Ramsey numbers for triangles versus almost-complete graphs

To the fond memory of Paul Erdős

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Abstract.

We show that, in any coloring of the edges of K_{38} with two colors, there exists a triangle in the first color or a monochromatic $K_{10}-e$ (K_{10} with one edge removed) in the second color, and hence we obtain a bound on the corresponding Ramsey number, $R(K_3, K_{10}-e) \leq 38$. The new lower bound of 37 for this number is established by a coloring of K_{36} avoiding triangles in the first color and $K_{10}-e$ in the second color. This improves by one the best previously known lower and upper bounds. We also give the bounds for the next Ramsey number of this type, $42 \leq R(K_3, K_{11}-e) \leq 47$.

1. Introduction.

We shall only consider graphs without multiple edges or loops. For graphs G and H, the two-color Ramsey number $R(G, H)$ is the smallest integer n such that, for any graph F on n vertices, either F contains G or $\overline{F}$ contains H, where $\overline{F}$ denotes the complement of F. In this paper we consider the case where $G = K_3$ and H is a complete graph with one edge deleted.

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A graph F is called a *k-graph* if F is triangle-free and $\overline{F}$ does not contain a $K_k{-}e$, where $K_k{-}e$ is a complete graph on k vertices with one edge deleted. Such graphs F can be interpreted as colorings of the edges of the complete graph without a triangle in the first color and without a $K_k{-}e$ in the second color. We define a (k,n)-graph as a k-graph on n vertices, and a (k,n,e)-graph as a (k,n)-graph with e edges. Let $\mathcal{R}(k)$, $\mathcal{R}(k,n)$ and $\mathcal{R}(k,n,e)$ denote the set of all k-graphs, (k,n)-graphs and (k,n,e)-graphs, respectively. Note that $R(K_3,K_k{-}e)$ is the least $n>0$ such that there is no (k,n)-graph. Any $(k,R(K_3,K_k{-}e){-}1)$-graph will be called *critical* for the corresponding Ramsey number, or simply *k-critical.* Let us also note that the definition of Ramsey numbers directly implies $R(K_3,K_{k-1}) \leq R(K_3,K_k{-}e) \leq R(K_3,K_k)$, for all $k \geq 2$.

Table I presents all known values (and lower and upper bounds for $k \leq 11$) of this type of Ramsey number. The main result of this paper is an improvement over the bounds $36 \leq R(K_3,K_{10}{-}e) \leq 39$ established in [12]. In addition, for $k=8$ we enumerate all critical graphs, and for $k=9$ we find five new ones, though there might be more. The uniqueness of the critical graph for $k=7$ was shown in [12]. The upper bound for $k=11$ is discussed for the first time here. The census of Ramsey numbers $R(K_3,G)$ for all connected graphs of orders up to 9 was completed by Brandt, Brinkmann and Harmuth in [1]. For values and bounds on classical and other types of Ramsey numbers, see the regularly updated dynamic survey by the third author [13].

year	reference	k	$R(K_3,K_k{-}e)$	critical graphs
-	trivial	3	5	1, C_4
1972	[2]	4	7	4, including $K_{3,3}$
1977	[3]	5	11	2, including Petersen graph
1980	[4]	6	17	1, Kalbfleisch graph
1982	[6]	7	21	1
1990	[12]	8	25	9
1990	[12]	9	31	≥ 6
2001	this work	10	37-38	unknown
1998	[16], this work	11	42-47	unknown

Table I. Progress on computing $R(K_3,K_k{-}e)$

2. Graph Extensions and Upper Bound on $R(K_3, K_{10}-e)$.

Let $G \in \mathcal{R}(k,n)$ and $v \in V(G)$. Since the neighborhoods of vertices of G are independent sets, the maximum degree of G is at most $k-1$. Furthermore, if we denote by G_v^- the subgraph of G induced by the set of vertices $V(G) - N_v(G) - \{v\}$, then it is clear that $G_v^- \in \mathcal{R}(k-1, \hat{n})$ for $\hat{n} = n - deg_G(v) - 1 \geq n-k$. Denote by $e(k,n)$ and $E(k,n)$ the minimum and maximum number of edges in any (k,n)-graph, respectively.

A collection of algorithms and their implementations to construct all graphs in $\mathcal{R}(k,n,e)$ was described in [11, 14] and used extensively in [12, 15]. This technique requires previous knowledge of all $(k-1, \hat{n}, \hat{e})$-graphs, for $\hat{n}$ and $\hat{e}$ ranging over a set of values which can be determined by the the above observations and by the method of Graver and Yackel [5]. The principle of the latter is contained in the following variation of proposition 4 of [5].

Lemma 1. *For any graph $G \in \mathcal{R}(k,n,e)$,*

$$ne - \sum_{i=0}^{k-1} n_i(e(k-1, n-i-1) + i^2) \geq 0,$$

where n_i is the number of vertices of degree i in G, and

$$n = \sum_{i=0}^{k-1} n_i, \qquad 2e = \sum_{i=1}^{k-1} i n_i.$$

Lemma 1 gives reasonable lower bounds for $e(k,n)$ provided good lower bounds for $e(k-1, n-i-1)$ are given. Furthermore, it permits the design of efficient extension algorithms reconstructing the graph G from G_v^- for some v, following the ideas originated by Grinstead and Roberts [7] (where they used them to evaluate $R(K_3, K_9) = 36$), and those employed by the authors in computations estimating other triangle Ramsey numbers [11, 12, 14, 15]. For this work we have implemented similar algorithms for the case of k-graphs, and they have produced the results gathered below. In this paper, by an *extension algorithm* we will understand an application of the methods referenced in this paragraph, which construct graphs G from H, such that there exists a vertex $v \in V(G)$ for which G_v^- is isomorphic to H. Clearly, if $G \in \mathcal{R}(k,n)$ and $deg_G(v) = d$, then $H \in \mathcal{R}(k-1, n-d-1)$.

Our starting set of graphs consisted of $\mathcal{R}(6)$, which contains 5017 nonisomorphic graphs as found in [12]. By extending appropriate subfamilies of $\mathcal{R}(6)$, we first obtained $\mathcal{R}(7,n)$ for all $n \geq 16$. Table II includes some data about these constructed graphs, confirming and extending the previous results about $\mathcal{R}(7)$ presented in [12]. Because of the very large number of graphs in $\mathcal{R}(7,15)$, we did not attempt their exhaustive generation.

n	16	17	18	19	20
$\|R(7,n)\|$	158459	4853	225	1	1
$e(7,n)$	30	37	43	54	60
$E(7,n)$	48	50	51	54	60

Table II. Statistics of $(7,n)$-graphs for $n \geq 16$

The next stage consisted of the generation of complete sets $\mathcal{R}(8,n)$ for $n \geq 23$ and $\mathcal{R}(8,22,e)$ for $e \leq 65$. Any $(8,23)$- or $(8,24)$-graph must be an extension of a $(7,\hat{n})$-graph for some $\hat{n} \geq 16$. Using simple vertex degree counting arguments and/or Lemma 1 one can easily see that the graphs reported in Table II are also sufficient to obtain all $(8,22,e)$-graphs, for $e \leq 65$, by applying an extension algorithm.

n	22	23	24
$\|R(8,n)\|$	*80219	36	9
$e(8,n)$	59	70	80
$E(8,n)$	74	77	84

*only graphs with $e \leq 65$

Table III. Statistics of $(8,n)$-graphs for $n \geq 22$

The computations produced graphs reported in Table III, which again confirmed and extended the previous results about $\mathcal{R}(8)$ presented in [12]. In particular, now we claim that there are exactly 9 nonisomorphic 8-critical graphs; one with 80 edges, and eight 7-regular 8-critical graphs in $\mathcal{R}(8,24,84)$. The one with the largest automorphism group, with 84 edges, and one with 80 edges, have been previously reported in [12].

For $k = 8$ and lower values on n we have found the sets of graphs with the minimum number of edges, namely: $|\mathcal{R}(8,19,37)| = 20$ for $e(8,19) = 37$, $|\mathcal{R}(8,20,44)| = 169$ for $e(8,20) = 44$, and $|\mathcal{R}(8,21,51)| = 7$ for $e(8,21) = 51$. We have also obtained $e(8,17) = 25$ and $e(8,18) = 30$.

Lemma 2. *The following bounds on the minimum number of edges hold:*

$$71 \le e(9,26) \le 73,$$

$$81 \le e(9,27) \le 83,$$

$$91 \le e(9,28) \le 95,$$

$$101 \le e(9,29) \le 106,$$

$$113 \le e(9,30) \le 117.$$

Proof. The lower bounds 71, 80, 90, 100 and 111, respectively, can be obtained by applying only Lemma 1 to the known values of $e(8,\hat{n})$, as in [12]. We can improve all but the first bound by performing further computations which reconstruct graphs G from known $G_v^- \in R(8,\hat{n})$ with minimal number of edges. For each $\hat{n} \ge 19$, we computed all extensions of $(8,\hat{n},e(8,\hat{n}))$-graphs to $(9,n)$-graphs, and none with the number of edges smaller than a corresponding lower bound of Lemma 2 was found. Now, if we apply Lemma 1 with $k = 9$ and $27 \le n \le 30$, but using the values of $e(8,\hat{n})$ each increased by 1, then in all cases there are no integral solutions with e smaller than the corresponding lower bound. Thus the lower bounds for $n \ge 27$ are valid. The bound $71 \le e(9,26)$ follows from an application of Lemma 1 to the true values of $e(8,\hat{n})$. The upper bounds are equal to the smallest number of edges in the graphs constructed by running an extension algorithm on graphs other than edge-minimal (3,8)-graphs, and by some heuristic searches. ∎

The first 9-critical graph, an 8-regular circulant graph on 30 vertices and 120 edges, with the vertices connected iff they are in distance 1, 3, 9 or 14, was constructed in [12]. During the computations described above five further $(9,30)$-graphs were found, three with 117, 118, and 119 edges, respectively, and two regular of degree 8 having automorphism groups with 2 orbits.

Theorem 1. $R(K_3, K_{10}-e) \le 38$.

Proof. As observed in [12], the facts that $90 < e(9,28)$, the lower bounds on $e(9,n)$ for $n \ge 27$ listed at the start of the proof of Lemma 2, and Lemma 1, imply that

$R(K_3, K_{10}-e) \leq 38$. The theorem follows since in Lemma 2 we have established that $91 \leq e(9, 28)$. ∎

Further improvement of the upper bound on $R(K_3, K_{10}-e)$ would require very large scale computations, despite the fact that the data we gathered here about $\mathcal{R}(8)$ and bounds in Lemma 2 are well beyond what was needed in the proof of Theorem 1. It is possible, however, that some method similar to the linear programming techniques used in other Ramsey computations by the authors [10] could lead to a better algorithm in this case, and thus to the exact value of $R(K_3, K_{10}-e)$, which we conjecture to be 37.

3. Construction.

In order to improve the lower bound on $R(K_3, K_{10}-e)$ we have applied repeatedly the following reduction/extension process. Given a $(10, n)$-graph G we first delete one or more vertices then extend it in all possible ways to larger 10-graphs. Using as a starting graph the only known critical graph for $R(K_3, K_9)$ on 35 vertices, which clearly is a $(10, 35)$-graph, after considerable computations we have constructed 40 nonisomorphic graphs in $\mathcal{R}(10, 36)$. This improves by one the easy lower bound $R(K_3, K_{10}-e) \geq 36$ implied by $R(K_3, K_9) = 36$. The number of edges in the constructed graphs is ranging from 156 to 162. Six of them are 9-regular graphs of which four are vertex transitive. One of the latter is presented in Figure 1.

Theorem 2. *$R(K_3, K_{10}-e) \geq 37$.*

Proof. The lower bound is established by a (10,36,162)-graph whose adjacency lists are presented in Figure 1. ∎

```
 0:  1 2 4 8 12 18 20 23 24          1:  0 3 5 9 13 19 21 22 25
 2:  0 3 6 10 14 16 21 22 26         3:  1 2 7 11 15 17 20 23 27
 4:  0 5 7 10 13 19 27 32 34         5:  1 4 6 11 12 18 26 33 35
 6:  2 5 7 8 15 17 25 32 34          7:  3 4 6 9 14 16 24 33 35
 8:  0 6 9 11 13 19 27 33 35         9:  1 7 8 10 12 18 26 32 34
10:  2 4 9 11 15 17 25 33 35        11:  3 5 8 10 14 16 24 32 34
12:  0 5 9 13 15 16 27 28 31        13:  1 4 8 12 14 17 26 29 30
14:  2 7 11 13 15 18 25 28 31       15:  3 6 10 12 14 19 24 29 30
16:  2 7 11 12 17 19 25 29 30       17:  3 6 10 13 16 18 24 28 31
18:  0 5 9 14 17 19 27 29 30        19:  1 4 8 15 16 18 26 28 31
20:  0 3 21 25 26 29 31 32 35       21:  1 2 20 24 27 28 30 33 34
22:  1 2 23 24 27 29 31 32 35       23:  0 3 22 25 26 28 30 33 34
24:  0 7 11 15 17 21 22 25 26       25:  1 6 10 14 16 20 23 24 27
26:  2 5 9 13 19 20 23 24 27        27:  3 4 8 12 18 21 22 25 26
28:  12 14 17 19 21 23 29 32 35     29:  13 15 16 18 20 22 28 33 34
```

30:	13 15 16 18 21 23 31 32 35	31:	12 14 17 19 20 22 30 33 34
32:	4 6 9 11 20 22 28 30 33	33:	5 7 8 10 21 23 29 31 32
34:	4 6 9 11 21 23 29 31 35	35:	5 7 8 10 20 22 28 30 34

Figure 1. An example of a vertex-transitive 9-regular (10,36,162)-graph

4. Bounds on $R(K_3, K_{11}-e)$.

We conclude this paper with a section noting the bounds on the next Ramsey number of this type, $R(K_3, K_{11}-e)$. First we need an estimate on the number of edges in $(10, 36)$- and $(10, 37)$-graphs.

Lemma 3. $146 \leq e(10, 36)$*, and if a* $(10, 37)$*-graphs exist, then* $160 \leq e(10, 37)$.

Proof. Using lower bounds from Lemma 2, the inequality in Lemma 1 for $k = 10$ has no solutions for $n = 37$ and $e < 160$, nor any solutions for $n = 36$ and $e < 146$. Hence the lower bounds hold. ∎

Theorem 3. $42 \leq R(K_3, K_{11}-e) \leq 47$.

Proof. The lower bound was established by Wang, Wang and Yan in [16], who constructed a cyclic graph with the vertex set Z_{41}, in which 1, 4, 10, 16, 18 are the distances between vertices connected by edges. For the upper bound, assume $G \in \mathcal{R}(11, 47)$. Note that G can have only vertices of degree 9 and 10. Applying bounds from Lemma 3 in Lemma 1 leads to a contradiction. ∎

Further improvements of the upper bound in Theorem 3 are possibly not very hard to obtain with the methods of this paper, in contrast to the current bounds on $R(K_3, K_{10}-e)$, which would require a new approach or a much larger computational effort.

For completeness, we note the current state of knowledge for the next few values of k. We found many thousands of $(12, 45)$-graphs, so $R(K_3, K_{12}-e) \geq 46$. The paper [16] contains circulant $(13, 53)$-graphs, $(14, 58)$-graphs and $(15, 68)$-graphs. We found an additional 14 $(14, 58)$-graphs, all of them 13-regular, but did not improve any of these lower bounds.

All the computations needed for this work were done at least twice, by algorithms implemented independently by different authors. Two general utility programs for combinatorial computing, written by the first author, were used extensively: *nauty* [8] for graph isomorph rejection, and *autoson* [9] for distributing jobs over local area networks.

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