VERTEX-TRANSITIVE GRAPHS WHICH ARE NOT CAYLEY GRAPHS, I

Brendan D. McKay
Computer Science Department, Australian National University
GPO Box 4, ACT 2601, Australia

Cheryl E. Praeger
Department of Mathematics, University of Western Australia
Nedlands, WA 6907, Australia

Abstract.

The Petersen graph on 10 vertices is the smallest example of a vertex-transitive graph which is not a Cayley graph. We consider the problem of what orders such graphs have. In this, the first of a series of papers, we present a sequence of constructions which solve the problem for many orders. In particular, such graphs exist for all orders divisible by a fourth power, and all even orders which are divisible by a square.

1. Introduction.

Unless otherwise indicated, our graph-theoretic terminology will follow [3], and our group-theoretic terminology will follow [18].

If Γ is a graph, then $V\Gamma$, $E\Gamma$ and $\mathrm{Aut}(\Gamma)$ will denote its vertex-set, its edge-set, and its automorphism group, respectively. The cardinality of $V\Gamma$ is called the *order* of Γ, and Γ is called *vertex-transitive* if the action of $\mathrm{Aut}(\Gamma)$ on $V\Gamma$ is transitive.

For a group G and a subset $C \subset G$ such that $1_G \notin C$ and $C^{-1} = C$, the *Cayley graph* of G *relative to* C, $\mathrm{Cay}(G, C)$, is defined as follows. The vertex-set of $\mathrm{Cay}(G, C)$ is G, and two vertices $g, h \in G$ are adjacent in $\mathrm{Cay}(G, C)$ if and only if $gh^{-1} \in C$. It is easy to see that $\mathrm{Cay}(G, C)$ admits a copy of G acting regularly (by right multiplication) as a group of automorphisms, and so every Cayley graph is vertex-transitive. Conversely, every vertex-transitive graph which admits a regular group of automorphisms is (isomorphic to) a Cayley-graph of that group. However, there are vertex-transitive graphs which are not Cayley graphs, the smallest example being the well-known Petersen graph. Such a graph will be called a *non-Cayley* vertex-transitive graph, and its order will be called a *non-Cayley number.* Let *NC* be the set of all non-Cayley numbers.

In Table 1, we list, for $n \leq 26$, the total number t_n of vertex-transitive graphs of order n and the number u_n of vertex-transitive graphs of order n which are not Cayley graphs. These numbers are taken from [12], [13], [16] and [17]. It seems that, for small orders at least, the great majority of vertex-transitive graphs are Cayley

n	t_n	u_n	n	t_n	u_n	n	t_n	u_n
1	1	–	10	22	2	19	60	–
2	2	–	11	8	–	20	1214	82
3	2	–	12	74	–	21	240	–
4	4	–	13	14	–	22	816	–
5	3	–	14	56	–	23	188	–
6	8	–	15	48	4	24	15506	112
7	4	–	16	286	8	25	464	–
8	14	–	17	36	–	26	4236	132
9	9	–	18	380	4	27	1434	–

Table 1. The numbers of vertex-transitive graphs.

graphs. We expect this trend to continue to larger orders, but do not know how to prove it.

The problem of determining NC was posed by Marušič [8]. Since the union of finitely many copies of a vertex-transitive graph Γ is a Cayley graph if and only if Γ is a Cayley graph, we see that any multiple of a member of NC is also in NC. Thus, it will suffice to find those members of NC whose non-trivial divisors are not members of NC. The most important previous results on this problem can be summarised as follows.

Theorem 1. *Let p and q be distinct primes. Then*
(a) $p, p^2, p^3 \notin NC$,
(b) $2p \in NC$ *if and only if* $p \equiv 1 \pmod 4$,
(c) $pq \in NC$ *if* $p \equiv 1 \pmod{q^2}$,
(d) $\binom{m}{r} \in NC$ *if* $r \geq 2$ *and* $m \geq 2r+1$, *except possibly if* $r = 2$ *and* m *is a prime power of the form* $4k+3$.
(e) $12, 21 \notin NC$, *and*
(f) $15, 16, 18, 20, 24, 28, 56, 84, 102 \in NC$. ∎

Part (a) is proved in [9]. A non-Cayley vertex-transitive graph of order $2p$, $p \equiv 1 \pmod 4$, was constructed in [4]. On the other hand, it was shown in [2] that all vertex-transitive graphs of order $2p$, $p \equiv 3 \pmod 4$, are Cayley graphs, provided that the only simply primitive permutation groups of degree $2p$ are A_5 and S_5 of degree 10. This fact about primitive groups was verified in [6] using the finite simple group classification, thus proving part (b). Parts (c) and (d) were proved in [1] and [5] respectively by constructions of non-Cayley vertex-transitive graphs of the relevant orders. (The other exceptional cases given in [5] are covered by part (f).) The results of parts (e) and (f) are reported in [7], [12], [13], [15], and [17].

In the paper [9], a construction was proposed for a non-Cayley vertex-transitive graph of order p^k, $k \geq 4$. However, we believe that the construction as given is invalid, yielding a Cayley graph in at least some cases (for example, when $p^k = 3^4$). In Section 5 we will give a correct construction for such graphs of order p^4.

Our paper contains constructions of four families of non-Cayley vertex-transitive graphs: besides the p^4 construction, we produce such graphs of orders p^2q for certain primes p and q, and of orders $8m$ and $2m^2$ for most m. The implications of our constructions for the membership of NC can be summarised as follows.

Theorem 2.
(a) $m^4 \in NC$ *for all* $m \geq 2$.
(b) $p^2q \in NC$ *if* $p \geq 2$ *and* $q \geq 3$ *are distinct primes with* q *not dividing* $p^2 - 1$.
(c) For each $m \geq 7$, $2m \in NC$ *except possibly if* m *is the product of distinct primes of the form* $4k + 3$.
(d) $k^2m^2 \in NC$ *for all* $k, m \geq 2$. ∎

Part (a) follows from Theorem 1(f) if m is even and will be proved in Theorem 6 for odd m. Part (b) will be proved in Theorem 3. Suppose that $m \geq 7$. If m is even, then $2m \in NC$ by parts (a) and (b) above and Theorem 1(f). Also if m is divisible by a prime of the form $4k+1$, then $2m \in NC$ by Theorem 1(b), while if m is divisible by the square of a prime, then $2m \in NC$ by Theorems 3 and 5. Part (d) is a corollary of parts (a) and (b).

The $8m$ construction given in Theorem 4 is not actually needed for the proof of Theorem 2. We have included it because the construction is significantly different from our other constructions.

For integers r and s, we write $r \mid s$ if r is a divisor of s. For an integer $m > 0$, $\mathbb{Z}_m$ denotes the ring of integers modulo m, S_m denotes the symmetric group on m letters, and D_m denotes the dihedral group of order m.

In the second paper of this series, we will present some additional constructions of graphs with orders of the form p^kq for distinct primes p and q. We will also complete the classification, begun in [10], [11] and [15], of all non-Cayley vertex-transitive graphs of order pq, by computing the full automorphism groups of all these graphs. In [10], it is shown that such a graph is either metacirculant or belongs to a family of graphs admitting $SL(2, p-1)$ as a group of automorphisms, where p is a Fermat prime and q divides $p - 2$. The possible orders for the first family are determined in [1], whilst the second family is further investigated in [11]. The complete classification for the vertex-primitive case was done in [15].

2. Construction One.

Let p and q be distinct primes with $q \geq 3$. We investigate the graph $C = C(p,q,2)$ defined in [14], where

$$VC = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q \quad \text{and}$$
$$EC = \{(x,y,k)(z,x,k+1) \mid x,y,z \in \mathbb{Z}_p,\ k \in \mathbb{Z}_q\}.$$

It was shown in [14, Theorem 2.13] that the automorphism group of C is $A = \langle \rho, \eta, \sigma = \sigma(\sigma_0, \sigma_1, \ldots, \sigma_{q-1}) \mid \sigma_0, \sigma_1, \ldots, \sigma_{q-1} \in S_p \rangle = S_p \text{ wr } D_{2q}$, where

$$(x,y,k)^\rho = (x,y,k+1),$$
$$(x,y,k)^\eta = (y,x,-k), \quad \text{and}$$
$$(x,y,k)^\sigma = (x^{\sigma_k}, y^{\sigma_{k-1}}, k)$$

for all $(x,y,k) \in VC$. Since A acts transitively on VC, we see that C is vertex-transitive.

For $k \in \mathbb{Z}_q$, define $B_k = \{(x,y,k) \mid x,y \in \mathbb{Z}_p\}$, and let $B = \{B_0, B_1, \ldots, B_{q-1}\}$. It is clear that B is a block system preserved by A. We shall determine precisely when C is a Cayley graph. To do this we need the information in the following two lemmas.

Lemma 1. *Any element of A of order q which induces the same permutation of B as ρ does is conjugate to ρ in A.*

Proof. Such an element has the form $\rho\sigma$ for some $\sigma = \sigma(\sigma_0, \sigma_1, \ldots, \sigma_{q-1})$. Since $(\rho\sigma)^q = 1$, we have $\sigma_0\sigma_1 \cdots \sigma_{q-1} = 1$. Now define $\tau_0 = 1$ and $\tau_k = \sigma_0\sigma_1 \cdots \sigma_{k-1}$ for $k \geq 1$. Then $\rho\sigma = \rho^{\sigma(\tau_0, \ldots, \tau_{q-1})}$. ∎

Lemma 2. *A matrix $X = X(u,v)$ over* GF(p) *of the form*

$$\begin{pmatrix} u & v \\ 1 & 0 \end{pmatrix},$$

such that $X^q = 1$, exists if and only if $q \mid p^2 - 1$.

Proof. Since $|GL(2,p)| = p(p-1)(p^2-1)$, it is clear that X cannot exist unless $q \mid p^2 - 1$.

Suppose then that $q \mid p^2 - 1$, and let z be a primitive q-th root of 1 in GF(p^2). Set $u = z + z^{-1}$. If $q \mid p - 1$ then $z^p = z$, while if $q \mid p + 1$ then $z^p = z^{-1}$, and hence $u^p = z^p + z^{-p} = u$, so $u \in$ GF(p). Now consider $X = X(u,-1)$. Since X has characteristic polynomial $f(\lambda) = \lambda^2 - u\lambda + 1 = (\lambda - z)(\lambda - z^{-1})$, the polynomial $f(\lambda)$ is a divisor of $\lambda^q - 1$ and so $X^q = 1$. [Thanks to Peter Montgomery, Michael Larsen, Victor Miller and Carl Riehm.] ∎

Theorem 3. *Let p and q be distinct primes with $q \geq 3$. Then $C = C(p, q, 2)$ is vertex-transitive, and C is a Cayley graph if and only if $q \mid p^2 - 1$. Thus $p^2q \in NC$ if q does not divide $p^2 - 1$.*

Proof. Suppose that q does not divide $p^2 - 1$. If A has a regular subgroup R then R has a unique Sylow q-subgroup Q of order q, by Sylow's Theorem. Since $Q \trianglelefteq R$, the subgraphs of C induced on the orbits of Q must all be isomorphic. However it follows from Lemma 1 that Q is generated by some conjugate of ρ, and some orbits of $\langle \rho \rangle$ contain no edges while others induce a cycle of length q. This contradiction proves that C is a non-Cayley graph in this case.

Suppose instead that $q \mid p^2 - 1$. Let X be a matrix satisfying the conditions of Lemma 2 and let $\alpha \in S_p$ be the permutation $(0\ 1\ \cdots\ p{-}1)$. For $x, y \in \mathbb{Z}_p$ and $k \geq 0$, define $\binom{a_k(x,y)}{b_k(x,y)} = X^k \binom{x}{y}$. Then $H = \{\sigma(\alpha^{a_0(x,y)}, \alpha^{a_1(x,y)}, \ldots, \alpha^{a_{q-1}(x,y)}) \mid x, y \in \mathbb{Z}_p\}$ is a subgroup of A which fixes B blockwise and acts faithfully and regularly on each block. Moreover, $H^\rho = H$, so $\langle H, \rho \rangle$ is a regular subgroup of A. ∎

3. Construction Two.

Let $m \geq 2$. Define the graph $L = L(8m)$ of order $8m$ thus:

$$
\begin{aligned}
VL &= \{x_i, y_i \mid i \in \mathbb{Z}_{4m}\} \quad \text{and} \\
EL &= \{x_i x_{i+1}, y_i y_{i+1} \mid i \in \mathbb{Z}_{4m}\} \\
&\quad \cup \{x_i y_j \mid i \equiv j \equiv 0 \pmod 4 \text{ or } i \equiv j \equiv 3 \pmod 4 \\
&\qquad\qquad \text{or } i \equiv 1,\ j \equiv 2 \pmod 4 \text{ or } i \equiv 2,\ j \equiv 1 \pmod 4;\ i, j \in \mathbb{Z}_{4m}\}.
\end{aligned}
$$

It is easy to verify that the permutations γ and δ of VL, defined by

$$
\begin{aligned}
\gamma &= (x_0\ y_0)(x_1\ y_1)\cdots(x_{4m-1}\ y_{4m-1}) \quad \text{and} \\
\delta &= (x_0\ x_2\ x_4 \cdots x_{4m-2})(x_1\ x_3\ x_5 \cdots x_{4m-1})(y_0\ y_1)(y_2\ y_{4m-1})\cdots(y_{2m}\ y_{2m+1})
\end{aligned}
$$

are automorphisms of L. Moreover, $\langle \gamma, \delta \rangle$ is transitive, so L is vertex-transitive.

Lemma 3. $B = \{\{x_0, x_1, \ldots, x_{4m-1}\}, \{y_0, y_1, \ldots, y_{4m-1}\}\}$ *is a block system for* $\mathrm{Aut}(L)$.

Proof. The claim is easily verified directly for $m = 2$, so suppose $m > 2$. Consider the subgraph L' of L induced by those edges of L which lie in m or fewer 4-gons. A simple count shows that these are exactly those edges which join two x-vertices or two y-vertices. Hence the components of L' are the elements of B, which proves the lemma. ∎

Theorem 4. *Let $m \geq 2$. Then $L(8m)$ is vertex-transitive but not a Cayley graph. Thus $8m \in NC$ for $m \geq 2$.*

Proof. Suppose that $\mathrm{Aut}(L)$ contains a regular subgroup R. Then R has a subgroup of order $4m$ which fixes the two blocks of B setwise and acts regularly on each of them. Moreover, the subgraph of L induced by each of these blocks is a $4m$-gon, and so R contains an element of the form $(x_0\ x_2 \cdots x_{4m-2})(x_1\ x_3 \cdots x_{4m-1})(y_0\ y_2 \cdots y_{4m-2})^k$ $(y_1\ y_3 \cdots y_{4m-1})^k$, for some k with $(2m, k) = 1$. However, each permutation of this form maps the edge x_0y_0 onto the non-edge x_2y_{2k}. (Note that $2k \equiv 2 \pmod 4$.) This contradiction proves that L is a non-Cayley graph. ∎

4. Construction Three.

Let $m \geq 3$ be an integer. Define the graph $T = T(2m^2)$ of order $2m^2$ as follows:

$$\begin{aligned} VT &= \mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_2 \quad \text{and} \\ ET &= E_1 \cup E_2 \cup E_3, \end{aligned}$$

where

$$\begin{aligned} E_1 &= \{(x,y,0)(x+1,y,0), (x,y,1)(x,y+1,1) \mid x,y \in \mathbb{Z}_m\}, \\ E_2 &= \{(x,y,0)(x+1,y-1,0), (x,y,1)(x+1,y+1,1) \mid x,y \in \mathbb{Z}_m\} \quad \text{and} \\ E_3 &= \{(x,y,0)(x-1,y-1,1), (x,y,0)(x-1,y+1,1), \\ &\qquad (x,y,0)(x+1,y-1,1), (x,y,0)(x+1,y+1,1) \mid x,y \in \mathbb{Z}_m\}. \end{aligned}$$

It is easy to verify that the permutations α, β, γ defined by

$$\begin{aligned} (x,y,k)^\alpha &= (x+1,y,k), \\ (x,y,k)^\beta &= (x,y+1,k) \quad \text{and} \\ (x,y,k)^\gamma &= (-y,x,k+1) \end{aligned}$$

for all $(x,y,k) \in VT$, are automorphisms of T. Let $A = \langle \alpha, \beta, \gamma \rangle$ and, for $k \in \mathbb{Z}_2$, define $B_k = \{(x,y,k) \mid x,y \in \mathbb{Z}_m\}$. Then A has order $4m^2$, is transitive on VT, and has $\{B_0, B_1\}$ as a block system.

Lemma 4. *If $m = 3$ or $m \geq 5$, then* $\mathrm{Aut}(T(2m^2)) = A$.

Proof. The graph $T(18)$ appears in [12] as R147, and an explicit computation there showed that $\mathrm{Aut}(T(18)) = A$. Now consider $m \geq 5$. For distinct vertices $v, w \in VT$, define $f(v,w)$ to be the number of paths of length 3 from v to w in T. By direct enumeration of the possibilities, we find that

$$f(v,w) = \begin{cases} 6, & \text{if } vw \in E_1; \\ 8, & \text{if } vw \in E_2; \\ 7, & \text{if } vw \in E_3, \end{cases}$$

and so $\mathrm{Aut}(T)$ fixes the sets E_1, E_2 and E_3 setwise. The subgraph of T with edge-set $E_1 \cup E_2$ has components with vertex-sets B_0 and B_1, and so $\{B_0, B_1\}$ is a block system for $\mathrm{Aut}(T)$. Let G be the setwise stabiliser of B_0 in $\mathrm{Aut}(T)$.

From each $(x,y,0)$, the only vertex that can be reached in two distinct ways by taking an edge in E_2 followed by an edge in E_3 is $(x,y,1)$. Therefore, G acts faithfully on B_0. The subgraph induced by B_0 consists of a cartesian product of two polygons, with m disjoint m-gons of edges from E_1 orthogonal to m disjoint m-gons of edges from E_2. The full automorphism group of such an edge-coloured graph is isomorphic to $D_{2m} \times D_{2m}$. Thus $G \le D_{2m} \times D_{2m}$ and $|A \cap G| = 2m^2$. Hence, if G_0 is the stabiliser of $(0,0,0)$, then G_0, in its action on B_0, is a subgroup of $\langle g, h \rangle$, where $(x,y,0)^g = (-x-2y, y, 0)$ and $(x,y,0)^h = (x+2y, -y, 0)$ for every x, y. However, $f\big((1,0,0),(1,-1,0)\big) = 6$ whilst $f\big((1,0,0),(-1,1,0)\big) = 3$, so $h \notin G_0$. On the other hand $\gamma^2 \in G_0$ acts on B_0 in the same way that gh does, and it follows that $G_0 = \{1, \gamma^2\}$, whence $G = \langle \alpha, \beta, \gamma^2 \rangle$ and $\mathrm{Aut}(T) = A$. ∎

Theorem 5. *If $m = 3$ or $m \ge 5$, then $T = T(2m^2)$ is vertex-transitive but not a Cayley graph. Thus $2m^2 \in NC$ if $m = 3$ or $m \ge 5$.*

Proof. By Lemma 4, $\mathrm{Aut}(T) = A$. Since $\{B_0, B_1\}$ is a block system for A, it is a block system for any regular subgroup $R \le A$. Now, as γ^2 fixes $(0,0,0)$ and R is regular, $\gamma^2 \notin R$. But, as R has index 2 in A, R must contain the square of every element of A and hence $\gamma^2 \in R$, which is a contradiction. Thus T is not a Cayley graph. ∎

5. Construction Four.

Let p be an odd prime, and define $a = p+1$. Note that a has multiplicative order p in $\mathbb{Z}_{p^2}$ and multiplicative order p^2 in $\mathbb{Z}_{p^3}$.

Let $U = \mathbb{Z}_p \times \mathbb{Z}_{p^2}$. Define the permutations α and β of U by $(i,j)^\alpha = (i, j+1)$ and $(i,j)^\beta = (i+1, aj)$ for $(i,j) \in U$, and define $H = \langle \alpha, \beta \rangle$. The proof of the following lemma follows on noting that $\alpha^{p^2} = \beta^p = 1$ and $\alpha^\beta = \alpha^{p+1}$.

Lemma 5. *The group H is regular on U. Also, the elements of H with order p are exactly those of the form β^t for $1 \le t \le p-1$ or $\alpha^{up}\beta^t$ for $1 \le u \le p-1$ and $0 \le t \le p-1$.* ∎

Next, we define a Cayley graph F of H which will be used in our construction of a graph of order p^4. Define

$$VF = U, \quad \text{and}$$
$$EF = E_1 \cup E_2 \cup E_3,$$

with

$$E_1 = \{(i,j)(i,j') \mid (i,j),(i,j') \in U, j \neq j'\},$$
$$E_2 = \{(i,j)(i+1,j) \mid (i,j) \in U\} \quad \text{and}$$
$$E_3 = \{(i,j)(i+1,j+a^i) \mid (i,j) \in U\}.$$

Lemma 6. $\mathrm{Aut}(F) = H$.

Proof. It is easy to see that $H \le \mathrm{Aut}(F)$.

The graph F contains exactly p cliques $J_0, J_1, \ldots, J_{p-1}$ of order p^2, where $J_i = \{(i,j) \mid j \in \mathbb{Z}_{p^2}\}$ for $i \in \mathbb{Z}_p$. The edges they contain are exactly those in E_1. We observe that the only subset of $\{1, a, a^2, \ldots, a^{p-1}\}$ which sums to a multiple of p^2 is the empty subset. Therefore, the only cycles of length p in F which meet all the above p^2-cliques are those formed by the edges in E_2. We conclude that the edge-sets E_1, E_2 and E_3 are fixed setwise by $\mathrm{Aut}(F)$.

Suppose that $\mathrm{Aut}(F) \neq H$. Then there is an automorphism g of prime order which fixes $(0,0)$ but moves some vertex adjacent to $(0,0)$. Now, g fixes J_0 setwise, and either fixes J_1 and J_{p-1} setwise or interchanges them. If g fixes J_1 setwise, then g induces an automorphism of the subgraph consisting of the edges between J_0 and J_1. However, this subgraph is a $2p^2$-cycle with edges alternately in E_2 and E_3, and such an edge-coloured graph has no non-trivial automorphism which fixes a vertex, and hence g fixes $J_0 \cup J_1$ pointwise. A similar argument shows that g fixes J_{p-1} pointwise also, which is a contradiction. Alternatively, suppose that g has order 2 and interchanges J_1 and J_{p-1}. If we take $2k$ steps along the edges between J_0 and J_1, starting at vertex $(0,0)$ and using an edge from E_3 first, we finish at vertex $(0,k)$. The same procedure between J_0 and J_{p-1} takes us to vertex $(0, k(p-1))$. Hence g acts on J_0 as $(0,j)^g = (0,(p-1)j)$, for all j, contradicting the assumption that g has order 2. ∎

Now let $W = \mathbb{Z}_p \times \mathbb{Z}_{p^3}$, and define the graph $M = M(p^4)$ of order p^4 as follows:

$$VM = W, \quad \text{and}$$
$$EM = \{(i,j)(i,j+pk), (i,j)(i+1,j),$$
$$(i,j)(i+1,j+pa^i), (i,j)(i+1,j+a^{rp+i}) \mid (i,j) \in W, k \in \mathbb{Z}_{p^2}, r \in \mathbb{Z}_p\}.$$

Theorem 6. *If p is an odd prime, then $M = M(p^4)$ is vertex-transitive but not a Cayley graph. Thus $p^4 \in NC$ for all odd primes p.*

Proof. Define the permutations γ, δ of W by $(i,j)^\gamma = (i, j+1)$ and $(i,j)^\delta = (i+1, aj)$ for $(i,j) \in W$. It is easily verified that $\langle \gamma, \delta \rangle \le \mathrm{Aut}(M)$, and so M is vertex-transitive. (This group is the same as that used by Marušič in [9].)

The graph M contains exactly p^2 p^2-cliques, namely $J_{i,r} = \{(i, r+pk) \mid k \in \mathbb{Z}_{p^2}\}$ for $i, r \in \mathbb{Z}_p$. These must form a block system for $\mathrm{Aut}(M)$. Two such cliques, $J_{i,r}$ and $J_{i',r'}$, are joined by $2p^2$ edges if $|i - i'| = 1$ and $r = r'$, by p^3 edges if $i' - i = r' - r = \pm 1$, and by no edges otherwise. Therefore, $\{B_0, B_1, \ldots, B_{p-1}\}$ is also a block system for $\mathrm{Aut}(M)$, where $B_r = J_{0,r} \cup J_{1,r} \cup \cdots \cup J_{p-1,r}$ for $r \in \mathbb{Z}_p$. The mapping $\phi_r : B_r \to U$ defined by $(i, pj + r)\phi_r = (i,j)$ is an isomorphism from $\langle B_r \rangle$ to F. By Lemma 6, the group induced by $\mathrm{Aut}(M)$ on B_r is $H_r = \langle \alpha_r, \beta_r \rangle$, where $\alpha_r = \phi_r \alpha \phi_r^{-1}$ and $\beta_r = \phi_r \beta \phi_r^{-1}$.

Suppose $R \le \mathrm{Aut}(M)$ is regular, and let $g \in R$ take vertex $(0,0)$ to vertex $(1,0)$. Now R acts regularly on the set $\{B_0, \ldots, B_{p-1}\}$ and so g fixes $B_0, B_1, \ldots, B_{p-1}$ setwise. Thus we can write $g = g_0 g_1 \cdots g_{p-1}$, where $g_r \in H_r$ for $r \in \mathbb{Z}_p$. We know that H_0 is regular on B_0 and so $g_0 = \beta_0$ and g must have order p. By Lemma 5, we have $g_1 = \alpha_1^{up} \beta_1^t$ for some u, t. Since g_0 takes $(0,0)$ to $(1,0)$, g_1 must take W_0 onto W_1, where W_i is the neighbourhood of $(i,0)$ in B_1. Thus, in the graph F, $\alpha^{up}\beta^t$ must take $W_0\phi_1$ onto $W_1\phi_1$. However, $\alpha^{up}\beta^t$ takes $W_0\phi_1$ onto $\{(1+t, pa^t(r+u)) \mid r \in \mathbb{Z}_p\}$, whilst $W_1\phi_1 = \{(2, rp+1) \mid r \in \mathbb{Z}_p\}$. These two sets are not the same for any u and t, so there is no such element g in R. ∎

Finally, we note that F and W are metacirculant graphs in the treminology of [1]. The parameters are $(p, p^2, a, \{1, 2, \ldots, p^2 - 1\}, \{0, 1\}, \emptyset, \emptyset, \ldots, \emptyset)$ and $(p, p^3, a, \{pk \mid k \in \mathbb{Z}_{p^2}\}, \{0, 1, a^p, a^{2p}, \ldots, a^{(p-1)p}, p\}, \emptyset, \emptyset, \ldots, \emptyset)$, respectively.

References.

[1] B. Alspach and T. D. Parsons, A construction for vertex-transitive graphs, *Canad. J. Math.*, **34** (1982) 307–318.

[2] B. Alspach and R. J. Sutcliffe, Vertex-transitive graphs of order $2p$, *Annals New York Acad. Sci.* **319** (1979) 19–27.

[3] M. Behzad and G. Chartrand, Introduction to the theory of graphs (Allyn and Bacon, Boston, 1971).

[4] R. Frucht, J. Graver and M. Watkins, The groups of the generalized Petersen graphs, *Proc. Cambridge Philos. Soc.*, **70** (1971) 211–218.

[5] C. D. Godsil, More odd graph theory, *Discrete Math.*, **32** (1980) 205–217.

[6] M. W. Liebeck and J. Saxl, Primitive permutation groups containing an element of large prime order, *J. London Math. Soc.*, **31** (1985) 237–249.

[7] P. Lorimer, Trivalent symmetric graphs of order at most 120, *Europ. J. Combinatorics*, **5** (1984) 163–171.

[8] D. Marušič, Cayley properties of vertex symmetric graphs, *Ars Combinatoria*, **16B** (1983) 297–302.

[9] D. Marušič, Vertex-transitive graphs and digraphs of order p^k, *Annals of Disc. Math.*, **27** (1985) 115-128.

[10] D. Marušič and R. Scapellato, Characterising vertex-transitive pq-graphs with an imprimitive automorphism subgroup, *J. Graph Theory*, **16** (1992), 375-387.

[11] D. Marušič and R. Scapellato, Imprimitive representations of $SL(2, 2^\kappa)$, *J. Combinatorial Theory, Ser. B*, **58** (1993) 46–57.

[12] B. D. McKay, Transitive graphs with fewer than twenty vertices, *Math. Comp.*, **33** (1979) 1101-1121 & microfiche supplement.

[13] B. D. McKay and G. F. Royle, The transitive graphs with at most 26 vertices, *Ars Combinatoria*, **30** (1990) 161–176.

[14] C. E. Praeger and M. Y. Xu, A characterization of a class of symmetric graphs of twice prime valency, *European J. Combinatorics*, **10** (1989) 91-102.

[15] C. E. Praeger and M. Y. Xu, Vertex primitive graphs of order a product of two distinct primes, preprint (1990).

[16] G. F. Royle, Constructive enumeration of graphs, Ph. D. Thesis, Department of Mathematics, University of Western Australia (1987).

[17] G. F. Royle and C. E. Praeger, Constructing the vertex-transitive graphs of order 24, *J. Symbolic Computation*, **8** (1989) 309–326.

[18] H. Wielandt, Finite Permutation Groups (Academic Press, 1964).