ASYMPTOTIC ENUMERATION OF TOURNAMENTS WITH A GIVEN SCORE SEQUENCE

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Abstract.

We obtain the asymptotic number of labelled tournaments with a given score sequence in the case where each score is $n/2 + O(n^{3/4+\epsilon})$ for sufficiently small $\epsilon > 0$. Some consequences for the score sequences of random tournaments are also noted. The method used is integration in n complex dimensions.

1. Introduction.

A *tournament* is a digraph in which, for each pair of distinct vertices v and w, either (v, w) or (w, v) is an edge, but not both. A tournament is *regular* if the in-degree is equal to the out-degree at each vertex. Let $v_1, v_2, \ldots, v_n$ be the vertices of a labelled tournament and let d_j^-, d_j^+ be the in-degree and out-degree of v_j for $1 \le j \le n$. d_j^+ is also called the *score* of v_j. Define $\delta_j = d_j^+ - d_j^-$ and call $\delta_1, \delta_2, \ldots, \delta_n$ the *excess sequence* of the tournament. Let $NT(n; \delta_1, \ldots, \delta_n)$ be the number of labelled tournaments with n vertices and excess sequence $\delta_1, \ldots, \delta_n$. It is clear that $NT(n; \delta_1, \ldots, \delta_n) = 0$ unless all the excesses have different parity from n; we will assume this without further mention for the entire paper. As in [3], let $RT(n) = NT(n; 0, \ldots, 0)$ be the number of labelled regular tournaments with n vertices.

The first attack that we are aware of on the asymptotics of tournaments was due to Joel Spencer [6]. In particular, Spencer evaluated $RT(n)$ to within a factor of $(1 + o(1))^n$ and obtained the estimate

$$NT(n; \delta_1, \ldots, \delta_n) = RT(n) \exp\Bigl(\bigl(-\tfrac{1}{2} + o(1)\bigr) \frac{\sum_{j=1}^n \delta_j^2}{n}\Bigr)$$

for tournaments close to regular. The asymptotic value of $RT(n)$ as $n \to \infty$ was obtained by B. D. McKay [3]: for any $\epsilon > 0$,

$$RT(n) = \Bigl(\frac{2^{n+1}}{\pi n}\Bigr)^{(n-1)/2} n^{1/2} e^{-1/2} \bigl(1 + O(n^{-1/2+\epsilon})\bigr) \qquad (n \quad odd).$$

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We are concerned with the asymptotic value of $NT(n;\delta_1,\ldots,\delta_n)$. Since the generating function $\prod_{1\le j<k\le n}(x_j^{-1}x_k + x_j x_k^{-1})$ enumerates all tournaments by the excess at each vertex, $NT(n;\delta_1,\ldots,\delta_n)$ is the coefficient of $x_1^{\delta_1}\cdots x_n^{\delta_n}$. We will estimate this value by using the saddle-point method on the integral provided by Cauchy's Theorem.

The major results of this paper first appeared in the doctoral thesis of the second author [7], of which the first author was the supervisor.

2. An integral.

In this section, we will use the averaging method [5] to approximate the value of an n-dimensional integral we will need later. Define the real n-dimensional cube
$U_n(t) = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \mid |x_i| \le t, i = 1, 2, \ldots, n\}$

Theorem 2.1. *Let $\epsilon > 0$ be sufficiently small. Suppose $0 < t' \le t$ are constants, and that*

(i) *for $1 \le j,k \le n-1$, $B_k(n)$ and $C_{jk}(n)$ are purely imaginary functions which are uniformly $O(n^{-1/4+\epsilon})$,*

(ii) *for $1 \le j,k \le n-1$, $A(n)$, $D_{jk}(n)$, $E_k(n)$ and $F(n)$ are real functions such that $t' \le A(n) \le t$, $|D_{jk}(n)| \le t$, $|E_k(n)| \le t$ and $|F(n)| \le t$ for $n > 0$.*

Further suppose that $\delta > 0$ and that

$$\begin{aligned} f(\boldsymbol{x}) = \exp\Bigl(&-A(n)n\sum_{k=1}^{n-1} x_k^2 + n\sum_{k=1}^{n-1} B_k(n)x_k^3 + \sum_{j\ne k} C_{jk}(n)x_k^2 x_j \\ &+ \sum_{j\ne k} D_{jk}(n)x_k^3 x_j + n\sum_{k=1}^{n-1} E_k(n)x_k^4 + F(n)\Bigl(\sum_{k=1}^{n-1} x_k^2\Bigr)^2 + O(n^{-\delta})\Bigr)\end{aligned}$$

is integrable for $\boldsymbol{x} \in U_{n-1}(n^{-1/2+\epsilon})$. Then

$$\begin{aligned}&\int_{U_{n-1}(n^{-1/2+\epsilon})} f(\boldsymbol{x})\, d\boldsymbol{x} \\ &\quad = \Bigl(\frac{\pi}{A(n)n}\Bigr)^{(n-1)/2} \exp\Bigl(\frac{3\sum_{k=1}^{n-1} E_k(n)}{4A(n)^2 n} + \frac{F(n)}{4A(n)^2} + O(n^{-1/4+\epsilon} + n^{-\delta})\Bigr).\end{aligned}$$

Proof. Define $\mu_2 = \sum_{k=1}^{n-1} x_k^2$ and $W_{n-1}(\rho) = U_{n-1}(n^{-1/2+\epsilon}) \cap \{\boldsymbol{x} \mid \mu_2 = \rho^2\}$ for $\rho \ge 0$. We approach the integral by considering integration first over $W_{n-1}(\rho)$ and then over ρ, although this is not the way we obtain the final estimate. Note first that $W_{n-1}(\rho) = \emptyset$ if $\rho > n^{\epsilon}$.

For $\boldsymbol{x} \in W_{n-1}(\rho)$ and $\rho \le n^\epsilon$, we have

$$\Big| n \sum_{k=1}^{n-1} B_k(n) x_k^3 \Big| \le t\rho^2 n^{1/2+2\epsilon}, \qquad \Big| \sum_{j \ne k} C_{jk}(n) x_k^2 x_j \Big| \le t\rho^2 n^{1/2+2\epsilon},$$

$$\Big| \sum_{j \ne k} D_{jk}(n) x_k^3 x_j \Big| \le t\rho^2 n^{2\epsilon}, \qquad \Big| F(n) \Big(\sum_{k=1}^{n-1} x_k^2 \Big)^2 \Big| \le t\rho^2 n^{2\epsilon}.$$

$$\Big| n \sum_{k=1}^{n-1} E_k(n) x_k^4 \Big| \le t\rho^2 n^{2\epsilon},$$

We now divide the region of integration into three parts. Let $0 < \varDelta < 1/4 - \epsilon/2$ and define

$$\begin{aligned} K_1 &= U_{n-1}(n^{-1/2+\epsilon}) \cap \big\{ \boldsymbol{x} \bigm| 0 \le \rho < (2A(n))^{-1/2}(1 - n^{-\varDelta}) \big\}, \\ K_2 &= U_{n-1}(n^{-1/2+\epsilon}) \cap \\ &\qquad \big\{ \boldsymbol{x} \bigm| (2A(n))^{-1/2}(1 - n^{-\varDelta}) \le \rho \le (2A(n))^{-1/2}(1 + n^{-\varDelta}) \big\}, \text{ and} \\ K_3 &= U_{n-1}(n^{-1/2+\epsilon}) \cap \big\{ \boldsymbol{x} \bigm| (2A(n))^{-1/2}(1 + n^{-\varDelta}) < \rho \le n^\epsilon \big\}. \end{aligned}$$

The integral over K_1 can be bounded by multiplying the range of ρ by the maximum value of the integrand in that range. Using the fact that the surface area of an n-dimensional sphere of radius ρ is $2\pi^{n/2}\rho^{n-1}/\varGamma(n/2)$, we find

$$\Big| \int_{K_1} f(\boldsymbol{x})\, d\boldsymbol{x} \Big| \le \Big(\frac{\pi}{A(n)n} \Big)^{(n-1)/2} \exp\big(-n^{1-2\varDelta} + O(n^{1/2+\epsilon}) \big),$$

and similarly for the integral over K_3. Both of these integrals will turn out to be negligible compared to that over K_2, which we now consider.

The function $f(\boldsymbol{x})$ shows a lot of variation on $W_{n-1}(\rho)$, $\rho \approx (2A(n))^{-1/2}$, making direct estimation of the integral difficult. Instead, we take advantage of the the fact that an integral over a region symmetrical about the origin is invariant under averaging of its integrand over sign changes of the arguments.

For $1 \le m \le n$, define

$$\begin{aligned} \psi_m(\boldsymbol{x}) = \exp\Big(&-A(n)n \sum_{k=1}^{n-1} x_k^2 + n \sum_{k=1}^{n-1} E_k(n) x_k^4 + F(n) \Big(\sum_{k=1}^{n-1} x_k^2 \Big)^2 + n \sum_{k=m}^{n-1} B_k(n) x_k^3 \\ &+ \sum_{k=1}^{n-1} \sum_{j=m}^{n-1} C_{jk}(n) x_k^2 x_j + \sum_{k=m}^{n-1} \sum_{j=m}^{n-1} D_{jk}(n) x_k^3 x_j + \frac{n^2}{2} \sum_{k=1}^{m-1} B_k(n)^2 x_k^6 \Big) \end{aligned}$$

and, for $1 \le m \le n-1$,

$$\bar\psi_m(\boldsymbol{x}) = \tfrac12 \big(\psi_m(\boldsymbol{x}) + \psi_m(x_1, \ldots, x_{m-1}, -x_m, x_{m+1}, \ldots, x_{n-1}) \big).$$

Further define $\eta = \frac32 - 8\epsilon$. Then we have

$$\int_{U_{n-1}(n^{-1/2+\epsilon})} \bar\psi_m(\boldsymbol{x})\, d\boldsymbol{x} = \int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_m(\boldsymbol{x})\, d\boldsymbol{x}.$$

For $\boldsymbol{x} \in U_{n-1}(n^{-1/2+\epsilon})$, since

$$\psi_m(\boldsymbol{x}) = \psi_{m+1}(\boldsymbol{x}) \exp\Big(B_m(n)nx_m^3 + \sum_{k=1}^{n-1} C_{mk}(n)x_k^2 x_m - \tfrac{1}{2}B_m(n)^2 n^2 x_m^6 + \sum_{k=m+1}^{n-1} D_{mk}(n)x_k^3 x_m + \sum_{k=m+1}^{n-1} D_{jm}(n)x_m^3 x_j + D_{mm}(n)x_m^4\Big)$$

and $\frac{1}{2}(e^x + e^{-x}) = \exp\big(\frac{1}{2}x^2 + O(x^4)\big)$ for small x, we have $\bar{\psi}_m(\boldsymbol{x}) = \psi_{m+1}(\boldsymbol{x})\exp\big(O(n^{-\eta})\big)$ uniformly over m. Furthermore, $f(\boldsymbol{x}) = \psi_1(\boldsymbol{x})\exp\big(O(n^{-\delta} + n^{-1/2+3\epsilon})\big)$ and

$$\psi_n(\boldsymbol{x}) = \exp\Big(-A(n)n\sum_{k=1}^{n-1} x_k^2 + n\sum_{k=1}^{n-1} E_k(n)x_k^4 + F(n)\Big(\sum_{k=1}^{n-1} x_k^2\Big)^2 + \frac{1}{2}\sum_{k=1}^{n-1} B_k(n)^2 n^2 x_k^6\Big).$$

In K_2 we have $\mu_2 = (2A(n))^{-1}\big(1 + O(n^{-\varDelta})\big)$, so

$$\psi_n(\boldsymbol{x}) = \exp\Big(-A(n)n\sum_{k=1}^{n-1} x_k^2 + n\sum_{k=1}^{n-1} E_k(n)x_k^4 + \frac{1}{2}\sum_{k=1}^{n-1} B_k(n)^2 n^2 x_k^6 + \frac{F(n)}{4A(n)^2} + O(n^{-\varDelta})\Big).$$

The integral of ψ_n over $U_{n-1}(n^{-1/2+\epsilon})$ differs from that over K_2 by at most

$$\Big(\frac{\pi}{A(n)n}\Big)^{(n-1)/2} \exp\big(-n^{1-2\varDelta} + O(n^{1/2+\epsilon})\big),$$

as in the estimation of the integral of f over $K_1 \cup K_3$. Furthermore,

$$\begin{aligned}
&\int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_n(\boldsymbol{x})\, d\boldsymbol{x} \\
&\quad = \exp\Big(\frac{F(n)}{4A(n)^2} + O(n^{-\varDelta})\Big) \\
&\qquad \times \prod_{k=1}^{n-1} \int_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \exp\big(-A(n)nx^2 + E_k(n)nx^4 + \tfrac{1}{2}B_k(n)^2 n^2 x^6\big)\, dx \\
&\quad = \Big(\frac{\pi}{A(n)n}\Big)^{(n-1)/2} \exp\Big(\frac{3\sum_{k=1}^{n-1} E_k(n)}{4A(n)^2 n} + \frac{F(n)}{4A(n)^2} + O(n^{-1/2+6\epsilon} + n^{-\varDelta})\Big). \qquad (2.1)
\end{aligned}$$

By the same argument as used in [5], we find that

$$\int f(\boldsymbol{x})\, d\boldsymbol{x} = \exp\big(O(n^{1-\eta} + n^{-\delta})\big) \int \psi_n(\boldsymbol{x})\, d\boldsymbol{x}. \qquad (2.2)$$

The theorem now follows from (2.1), (2.2), and the fact that the integral over $K_1 \cup K_3$ is negligible. ∎

3. The major part of the Cauchy integral.

In this section we will begin the estimation of $NT(n;\delta_1,\ldots,\delta_n)$ by approximating the Cauchy integral in the region from which the major contribution comes. In outline, our approach will be to expand the integrand in a Taylor series, eliminate the linear term by choice of contours, and diagonalise the quadratic term by linear transformations. The integrand will then be in the form required by Theorem 2.1.

By Cauchy's Theorem,

$$NT(n;\delta_1,\ldots,\delta_n) = \frac{1}{(2\pi i)^n} \oint \cdots \oint \frac{\prod_{1\le j<k\le n}(x_j^{-1}x_k + x_j x_k^{-1})}{x_1^{\delta_1+1}\cdots x_n^{\delta_n+1}} dx_1\cdots dx_n,$$

where each integration is around a simple closed contour encircling the origin once in the anticlockwise direction. Choosing the jth contour to be a circle of radius r_j by substituting $x_j = r_j e^{i\theta_j}$ for $1\le j\le n$, we obtain

$$NT(n;\delta_1,\ldots,\delta_n) = \frac{\prod_{1\le j<k\le n}\left(\frac{r_k}{r_j}+\frac{r_j}{r_k}\right)}{(2\pi)^n\prod_{1\le j\le n} r_j^{\delta_j}} I_1,$$

where

$$I_1 = \int_{U_n(\pi)} g(\boldsymbol{\theta})\, d\boldsymbol{\theta},$$

$$g(\boldsymbol{\theta}) = \exp\bigl(-i\sum_{1\le j\le n}(\delta_j\theta_j)\bigr)\prod_{1\le j<k\le n} T_{jk}(\boldsymbol{\theta}),$$

and

$$T_{jk}(\boldsymbol{\theta}) = \frac{r_k^2\exp\bigl(i(\theta_k-\theta_j)\bigr) + r_j^2\exp\bigl(i(\theta_j-\theta_k)\bigr)}{r_j^2+r_k^2}.$$

We will begin the evaluation of I_1 with the part of the domain which will turn out to give the major contribution. Let I_2 be the contribution to I_1 of those $\boldsymbol{\theta}$ such that either $|\theta_j-\theta_n|\le n^{-1/2+\epsilon}$ or $|\theta_j-\theta_n+\pi|\le n^{-1/2+\epsilon}$ for $1\le j\le n-1$, where θ_j values are taken mod 2π. Since the contributions to I_2 from different values of θ_n are the same, and the fact that translation of any θ_j by π leaves the integrand unchanged,

$$I_2 = 2^n\pi\int_{U_{n-1}(n^{-1/2+\epsilon})} g(\boldsymbol{\theta})\, d\boldsymbol{\theta}',$$

where $\boldsymbol{\theta}' = (\theta_1,\ldots,\theta_{n-1})$ with $\theta_n = 0$.

For $1\le j,k\le n$, define

$$a_{jk} = \frac{r_j^2-r_k^2}{r_j^2+r_k^2}.$$

Since we will later choose the radii r_j such that uniformly $r_j = 1+o(1)$, we can assume that $a_{jk} = o(1)$. For $\boldsymbol{\theta}'\in U_{n-1}(n^{-1/2+\epsilon})$, we can expand $g(\boldsymbol{\theta})$ using Taylor's Theorem to obtain

$$
\begin{aligned}
g(\boldsymbol{\theta}) = \exp\Big(& \sum_{1\le j\le n} \big(\sum_{1\le k\le n} a_{jk} - \delta_j \big) i\theta_j \\
& + \sum_{1\le j<k\le n} \big(-\tfrac{1}{2} + \tfrac{1}{2}a_{jk}^2\big)(\theta_k - \theta_j)^2 \\
& - \sum_{1\le j<k\le n} i\big(\tfrac{1}{3}a_{jk} - \tfrac{1}{3}a_{jk}^3\big)(\theta_k - \theta_j)^3 \\
& + \sum_{1\le j<k\le n} \big(-\tfrac{1}{12} + \tfrac{1}{3}a_{jk}^2 - \tfrac{1}{4}a_{jk}^4\big)(\theta_k - \theta_j)^4 \\
& + O\big(\sum_{1\le j<k\le n} |\theta_k - \theta_j|^5 \big) \Big).
\end{aligned} \tag{3.1}
$$

Our next task will be to choose r_j for $1 \le j \le n$ so that the coefficients of the linear terms in (3.1) vanish. That is, we need $r_1, \ldots, r_n$ such that

$$
\sum_{k=1}^{n} a_{jk} = \delta_j, \qquad (1 \le j \le n). \tag{3.2}
$$

Substitute $r_j^2 = (1 + b_j)/(1 - b_j)$ for $1 \le j \le n$ and consider the functions $f_1, f_2 \ldots, f_n$ defined by

$$
f_j(\boldsymbol{b}) = \frac{\delta_j}{n} - \frac{1}{n}\sum_{k=1}^{n} \frac{b_j b_k (b_j - b_k)}{1 - b_j b_k}, \qquad (1 \le j \le n).
$$

Further define $\boldsymbol{b}^{(0)} = (\delta_1/n, \ldots, \delta_n/n)$ and $b_j^{(i)} = f_j(\boldsymbol{b}^{(i-1)})$ for $i = 1, 2, \ldots$. Let $\|\cdot\|$ denote the maximum norm on $\mathbb{R}^n$, i.e., $\|(x_1, \ldots, x_n)\| = \max_{1\le k\le n}|x_k|$. Define $E = \|(\delta_1, \ldots, \delta_n)\|/n$ and assume that $E = O(n^{-1/4+\epsilon})$ and $E \le 1/100$. By a routine induction computation, we find that

$$
\|\boldsymbol{b}^{(i+1)} - \boldsymbol{b}^{(i)}\| \le 3E^{3+2i}15^i \tag{3.3}
$$

for $i \ge 0$. It follows that $\boldsymbol{b}^{(i)}$ converges to a vector $\boldsymbol{b}$ which satisfies (3.2). Now define $\boldsymbol{w}$ by

$$
\begin{aligned}
w_j = & \frac{\delta_j}{n} + \frac{\delta_j \sum_{k=1}^n \delta_k^2}{n^4} + \frac{-\delta_j^3 \sum_{k=1}^n \delta_k^2 + \delta_j^2 \sum_{k=1}^n \delta_k^3}{n^6} \\
& + \frac{3\delta_j \left(\sum_{k=1}^n \delta_k^2\right)^2}{n^7} + \frac{-\delta_j^4 \sum_{k=1}^n \delta_k^3 + \delta_j^3 \sum_{k=1}^n \delta_k^4}{n^8} \\
& + \frac{-6\delta_j^3 \left(\sum_{k=1}^n \delta_k^2\right)^2 + 6\delta_j^2 \sum_{k=1}^n \delta_k^2 \sum_{k=1}^n \delta_k^3 + 2\delta_j \left(\sum_{k=1}^n \delta_k^3\right)^2 - 2\delta_j \sum_{k=1}^n \delta_k^2 \sum_{k=1}^n \delta_k^4}{n^9} \\
& + \frac{12\delta_j \left(\sum_{k=1}^n \delta_k^2\right)^3}{n^{10}}
\end{aligned}
$$

for $1 \le j \le n$.

Lemma 3.1. $\|\boldsymbol{b}-\boldsymbol{w}\| = O(E^9)$.

Proof. The vector $\boldsymbol{w}$ is the same as $\boldsymbol{b}^{(4)}$ except that terms which are $O(E^9)$ have been rejected. By (3.3),

$$\|\boldsymbol{b}-\boldsymbol{b}^{(4)}\| \le \|\boldsymbol{b}^{(5)}-\boldsymbol{b}^{(4)}\| + \|\boldsymbol{b}^{(6)}-\boldsymbol{b}^{(5)}\| + \cdots = O(E^{11}),$$

so the lemma follows. ∎

We can now continue our estimation of I_2 by substituting the radii corresponding to $\boldsymbol{b}$, that is $r_j = \big((1+b_j)/(1-b_j)\big)^{1/2}$, into (3.1), causing the linear term to vanish. Note that $a_{jk} = (b_j - b_k)/(1-b_jb_k)$.

Our next step will be to apply a linear transformation which diagonalises the quadratic term in (4.3). This will be comprised of the transformation used in [3], which is exact for regular tournaments ($a_{jk} \equiv 0$), followed by a second which corrects for the error in the first.

Define $V = U_{n-1}(n^{-1/2+\epsilon})$ and let $T : \mathbb{R}^{n-1} \longmapsto \mathbb{R}^{n-1}$ be the linear transformation defined by $T : \boldsymbol{\theta}' \longmapsto \boldsymbol{y} = (y_1, y_2, \ldots, y_{n-1})$, where

$$y_j = \theta_j - \sum_{k=1}^{n-1} \theta_k/(n+n^{1/2})$$

for $1 \le j \le n-1$. Let $V_1 = T(V)$ and $s = 1/(n^{1/2}+1)$. By straightforward calculations we have $\det(T) = n^{1/2}$ and

$$V_1 = \Big\{\boldsymbol{y} \;\Big|\; |y_j + s\sum_{k=1}^{n-1} y_k| \le n^{-1/2+\epsilon} \text{ for } 1 \le j \le n-1\Big\}.$$

Applying the transformation yields

$$\begin{aligned}
g(\boldsymbol{\theta}) = \exp\Big(&\sum_{k=1}^{n-1}\Big(-\frac{n}{2} + \tfrac{1}{2}\sum_{j=1}^{n} a_{kj}^2 + sa_{nk}^2 + \tfrac{1}{2}s^2\sum_{l=1}^{n-1} a_{nl}^2\Big)y_k^2 \\
&+ \sum_{j\ne k}\Big(-\tfrac{1}{2}a_{kj}^2 + sa_{nj}^2 + \tfrac{1}{2}s^2\sum_{l=1}^{n-1} a_{nl}^2\Big)y_jy_k \\
&+ \sum_{k=1}^{n-1}\Big(\tfrac{1}{3}\sum_{j=1}^{n} a_{kj} - \tfrac{1}{3}\sum_{j=1}^{n-1} a_{kj}^3\Big)iy_k^3 \\
&+ \sum_{j\ne k}\big(-a_{kj} + O(E^2)\big)iy_k^2y_j \\
&+ \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}\big(-\tfrac{1}{12} + \tfrac{1}{3}a_{kj}^2 - \tfrac{1}{4}a_{kj}^4\big)\Big)y_k^4 \\
&+ \sum_{j\ne k} O(1)y_k^3y_j \\
&- \sum_{j\ne k} \tfrac{1}{4}y_k^2y_j^2 \\
&+ O(n^{-1/2+6\epsilon})\Big),
\end{aligned} \tag{3.4}$$

where each $O(\ )$ term is uniform over the subscript set of the sum involved, and each such term except the last is independent of $\boldsymbol{y}$.

Denote

$$u_k = -\frac{n}{2} + \tfrac{1}{2}\sum_{j=1}^{n} a_{kj}^2 + s a_{nk}^2 + \tfrac{1}{2}s^2 \sum_{l=1}^{n-1} a_{nl}^2,$$

$$v_{kj} = -\tfrac{1}{2}a_{kj}^2 + \tfrac{1}{2}(a_{nj}^2 + a_{nk}^2)s + \tfrac{1}{2}s^2 \sum_{l=1}^{n-1} a_{nl}^2, \qquad \text{for } k < j,$$

$v_{kj} = v_{jk}$ for $k > j$, and $v_{kk} = u_k$. Let A be the diagonal matrix with entries $u_1, \ldots, u_{n-1}$, V be the $n-1 \times n-1$ matrix with entries v_{jk}, and $B = V - A$.

Define a linear transformation from $\boldsymbol{y}$ to $\boldsymbol{x}$ by

$$\boldsymbol{y} = \operatorname{diag}\Big(\frac{n}{-2u_k}\Big)^{1/2} (I + A^{-1}B)^{-1/2} \boldsymbol{x}.$$

This transformation exactly diagonalises the quadratic terms in (3.4), as can be seen from the following lemma, which can be proved by series expansion.

Lemma 3.2. *Let A and B be square matrices of the same order, such that A^{-1} exists, $\|BA^{-1}\| < 1$ and $\|A^{-1}B\| < 1$ for some matrix norm. Then*

$$(I + BA^{-1})^{-1/2}(A + B)(I + A^{-1}B)^{-1/2} = A,$$

where the fractional powers are defined by the usual Taylor series. ∎

Expanding $(I + BA^{-1})^{-1/2}$ in a Taylor series, we finally have

$$\begin{aligned}
g(\boldsymbol{\theta}) = \exp\Big(&\sum_{k=1}^{n-1} -\frac{n}{2}x_k^2 \\
&+ \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}\big(\tfrac{1}{3}a_{kj} - \tfrac{1}{3}a_{kj}^3\big)\big(\frac{n}{-2u_k}\big)^{3/2} + O(E^2)\Big) i x_k^3 \\
&+ \sum_{1\le j\ne k\le n-1} \big(-a_{kj} + O(E^2)\big) i x_k^2 x_j \\
&+ \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}\big(-\tfrac{1}{12} + \tfrac{1}{3}a_{kj}^2 - \tfrac{1}{4}a_{kj}^4\big)\big(\frac{n}{-2u_k}\big)^2\Big) x_k^4 \\
&+ \sum_{1\le j\ne k\le n-1} O(1) x_k^3 x_j \\
&- \sum_{1\le j\ne k\le n-1} \tfrac{1}{4}x_k^2 x_j^2 \\
&+ O(n^{-1/4+6\epsilon})\Big).
\end{aligned} \tag{3.5}$$

Using the identity $\det\big((I+A^{-1}B)^{-1/2}\big) = \exp\big(-\frac{1}{2}\mathrm{tr}\log(I+A^{-1}B)\big)$, we find that the determinant of this transformation is

$$\big(1+O(E^2)\big)\Big(\frac{n}{2}\Big)^{(n-1)/2}\prod_{k=1}^{n-1}(-u_k)^{-1/2}.$$

Let $T' : \boldsymbol{\theta}' \longmapsto \boldsymbol{x}$ be the transformation involved in this section and $V' = T'(V)$. We know that $V' \subseteq U_{n-1}(3n^{-1/2+\epsilon})$. The asymptotic value of the integral of $f(\boldsymbol{x})$ over $U_{n-1}(3n^{-1/2+\epsilon})$ will be the same with that over $U_{n-1}(n^{-1/2+\epsilon})$. Furthermore, similar argument to that of [3, Theorem 2.1] shows that the asymptotic value of the integral of $f(\boldsymbol{x})$ over $U_{n-1}(3n^{-1/2+\epsilon})\backslash V'$ is negligible. Therefore, we still keep the region as $U_{n-1}(n^{-1/2+\epsilon})$.

We can now obtain the following estimate for I_2.

Lemma 3.3. *Suppose* $\max\{|\delta_1|,\ldots,|\delta_n|\} = O(n^{3/4+\epsilon})$ *where* $\epsilon > 0$ *is sufficiently small. Then*

$$I_2 = 2^n \pi n^{1/2}\Big(\frac{2\pi}{n}\Big)^{(n-1)/2}\exp\big(-\tfrac{1}{2}+O(n^{-1/4+6\epsilon})\big)\prod_{k=1}^{n-1}\Big(1-\frac{1}{n}\sum_{j=1}^{n}a_{kj}^2\Big)^{-1/2}.$$

Proof. Apply Theorem 2.1 to (3.5), then note that

$$\frac{n}{-2u_k} = \Big(1-\frac{1}{n}\sum_{j=1}^{n}a_{kj}^2\Big)^{-1} + O(n^{-1}E^2). \quad \blacksquare$$

4. The main results.

In order to complete the estimation of $NT(n;\delta_1,\ldots,\delta_n)$, we need only to show that I_2 contributes almost all of I_1. We begin with a technical lemma which can be proved using Taylor series for small x and simple bounds for larger x.

Lemma 4.1. *For sufficiently small* $\epsilon > 0$,

$$|1-\lambda+\lambda\cos(x)| \le \exp(-\tfrac{1}{2}\lambda x^2)$$

whenever $|\lambda - 1/2| \le \epsilon$ *and* $|x| \le \pi$. $\blacksquare$

Lemma 4.2. *Let* $m \ge 2$. *For* $1 \le j < k \le m$, *let* $\gamma_{jk} = \frac{1}{2} + O(n^{-1/2+\epsilon})$ *uniformly, where* $\epsilon > 0$ *is sufficiently small. Then*

$$\int_{\mathbb{R}^{m-1}} \exp\Big(-\sum_{1\le j<k\le m}\gamma_{jk}(\phi_j-\phi_k)^2\Big)\,d\boldsymbol{\phi}' = \frac{(2\pi)^{(m-1)/2}}{m^{m/2-1}}\exp\big(O(mn^{-1/2+2\epsilon})\big),$$

where the integration is over $\boldsymbol{\phi}' = (\phi_1, \ldots, \phi_{m-1})$ *with* $\phi_m = 0$. *Moreover, for any* $x \geq m^{-1/2}$, *the integral over* $\boldsymbol{\phi}' \in U_{m-1}(x)$ *differs from that over* $\mathbb{R}^{m-1}$ *by a factor of at worst* $1 - 2m\exp(-cx^2m)$ *for some* $c > 0$.

Proof. Define $\gamma_{jj} = \frac{1}{2}$ and $\gamma_{kj} = \gamma_{jk}$ for $1 \leq j < k \leq m$. Then the integrand is $\exp\bigl(-\frac{1}{2}\boldsymbol{\phi}'Q\boldsymbol{\phi}'^T\bigr)$, where $Q = (q_{jk})$ with $q_{jj} = -1 + 2\sum_{k=1}^{m}\gamma_{jk}$ and $q_{jk} = -2\gamma_{jk}$ $(j \neq k)$. By Theorem III.6.3 of [1], the integral over $\mathbb{R}^{m-1}$ is $(2\pi)^{(m-1)/2}|Q|^{-1/2}$. Using J to represent a matrix with every entry one, we note that $(I+J)Q$ is nearly diagonal and obtain $|Q| = m^{m-2}\exp\bigl(O(mn^{-1/2+2\epsilon})\bigr)$. This gives the first claim.

To obtain the second claim, note from [1] that, apart from a constant, the integrand is the density of the $(m-1)$-dimensional normal variate $(X_1, \ldots, X_{m-1})$ with mean zero and covariance matrix Q^{-1}. The variance of X_j is the jth diagonal entry of Q^{-1}, which is seen to be $m^{-1}\bigl(1 + O(n^{-1/2+2\epsilon})\bigr)$ by using $Q^{-1} = \bigl((I+J)Q\bigr)^{-1}(I+J)$. Hence $\mathrm{Prob}(|X_j| > x) \leq 2\exp\bigl(-\frac{1}{2}x^2m(1 + O(n^{-1/2+2\epsilon})\bigr)$. ∎

Define the regions $A = \{\theta \mid |\theta| \leq \frac{1}{8}\pi\}$, $B = \{\theta \mid \frac{1}{8}\pi \leq |\theta| \leq \frac{1}{4}\pi\}$ and $C = \{\theta \mid \frac{1}{4}\pi \leq |\theta| \leq \frac{1}{2}\pi\}$. Using symmetry and translations by π as before, we lose a factor of at most 2^{n+3} from I_1 if we assume that all θ_j lie in $A \cup B \cup C$, and at least $n/8$ lie in C. Now define

$$I_3 = \int_{U_n(\pi/2)} \bigl|g(\boldsymbol{\theta})\bigr|\, d\boldsymbol{\theta}$$

subject to those conditions, and let $I_3(t)$ be the contribution from those $\boldsymbol{\theta}$ for which exactly t of the θ_j lie in C $(0 \leq t \leq \frac{7}{8}n)$.

Note that

$$\bigl|T_{jk}(\boldsymbol{\theta})\bigr| = \bigl(1 - \lambda_{jk} + \lambda_{jk}\cos(2(\theta_j - \theta_k))\bigr)^{1/2},$$

where $\lambda_{jk} = \frac{1}{2} - \frac{1}{2}a_{jk}^2 = \frac{1}{2} + O(n^{-1/2+2\epsilon})$ for $1 \leq j,k \leq n$. If θ_j and θ_k lie in A and C, respectively, we have $\bigl|T_{jk}(\boldsymbol{\theta})\bigr| \leq \alpha = |1 - \lambda + \lambda\cos(\frac{1}{8}\pi)| < 1$, where λ is the least value of λ_{jk}. If θ_j and θ_k both lie in $B \cup C$, $\bigl|T_{jk}(\boldsymbol{\theta})\bigr| \leq \exp\bigl(-\frac{1}{2}\lambda_{jk}(\theta_j - \theta_k)^2\bigr)$, by Lemma 4.1. In all other cases, $\bigl|T_{jk}(\boldsymbol{\theta})\bigr| \leq 1$. Thus,

$$I_3(t) \leq \tfrac{1}{4}\pi\binom{n}{t}\alpha^{tn/8}(\pi/2)^t I^{(n-t)},$$

where $I^{(n-t)}$ is an integral of the form of Lemma 4.2 with $m = n - t$. Applying Lemma 4.2, we find that $I_3(t) \leq \exp(-ctn)I_3(0)$ for some constant $c > 0$ independent of t, and so $\sum_{t=1}^{7n/8} I_3(t) \leq \exp(-c'n)I_3$ for $c' > 0$. In $I_3(0)$, we can apply Lemma 4.1 to every $T_{jk}(\boldsymbol{\theta})$. The integrand then just consists of the quadratic term of (3.1), which contributed all but a constant to I_2. Applying Lemma 4.2 once more, we find that $I_1 = \bigl(1 + O(\exp(-c''n^{2\epsilon}))\bigr)I_2$ for some $c'' > 0$. Thus we have the following theorem.

Theorem 4.3. *For $\epsilon > 0$ sufficiently small, suppose* $\max\{|\delta_1|,\ldots,|\delta_n|\} = O(n^{3/4+\epsilon})$. *Choose $r_1,\ldots,r_n$ as above. Then*

$$NT(n;\delta_1,\ldots,\delta_n) = n^{1/2}\Bigl(\frac{2}{n\pi}\Bigr)^{(n-1)/2}\prod_{k=1}^{n-1}\Bigl(1-\frac{1}{n}\sum_{j=1}^{n}a_{kj}^2\Bigr)^{-1/2}$$
$$\times \prod_{1\le j<k\le n}\Bigl(\frac{r_k}{r_j}+\frac{r_j}{r_k}\Bigr)\prod_{1\le j\le n} r_j^{-\delta_j}\exp\bigl(-\tfrac{1}{2}+O(n^{-1/4+6\epsilon})\bigr). \quad \blacksquare$$

From Lemma 3.1, we have $r_j = \bigl(1+O(E^9)\bigr)\bigl((1+w_j)/(1-w_j)\bigr)^{1/2}$. Strengthening the conditions slightly, we can recast Theorem 4.3 in the following more explicit form.

Theorem 4.4. *Suppose* $\delta = \max\{|\delta_1|,\ldots,|\delta_n|\} = o(n^{3/4})$. *Then, for any $\epsilon > 0$,*

$$NT(n;\delta_1,\ldots,\delta_n) = n^{1/2}\Bigl(\frac{2^{n+1}}{n\pi}\Bigr)^{(n-1)/2}\exp\Bigl(-\frac{1}{2}-\frac{1}{2n}\sum_{j=1}^{n}\delta_j^2+\frac{1}{n^2}\sum_{j=1}^{n}\delta_j^2$$
$$-\frac{1}{12n^3}\sum_{j=1}^{n}\delta_j^4-\frac{1}{4n^4}\Bigl(\sum_{j=1}^{n}\delta_j^2\Bigr)^2-\frac{1}{30n^5}\sum_{j=1}^{n}\delta_j^6$$
$$-\frac{1}{6n^6}\Bigl(\sum_{j=1}^{n}\delta_j^3\Bigr)^2-\frac{1}{2n^7}\Bigl(\sum_{j=1}^{n}\delta_j^2\Bigr)^3+O\Bigl(\frac{\delta^4}{n^3}+n^{-1/4+\epsilon}\Bigr)\Bigr). \quad \blacksquare$$

In both of the preceding theorems we assume the obvious conditions that $\delta_1,\ldots,\delta_n$ are integers, of opposite parity to n, that sum to zero.

Theorem 4.4 has an obvious application to the excess sequences of random tournaments. If $P(n;\delta_1,\ldots,\delta_n)$ is the probability that a random (labelled) tournament has excesses $\delta_1,\ldots,\delta_n$, then clearly $P(n;\delta_1,\ldots,\delta_n) = NT(n;\delta_1,\ldots,\delta_n)/2^{\binom{n}{2}}$. It is instructive to compare these values to a simpler probability space. Let $D_1,\ldots,D_n$ be random variables with the binomial distribution $\mathrm{Binom}(n-1,1/2)$, independent subject only to have sum $n(n-1)/2$. We will call this the *$\mathcal{D}$-model.* Except for some additional dependence, these conditions apply to the out-degrees of a random tournament. Let $P_{\mathcal{D}}(n;\delta_1,\ldots,\delta_n)$ denote the probability that $2D_j = n-1+\delta_j$ for $1\le j\le n$. Then direct computation gives the following.

Theorem 4.5. *Suppose* $\delta = \max\{|\delta_1|,\ldots,|\delta_n|\} = o(n^{3/4})$. *Then, for any $\epsilon > 0$,*

$$NT(n;\delta_1,\ldots,\delta_n) = 2^{\binom{n}{2}}P_{\mathcal{D}}(n;\delta_1,\ldots,\delta_n)\exp\Bigl(-\frac{3}{4}+\frac{1}{n^2}\sum_{j=1}^{n}\delta_j^2-\frac{1}{4n^4}\Bigl(\sum_{j=1}^{n}\delta_j^2\Bigr)^2$$
$$-\frac{1}{6n^6}\Bigl(\sum_{j=1}^{n}\delta_j^3\Bigr)^2-\frac{1}{2n^7}\Bigl(\sum_{j=1}^{n}\delta_j^2\Bigr)^3+O\Bigl(\frac{\delta^4}{n^3}+n^{-1/4+\epsilon}\Bigr)\Bigr). \quad \blacksquare$$

Interestingly, the argument of the exponential is close to zero for the excess sequences of almost all tournaments on n vertices. Precisely, with probability $1-O(n^{-k})$ for any k, a random tournament has $\sum \delta_j^2 = \bigl(1+O(n^{-1/2+\epsilon})\bigr)n^2$, $\sum \delta_j^3 = O(n^{5/2+\epsilon})$ and $\delta = O(n^{1/2+\epsilon})$, and the same is true of the $\mathcal{D}$-model.

As an example of how this can be useful, we have the following theorem.

Theorem 4.6. *Let $f(\delta_1,\ldots,\delta_n)$ be any function such that $\bigl|f(\delta_1,\ldots,\delta_n)\bigr| = O(n^k)$ for some fixed k, uniformly for the excess sequences of tournaments. Let $E_T(f,n)$ and $E_{\mathcal{D}}(f,n)$ denote the expectations of $f(\delta_1,\ldots,\delta_n)$ for random tournaments and for the $\mathcal{D}$-model, respectively. Then $E_T(f,n) = O(n^{-t}) + \bigl(1+O(n^{-1/4+\epsilon})\bigr)E_{\mathcal{D}}(f,n)$ for any $\epsilon, t > 0$.* ∎

In closing, we mention a checking calculation that adds confidence to Theorem 4.4. Let $0 < \alpha < 1$ be constant, $\varDelta = \varDelta(n) = o(n^{3/4})$ and $\bar\alpha = 1-\alpha$. Define $T(\varDelta,\alpha,n)$ to be the set of all tournaments on vertices $\{v_1, v_2, \ldots, v_n\}$ such that $v_1,\ldots,v_{\alpha n}$ have average excess $\bar\alpha\varDelta$ and $v_{\alpha n+1},\ldots,v_n$ have average excess $-\alpha\varDelta$. (Round to integer as necessary.) Since an equivalent characterisation is that there are $\frac12\alpha\bar\alpha(n+\varDelta)$ directed edges from $\{v_1,\ldots,v_{\alpha n}\}$ to $\{v_{\alpha n+1},\ldots,v_n\}$, we have

$$\begin{aligned}\bigl|T(\varDelta,\alpha,n)\bigr| &= 2^{n(n-1)/2-\alpha\bar\alpha n^2}\binom{\alpha\bar\alpha n^2}{\frac12\alpha\bar\alpha n(n+\varDelta)}\\ &= \frac{2^{(n^2-n+1)/2}}{n\sqrt{\alpha\bar\alpha\pi}}\exp\bigl(-\tfrac12\alpha\bar\alpha\varDelta^2 - \tfrac1{12}\alpha\bar\alpha\varDelta^4/n^2 - \tfrac1{30}\alpha\bar\alpha\varDelta^6/n^4 + o(1)\bigr).\end{aligned}$$

Alternatively, one can estimate $\bigl|T(\varDelta,\alpha,n)\bigr|$ by summing Theorem 4.4 over all relevant excess sequences. The result is precisely the same, and since each of the terms in the exponential in Theorem 4.4 contribute independent functions of α and $\varDelta$ to the answer, this is sufficient to check that every coefficient is correct provided that the general form is correct.

Further investigation of tournaments by similar methods will be reported in [2] and [4].

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