Nonhamiltonian 3-connected cubic planar graphs

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Abstract

We establish that every cyclically 4-connected cubic planar graph of order at most 40 is hamiltonian. Furthermore, this bound is determined to be sharp and we present all nonhamiltonian such graphs of order 42. In addition we list all nonhamiltonian cyclically 5-connected cubic planar graphs of order at most 52 and all nonhamiltonian 3-connected cubic planar graphs of girth 5 on at most 46 vertices. That all 3-connected cubic planar graphs on at most 176 vertices and with face size at most 6 are hamiltonian is also verified.

1 Introduction.

In this paper we describe an investigation (making much use of computation) of cyclically k-connected cubic planar graphs ($CkCP$s) for $k = 4, 5$ and report the results. We shall also have occasion to consider cubic 3-connected planar graphs with no restriction on cyclic connectivity; these we refer to as *C3CP*s. The investigation extends the work of Holton and McKay in [10] in which the smallest order of a nonhamiltonian C3CP was shown to be 38. We provide answers to two questions raised in that paper, namely:

(a) What is the smallest order of a nonhamiltonian C4CP?

(b) Is there more than one nonhamiltonian C5CP on 44 vertices?

The answer to the former question is determined to be 42 and all nonhamiltonian C4CPs of that order are presented. The latter question is answered in the negative and a complete list of all nonhamiltonian C5CPs on at most 52 vertices is presented.

Before proceeding, we include some definitions and results from the existing literature as we shall make use of them in the rest of the paper. By a *k-gon* we mean a face of a plane graph bounded by k edges. Note that a k-cycle is not necessarily a k-gon. By a *k-cut* we mean a set of k edges whose removal leaves the graph disconnected and of which no proper subset has that property. The two components (and clearly there are only two) formed by the removal of a k-cut are called *k-pieces*. A k-cut is *non-trivial* if each of its k-pieces contains a cycle and *essential* if it is non-trivial and each of its k-pieces contains more than k vertices. A cubic graph is *cyclically k-connected* if it has no non-trivial t-cuts for $0 \leq t \leq k-1$, and has *cyclic connectivity* k if in addition it has at least one non-trivial k-cut. We denote by $\lambda'(G)$ the value of k such that the C3CP G has cyclic connectivity k.

If G is a hamiltonian C3CP, then an *a-edge* is an edge which is present in every hamiltonian cycle in G, while a *b-edge* is absent from every hamiltonian cycle in G.

To focus our search for nonhamiltonian C4CPs of smallest order we shall make use of the following theorem which formed the main result in [10].

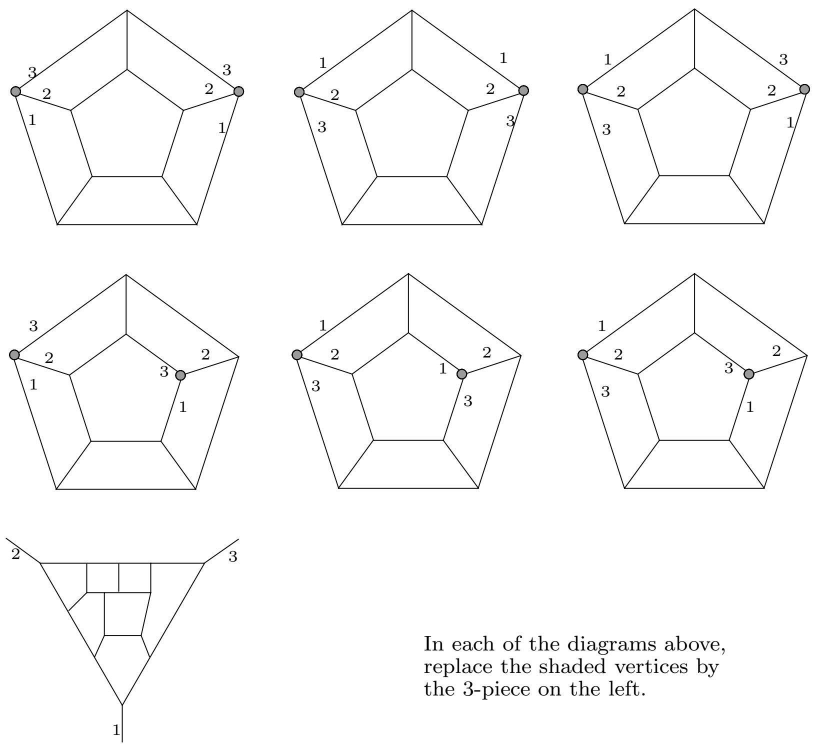


Figure 1. The 38-vertex nonhamiltonian C3CPs.

Theorem 1 *Every C3CP of order at most 36 is hamiltonian. Furthermore, if G is* *a nonhamiltonian C3CP with* $38 \leq |V(G)| \leq 42$*, then one of the following is true:*

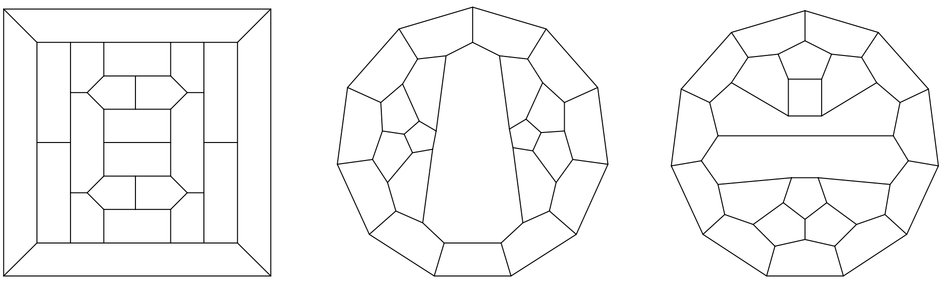
(a) G is one of the six C3CPs of order 38 and cyclic connectivity 3 shown in Figure 1;

(b) G has 40 or 42 vertices and has a non-trivial 3-cut for which one of the components has at most five vertices;

(c) $|V(G)| = 42$*,* $\lambda'(G) = 4$ *and G has an essential 4-cut, one 4-piece of which consists of a pair of adjacent 4-faces while the other is obtained from a non-hamiltonian C4CP on 38 vertices by deleting two adjacent vertices;*

(d) G is one of the two graphs NH42.b and NH42.c in Figure 2;

(e) $\lambda'(G) = 4$ *and G has no essential 4-cut.* ∎



NH42.a NH42.b NH42.c

Figure 2. The 42-vertex nonhamiltonian C4CPs.

Hence, if G is a nonhamiltonian C4CP of smallest order and not either of the graphs NH42.b, NH42.c, then $\lambda'(G) = 4$, G contains no essential 4-cut (i.e. the only non-trivial 4-cuts in G are the cocyles of 4-cycles and there must be at least one such 4-cut) and $|V(G)| \in \{38, 40, 42\}$. (The graph NH42.a in Figure 2 above is from Grünbaum [8] and is a nonhamiltonian C4CP on 42 vertices with no essential 4-cuts.)

2 The cyclically 4-connected case.

Throughout this section we shall assume that G is a nonhamiltonian C4CP of smallest order, other than NH42.b and NH42.c. In the light of the theorem above we know that $\lambda'(G) = 4$, G contains no essential 4-cut and $|V(G)| \in \{38, 40, 42\}$.

Subgraphs that G cannot contain are called *forbidden* subgraphs. Since G is cyclically 4-connected and contains no essential 4-cuts, triangles and adjacent 4-cycles are forbidden subgraphs. Similarly, cuts of sizes 1 and 2, non-trivial 3-cuts and essential 4-cuts are *forbidden cuts*.

Following the method of Okamura [14], we wish to show that none of the four subgraphs in Figure 3 can occur in G. For each subgraph H in Figure 3, the *reduction*

$H(G)$ of G *using* H is obtained by successive edge reduction (i.e. deleting an edge and suppressing the resulting degree 2 vertices) by the edges drawn more lightly. The resulting heavier edges are then edges in $H(G)$. Notice that if an edge marked with an asterisk corresponds to an edge in a hamiltonian cycle in $H(G)$, then G is hamiltonian.

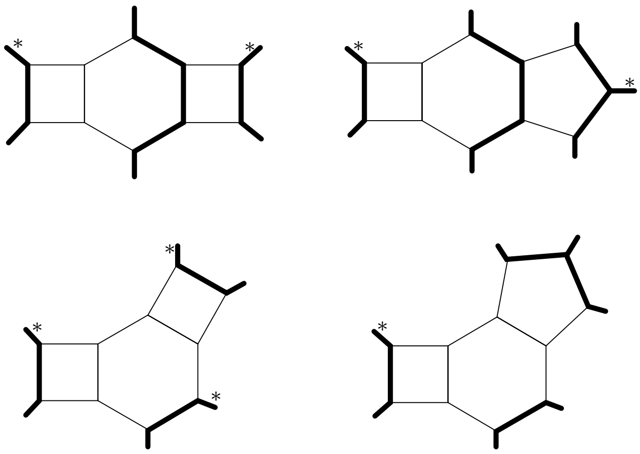


Figure 3. Some forbidden subgraphs.

Theorem 2 *If G is a nonhamiltonian C4CP of smallest order and contains a subgraph H, such that H is one of the subgraphs in Figure 3, then $H(G)$, the reduction of G using H, is a hamiltonian C3CP with a b-edge.*

Proof. We know that G must have at most 42 vertices and that the reduction of G by any of the subgraphs in Figure 3 must have ten fewer vertices than G. Thus $H(G)$ must be hamiltonian. By the observation above that a hamiltonian cycle through an edge in $H(G)$ corresponding to an edge with an asterisk lifts to a hamiltonian cycle of G, we see that $H(G)$ must contain a *b*-edge. The proof that $H(G)$ is 3-connected is as in [14]. ∎

Hence, if G contains one of the subgraphs in Figure 3, then G can be constructed from a smaller C3CP with *b*-edges using the inverse of some reduction by such a subgraph. Since the subgraphs in Figure 3 are derived from those of Okamura [14], we call such an inverse an *Okamura expansion.*

Our computation proceeded as follows.

Phase 1. Generate all 3-connected cubic planar graphs on up to 32 vertices which contain a *b*-edge but no triangle. The method used was that of Mohar [13], in conjunction with the graph isomorphism system described by McKay in [11].

Phase 2. Note that a C3CP H with a triangle has a *b*-edge if and only if the C3CP obtained from H by contracting the triangle has a *b*-edge. Thus the C3CPs on up to 32 vertices, with *b*-edges and triangles can be constructed from the graphs generated in Phase 1 on up to 30 vertices by successively expanding vertices to triangles. This was done, yielding graphs in the numbers indicated in Table 1. (Table 1 extends part of Table 1 of [10] where it should be noted that a typographical error was made indicating that the number of C4CPs with essential 4-cuts is 5863 when it is actually 5865.)

Order of graphs	Without triangles	With triangles	Total
24	1	0	1
26	7	24	31
28	26	483	509
30	146	6724	6870
32	776	79416	80192

Table 1. Numbers of C3CPs with b-edges.

Phase 3. Perform all possible Okamura expansions on the graphs obtained in Phases 1 and 2, discarding from the graphs so formed those with forbidden cuts.

Phase 4. Test the graphs yielded by Phase 3 for hamiltonicity and discard those with hamilton cycles. (All hamiltonicity checking in our computations was done using the method of McKay [12].)

No graphs remained at the completion of Phase 4, proving the following theorem.

Theorem 3 *Let G be a nonhamiltonian C4CP of order 38, 40 or 42. Then G contains none of the subgraphs in Figure 3.* ∎

Phase 5. Generate all of the C4CPs on at most 42 vertices with no pair of adjacent 4-gons, discarding those which contain any of the forbidden subgraphs in Figure 3. The small set of forbidden subgraphs provides an effective filter so that from approximately 1.5×10^{10} graphs generated, only about five million graphs are yielded in this phase.

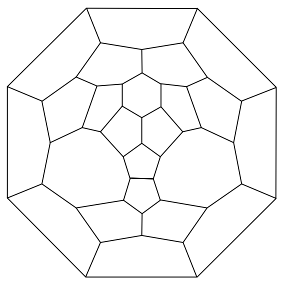
Phase 6. Test the graphs obtained from Phase 5 for hamiltonian cycles, discarding the hamiltonian graphs.

After Phase 6 there were three graphs left which were precisely those in Figure 2. Thus our computations establish the following result.

Theorem 4 *All C4CPs on at most 40 vertices are hamiltonian. Furthermore, the only nonhamiltonian C4CPs on 42 vertices are the three shown in Figure 2.* ∎

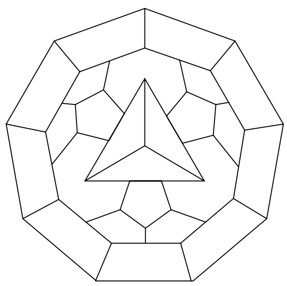
3 The cyclically 5-connected case.

In [7], Faulkner and Younger established that the smallest nonhamiltonian C5CP has 44 vertices. The nature of the search employed there was such that the graph known to exhibit this bound, NH44 of Figure 4, could not be guaranteed to be unique. We have now generated all C5CPs on up to 52 vertices using the generation procedure of Barnette [1] and Butler [6]. The graphs were then checked for hamiltonian cycles. This search confirms that NH44 is the unique nonhamiltonian C5CP of smallest order. It also yielded a complete list of all nonhamiltonian C5CPs on at most 52 vertices. These are shown in Figures 4, 5, 6 and 7.



NH44

Figure 4. The 44-vertex nonhamiltonian C5CP.



NH46

Figure 5. The 46-vertex nonhamiltonian C5CP.

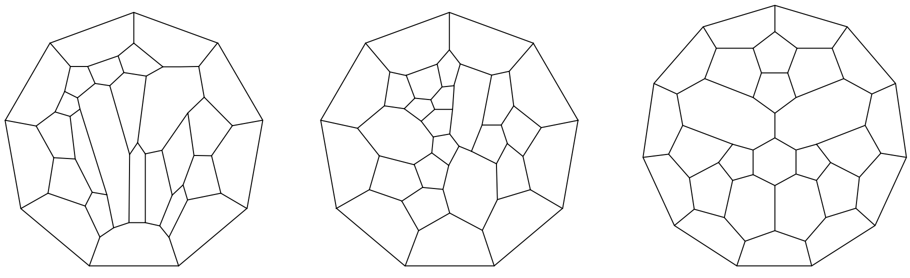


Figure 6. The 50-vertex nonhamiltonian C5CPs.

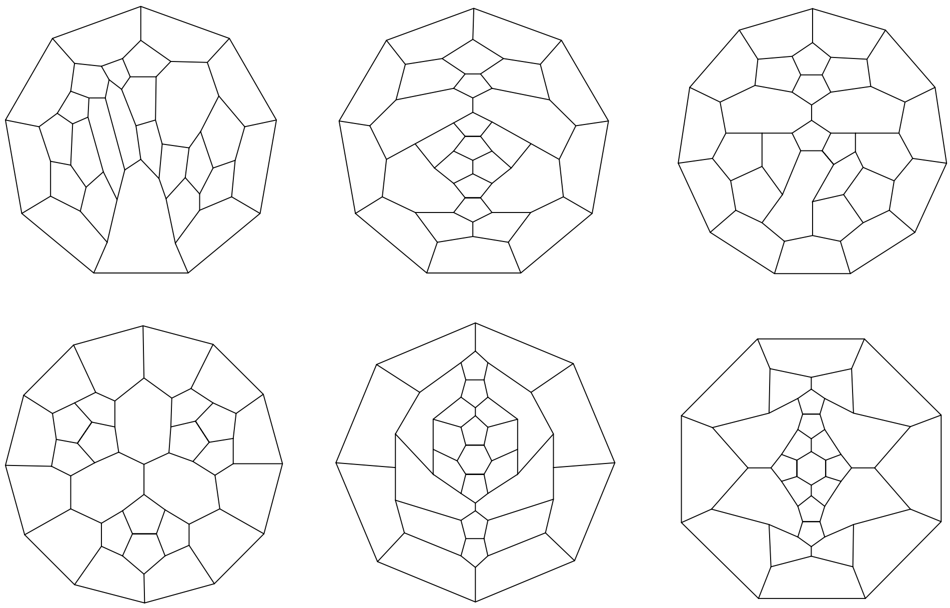


Figure 7. The 52-vertex nonhamiltonian C5CPs.

4 Some questions arising.

The graphs yielded in the above searches suggested further questions, some of which we have pursued partially. As was noted, no smallest nonhamiltonian cubic planar cyclically k-connected graph can contain a triangle. Thus all such graphs have girth 4 or 5, regardless of the value of k. Apart from the C5CPs, all of the minimal graphs determined through the above procedures have girth 4. This raises the question: What is the smallest order of a nonhamiltonian C3CP of girth 5?

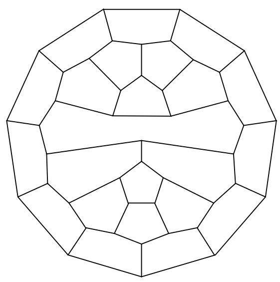


Figure 8. The other 44-vertex girth 5 nonhamiltonian C4CP.

We generated all girth 5 C3CPs up to 46 vertices and checked them for hamiltonicity. The only such graphs which were found to be nonhamiltonian and not cyclically 5-connected are the 44 and 46 vertex graphs in Figures 8 and 9. All of these graphs are cyclically 4-connected. Thus a smallest order nonhamiltonian C3CP of girth 5 with a non-trivial 3-cut is at least 48. We can easily construct a nonhamiltonian C3CP of girth 5 with a non-trivial 3-cut on 60 vertices using one of the 42 vertex graphs in Figure 2 and the dodecahedron.

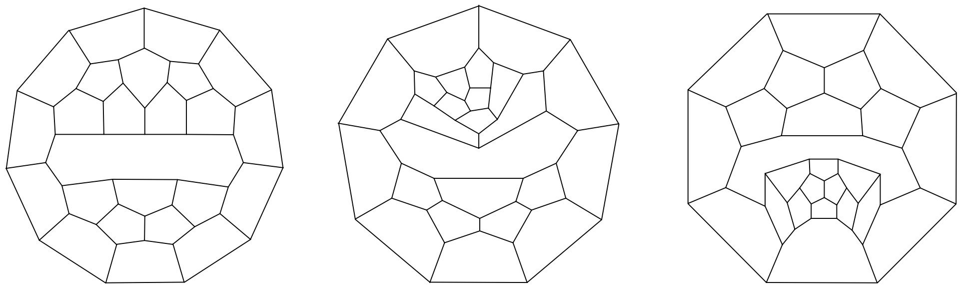


Figure 9. The other 46 vertex girth 5 nonhamiltonian C4CPs.

Each of the nonhamiltonian C3CPs found in our searches has at least one face of size 8 or more. Barnette conjectures that all C3CPs with maximum face size at most 6 are hamiltonian. In particular, this conjecture covers the fullerenes, C3CPs with 12 faces of size 5 and all other faces of size 6. Using generation techniques of Brinkmann and McKay [5] we generated all C3CPs with maximum face size at most 6 (note that all such graphs on more than 20 vertices must have faces of size 6) up to 176 vertices and tested them for hamiltonicity. All were found to be hamiltonian. This supports the conjecture and extends the known hamiltonicity of fullerenes beyond 150 vertices (established in Brinkmann and Dress [3]).

5 Some graph counts.

In the table below we include some numbers of the graphs produced in our computations. The numbers in the column headed n_1 are approximate total numbers of C3CPs of each of the indicated orders. We, of course, did not generate all such graphs but include them as an indication of the sparsity of nonhamiltonian C3CPs and of the effectiveness of the filters employed to make these computations feasible.

n	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8
38	4.53×10^{11}	6	0	0	192	0	187	0
40	3.58×10^{12}	37	191	0	651	0	627	0
42	2.86×10^{13}	274	5062	3	2070	0	1970	0
44	2.30×10^{14}	–	–	–	7290	1	6833	1
46	1.86×10^{15}	–	–	–	25381	3	23384	1
48	1.52×10^{16}	–	–	–	–	–	82625	0
50	1.25×10^{17}	–	–	–	–	–	292164	3
52	1.03×10^{18}	–	–	–	–	–	–	6

Table 2. Some numbers of graphs computed.

In the table above, the column headers are defined as follows:
n = order of G,
$n_1 \simeq$ number of C3CPs
n_2 = number of nonhamiltonian C3CPs with $\lambda'(G) = 3$ and girth ≥ 4,
n_3 =number of nonhamiltonian C3CPs with girth 3,
n_4 = number of nonhamiltonian C4CPs,
n_5 = number of C3CPs with girth 5,
n_6 = number of nonhamiltonian C4CPs with $\lambda'(G) = 4$ and girth 5,
n_7 = number of C5CPs and
n_8 = number of nonhamiltonian C5CPs.

Acknowledgement

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