tournaments with excess sequence* $\delta_1,\delta_2,\ldots,\delta_n$ *is asymptotically* $T(\delta_1,\delta_2,\ldots,\delta_n)/n!$.

Proof. Consider a random (labelled) tournament T with excess sequence $\delta_1,\delta_2,\ldots,\delta_n$. It suffices to prove that the expected number of automorphisms of T is asymptotically 1.

We know that $|\mathrm{Aut}(T)|$ is odd, because T is a tournament. Let g be a non-trivial permutation of V of odd order. Define $S = S(g)$ to be the set of vertices moved by g, and let $k = |S|$.

Consider the set E of pairs of distinct vertices defined by

$$E = \bigl\{\, \{i,j\}, \{i^g, j^g\} \;\bigm|\; i\in S, 1\le j-i\le 12\lceil \ln n\rceil \pmod{n} \,\bigr\}.$$

It is easy to see that E (considered as an undirected graph) has maximum degree at most $48\lceil \ln n\rceil$, and that $|E| = m$ for $6k\lceil \ln n\rceil \le m \le 48k\lceil \ln n\rceil$. Define a simple undirected graph $G = G(E,g)$ whose vertices are the elements of E and whose edges are the pairs $\{e, e^g\}$ for which both e and e^g are in E. From the definition of E, G has at most $m/2$ components.

Now consider digraphs H which are orientations of E. Within each component of G, there are only two orientations that are consistent with g being an automorphism of T, and so there are at most $2^{m/2}$ possibilities for H with that consistency. From Theorem 2, we have that each such H is a subgraph of T with probability less than $2^{-m}\exp(mn^{-1/3})$ for sufficiently large n. Consequently, the probability that g is an automorphism of T is at most

$$2^{-m/2}\exp(mn^{-1/3}) \leq n^{-2k}$$

for large n.

There are less than n^k permutations of V that move exactly k vertices, so the total expected number of nontrivial automorphisms of T is asymptotically at most

$$\sum_{k=3}^{n} n^{-k} = O(n^{-3}) = o(1).$$

This completes the proof. Note that the bound $O(n^{-3})$ is much larger than the real value; we have been content to find a bound tending to 0. ∎

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