Graceful and harmonious labellings of trees

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Abstract

We establish that all trees on at most 27 vertices admit graceful labellings and all trees on at most 26 vertices admit harmonious labellings.

A *graceful labelling* of a graph G with q edges is an injection $f : V(G) \to \{0, 1, 2, \ldots, q\}$ such that when each edge $xy \in E(G)$ is assigned the label, $|f(x) - f(y)|$, all of the edge labels are distinct. A graph which admits a graceful labelling is said to be *graceful.* This idea was introduced by Rosa [5] where it was shown that if all trees are graceful, then the Ringel-Kotzig conjecture is true. (Ringel [4] conjectured that K_{2n+1} can be decomposed into $2n + 1$ subgraphs which are isomorphic to a given tree with n edges. Kotzig conjectured K_{2n+1} can be cyclically decomposed into $2n + 1$ subgraphs which are isomorphic to a given tree with n edges.) In the same paper, Rosa showed that several families of trees are graceful and also that all trees on at most 16 vertices are graceful. Since this paper many papers have been written about graceful graphs and in particular graceful trees (see [1],[2]) but, apart from several more families, there has been no advance from 16 on the maximum order for which all trees are known to be graceful. In this note we describe a computer search by which we have been able to establish the following result.

Theorem 1. *All trees on at most 27 vertices are graceful.*

Before describing the method by which the above result is obtained, we introduce a similar problem of graph labelling to which we have been able to apply the same method.

In [3], Graham and Sloan made the following definition. An *harmonious labelling* of a given a graph G with q edges is an assignment to each vertex $x \in V(G)$, a distinct element $\lambda(x)$, of the group of integers modulo q, so that when each edge xy is labelled $\lambda(x) + \lambda(y)$, the edge labels are all distinct. In the case of a tree with q edges, one element of the group is assigned to two vertices while all others are used precisely once. A graph which admits an harmonious labelling is said to be *harmonious.*

In the same paper it is verified that all trees on at most 10 vertices are harmonious and it is conjectured that all trees are harmonious. Employing a computer search similar to that used to find graceful labellings for trees we have established the following theorem.

Theorem 2. *All trees on at most 26 vertices are harmonious.*

Graceful and Harmonious alg.

For a given tree T and labelling L of the vertices, let z(T,L) be the number of distinct edge labels.

For n=—V(T)—, the aim is to find L such that z(T,L)=n-1.

If L is a labelling and $v, w \in V(T)$, define $S_w(L; v, w)$ to be the labelling got from L by swapping the labels on v and w.

The method is like this, using a parameter M:

1. Start with any labelling of $V(T)$.
2. If $z(T, L) = n - 1$, stop.
3. For each pair $\{v, w\}$, replace L by $L' = S_w(L; v, w)$ if $z(T, L') > z(T, L)$.
4. If step 3 finishes with L unchanged, replace L by $S_w(L; v, w)$, where $\{v, w\}$ is chosen at random from the set of all $\{v, w\}$ such that

   (a) $\{v, w\}$ has not been chosen during the most recent M times this step has been executed.

   (b) $S_w(L; v, w)$ is maximal subject to (a).
5. Repeat from step 2.

This method can be described as a combination of hill-climbing and tabu search. Sometimes it appears to "get stuck" and needs to be restarted from step 1 with a new labelling chosen at random.

A value of $M = 10$ seems ok for small trees, but a slightly larger value seems to be needed for larger trees. The purpose of M is to prevent the algorithm from repeatedly cycling around within some small set of labellings.

Since the generation algorithm produces trees in an order such that most trees are very similar to the previous tree, it proved advantageous to use the graceful/harmonious labelling of each tree as the starting point for the next tree.

References

[1] I. Cahit, Status of the graceful tree conjecture in 1989, in *Topics in Cominatorics and Graph Theory,* R. Bodendiek and R. Henn (eds), *Physica-Verlag,* Heidelberg, 1990.

[2] J. A. Gallian, A dynamic survey of graph labeling, submitted.

[3] R. L.Graham and N. J. A. Sloane, On additive bases and harmonious graphs, *SIAM J. Alg. Disc. Meth.,* **1**, (1980), 382-404.

[4] G. Ringel, Problem 25, in *Theory of Graphs and its Applications, Proc. Symposium SSmolenice 1963,* Prague, 1963.

[5] A. Rosa, On certain valuations of the vertices of a graph, in *Theory of Graphs (Internat. Symposium, Rome, July, 1966,* Gordon and Breach, N. Y. and Dunod Paris, (1967), 349-355.

[6] R. A. Wright, L. B. Richmond, A. Odlyzko and B. D. McKay, Constant time generation of free trees, *SIAM J. Computing,* **15**, (1986), 540-548.