SUBGRAPHS OF RANDOM GRAPHS WITH SPECIFIED DEGREES

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Let $\boldsymbol{g}$ be a graphical degree sequence and let $H \subseteq J \subseteq L$ be (the edge-sets of) graphs of the same order as $\boldsymbol{g}$. Using an elementary technique, we derive bounds on the probability that $G \cap L = J$ given that $G \cap L \supseteq H$, when G is chosen at random from the set of all labelled graphs with degree sequence $\boldsymbol{g}$. Similar results are also given for bicoloured graphs. An application to the enumeration of spanning trees is mentioned.

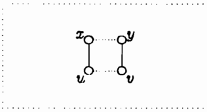
1. INTRODUCTION

For $n \geq 1$ define $K_n = \{\{u,v\} \mid 1 \leq u < v \leq n\}$. For notational clarity, an element $\{u,v\} \in K_n$ will be written as uv. A *graph of order* n is a subset $G \subseteq K_n$. The *degree sequence* of G is the n-tuple $\boldsymbol{d}(G) = (g_1, g_2, \ldots, g_n)$, where $g_u = |\{v \mid uv \in G\}|$.

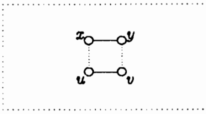
Let $\boldsymbol{g}$ be an n-tuple of natural numbers and let $H \subseteq L \subseteq K_n$. Define $\mathcal{G}(\boldsymbol{g}, L, H)$ to be the set of all graphs G such that $G \cap L = H$, and denote the cardinality of $\mathcal{G}(\boldsymbol{g}, L, H)$ by $N(\boldsymbol{g}, L, H)$. In this paper we will employ an elementary technique to investigate the relative values of $N(\boldsymbol{g}, L, H)$ when H varies, with $\boldsymbol{g}$ and L fixed.

Two graphs G_1, $G_2 \in K_n$ are said to be *closely related* if $\boldsymbol{d}(G_1) = \boldsymbol{d}(G_2)$ and $|G_1 \setminus G_2| = 2$. It is easy to see that G_1 and G_2 are closely related if and only if there are distinct u, v, x, $y \in \{1, 2, \ldots, n\}$ such that ux, $vy \in G_1 \setminus G_2$ and uv, $xy \in G_2 \setminus G_1$. This situation is depicted in Figure 1. The operation which takes G_1 onto G_2 or vice versa is sometimes called a *switching*, and was first employed by Senior [11]. Any graph can be obtained from any other with the same degree sequence by applying a sequence of switchings (see Havel[6] and Eggleton[4]). Similar results were proved for bicoloured graphs (in effect) by Ryser [10] and for multigraphs by Hakimi [5]. Our method does not depend on these results.

CONGRESSUS NUMERANTIUM, Vol. 33 (1981), pp. 213-223.



G_1



G_2

Figure 1. — Two closely related graphs

2. THE MAIN RESULTS

Let $\boldsymbol{g}$ be an n-tuple of non-negative integers and let $H \subseteq L \subseteq K_n$. Define $(h_1, h_2, \ldots, h_n) = \boldsymbol{d}(H)$ and $(l_1, l_2, \ldots, l_n) = \boldsymbol{d}(L)$. Suppose that $uv \in L \setminus H$. We will investigate the relative values of $N(\boldsymbol{g}, L, H)$ and $N(\boldsymbol{g}, L, H \cup \{uv\})$ by finding bounds on M, the number of closely related pairs (G_1, G_2) for which $G_1 \in \mathcal{G}(\boldsymbol{g}, L, H)$ and $G_2 \in \mathcal{G}(\boldsymbol{g}, L, H \cup \{uv\})$. We are not assuming that $M \neq 0$.

Define $g_{\max} = \max\{\, g_i \mid 1 \leq i \leq n \,\}$ and $l_{\max}$ similarly.

2·1 Lemma $M \leq (g_u - h_u)(g_v - h_v)N(\boldsymbol{g}, L, H)$.

Proof: Let G_1 be an arbitrary element of $\mathcal{G}(\boldsymbol{g}, L, H)$. To obtain a closely related graph $G_2 \in \mathcal{G}(\boldsymbol{g}, L, H \cup \{uv\})$ by means of a switching as in Figure 1, we need only to choose $ux \in G_1 \setminus H$ and $vy \in G_1 \setminus H$. The first choice can be made in exactly $g_u - h_u$ ways, then the second can be made in at most $g_v - h_v$ ways. □

2·2 Lemma

$$M \geq \Big(\sum (g_i - h_i) - (g_{\max} + 1)(g_u + g_v) + (g_{\max} + 2)(h_u + h_v) - g_{\max}(l_u + l_v) + 2g_{\max}\Big) N(\boldsymbol{g}, L, H \cup \{uv\})$$

Proof: Take an arbitrary graph $G_2 \in \mathcal{G}(\boldsymbol{g}, L, H \cup \{uv\})$. Referring to Figure 1, we need to choose an ordered pair (x, y) which satisfies the requirements

(i) $|\{u, v, x, y\}| = 4$, and

(ii) $xy \in G_2 \setminus H$, and $ux,\ vy \notin G_2 \cup L$.

We can choose $\sum_{i=1}^{n}(g_i - h_i)$ ordered pairs (x, y) such that $xy \in G_2 \setminus H$. However we cannot choose (u, v) or (v, u). Similarly we must not choose pairs of the form (u, w) or (v, w), nor pairs (w, z) for which $uw \in G_2$ or $vz \in G_2$. Bounding the number of pairs excluded in each case, we find the number of choices possible is at least

$$\sum_{i=1}^{n}(g_i - h_i) - (g_{\max} + 1)(g_u + g_v) + (g_{\max} + 2)(h_u + h_v) - g_{\max}(l_u + l_v) + 2g_{\max}. \qquad \square$$

To simplify the subsequent calculations we will use the following weaker form of Lemma 2·2.

2·3 **Lemma** *Let $\Delta = g_{\max}(g_{\max} + l_{\max})$. Then*

$$M \geq \Big(\sum_{i=1}^{n}(g_i - h_i) - 2\Delta\Big)N(\boldsymbol{g}, L, H \cup \{uv\}). \qquad \square$$

2·4 **Corollary**

$$\Big(\sum_{i=1}^{n}(g_i - h_i) - 2\Delta\Big)N(\boldsymbol{g}, L, H \cup \{uv\}) \leq (g_u - h_u)(g_v - h_v)N(\boldsymbol{g}, L, H).$$

Proof: This is an immediate consequence of Lemmas 2·1 and 2·3. $\square$

The bound on $N(\boldsymbol{g}, L, H \cup \{uv\})$ given by Corollary 2·4 can be generalised to a bound on $N(\boldsymbol{g}, L, H \cup J)$, where $H \cap J = \emptyset$ and $H \cup J \subseteq L$, by simply applying Corollary 2·4 $|J|$ times. Let $(j_1, j_2, \ldots, j_n) = \boldsymbol{d}(J)$, and define $E_j = \frac{1}{2}\sum_{i=1}^{n} j_i$, $E_g = \frac{1}{2}\sum_{i=1}^{n} g_i$ and $E_h = \frac{1}{2}\sum_{i=1}^{n} h_i$. For $x, m \geq 0$ define $x^{[m]} = x(x-1)\cdots(x-m+1)$. In particular, $x^{[0]} = 1$.

2·5 **Theorem** *If $E_g - E_h - \Delta \geq 0$, then*

$$2^{E_j}(E_g - E_h - \Delta)^{[E_j]}N(\boldsymbol{g}, L, H \cup J) \leq \prod_{i=1}^{n}(g_i - h_i)^{[j_i]}N(\boldsymbol{g}, L, H). \qquad \square$$

The bound given by Theorem 2·5 is of course trivial if $E_g - E_h - \Delta < E_j$.

In order to obtain a bound in the opposite direction to that in Theorem 2·5 we can use similar methods. A bound complementary to Lemma 2·2 is easy.

2·6 **Lemma** $M \le 2(E_g - E_h - 1)N(g, L, H \cup \{uv\})$. □

In order to obtain a bound complementary to Lemma 2·1 we need to use a slightly different argument. The reason for this is that an element of $\mathcal{G}(g, L, H)$ might be closely related to anything from 0 to $(g_u - h_u)(g_v - h_v)$ elements of $\mathcal{G}(g, L, H \cup \{uv\})$ — there is no useful uniform lower bound.

2·7 **Lemma** *Let* $\bar{\Delta} = g_{\max}(g_{\max} + l_{\max} + 2)$. *If* $E_g - E_h > \bar{\Delta} + 1$ *then*

$$M \ge (g_u - h_u)(g_v - h_v)\left(1 - \frac{g_{\max}(l_{\max} + 1)}{2(E_g - E_h - \bar{\Delta} - 1)}\right)N(g, L, H).$$

Proof: We must reconsider the proof of Lemma 2·1 and bound the amount by which $(g_u - h_u)(g_v - h_v)N(g, L, H)$ overcounts M. First, note that the lemma is trivially true if $N(g, L, H) = 0$, so from now on we will assume $N(g, L, H) > 0$.

As in the proof of Lemma 2·1, we can always choose x in $g_u - h_u$ ways. However in the choice of y such that $vy \in G_1 \setminus H$, two things that can go wrong are (i) $y = x$ and (ii) $xy \in L$.

For a random choice of $G_1 \in \mathcal{G}(g, L, H)$ and of x such that $ux \in G_1 \setminus H$, the probability that $vx \in G_1$ is at most

$$\frac{N(g, L \cup \{ux, vx\}, H \cup \{ux, vx\})}{N(g, L \cup \{ux\}, H \cup \{ux\})} \le \frac{N(g, L \cup \{ux, vx\}, H \cup \{ux, vx\})}{N(g, L \cup \{ux, vx\}, H \cup \{ux\})}$$
$$\le \frac{g_{\max}(g_v - h_v)}{2(E_g - E_h - \bar{\Delta} - 1)}, \quad \text{by Corollary 2·4.}$$

Therefore, the total number of choices of G_1, x and y which encounter problem (i) is at most

$$\frac{g_{\max}(g_u - h_u)(g_u - h_u)N(g, L, H)}{2(E_g - E_h - \bar{\Delta} - 1)}.$$

By a similar argument, for a random choice of $G_1 \in \mathcal{G}(g, L, H)$, x such that $ux \in G_1 \setminus H$ and y such that $xy \in L$, the probability that $vy \in G_1$ is at most

$$\frac{g_{\max}(g_v - h_v)}{2(E_g - E_h - \Delta' - 1)},$$

where $\Delta' = g_{\max}(g_{\max} + l_{\max} + 1)$. Therefore the total number of choices of G_1, x and y which encounter problem (ii) is at most

$$\frac{g_{\max} l_{\max}(g_u - h_u)(g_v - h_v)N(g, L, H)}{2(E_g - E_h - \Delta' - 1)}.$$

The required inequality now follows immediately. □

Combining Lemmas 2·6 and 2·7 we obtain bounds complementary to Corollary 2·4 and Theorem 2·5.

2·8 Lemma *If $E_g - E_h > \bar{\Delta} + 1$ then*

$$2(E_g - E_h - 1)N(g, L, H \cup \{uv\}) \geq (g_u - h_u)(g_v - h_v)\left(1 - \frac{g_{\max}(l_{\max}+1)}{2(E_g - E_h - \bar{\Delta} - 1)}\right)N(g, L, H). \qquad \square$$

2·9 Theorem *If $E_g - E_h - E_j - \bar{\Delta} > g_{\max}(l_{\max}+1)$, then*

$$2^{E_j}(E_g - E_h - 1)^{[E_j]} N(g, L, H \cup J) \geq \prod_{i=1}^{n}(g_i - h_i)^{[j_i]}\left(1 - \frac{g_{\max}(l_{\max}+1)}{2(E_g - E_h - E_j - \bar{\Delta})}\right)^{E_j} N(g, L, H). \qquad \square$$

Recall that $\mathcal{G}(g, L, L)$ is the set of all graphs $G \subseteq K_n$ such that $d(G) = g$ and $L \subseteq G$. If $H \subseteq L$ then the ratio $N(g, L, L)/N(g, H, H)$ is thus the probability that $G \in K_n$ contains L, given that $d(G) = g$ and G contains H. Bounds on this ratio can be obtained with the aid of Theorems 2·5 and 2·9.

2·10 Theorem *Let $H \subseteq L \subseteq K_n$, and $J = L \setminus H$. Let $g = (g_1, g_2, \ldots, g_n)$ be an n-tuple such that $\mathcal{G}(g, L, H) \neq \emptyset$. Define $g_{\max}$, $l_{\max}$, Δ, $\bar{\Delta}$, E_g, E_h, E_j as above.*

(a) *If $E_g - E_h - E_j \geq \Delta$ then*

$$\frac{N(g, L, L)}{N(g, H, H)} \leq \frac{\prod_{i=1}^{n}(g_i - h_i)^{[j_i]}}{2^{E_j}(E_g - E_h - \Delta)^{[E_j]}}.$$

(b) *If $E_g - E_h - E_j \geq \bar{\Delta} + g_{\max}(l_{\max}+1)$ then*

$$\frac{N(g, L, L)}{N(g, H, H)} \geq \frac{\prod_{i=1}^{n}(g_i - h_i)^{[j_i]}}{2^{E_j}(E_g - E_h - 1)^{[E_j]}} \times \left[\left(1 - \frac{g_{\max}(l_{\max}+1)}{2(E_g - E_h - E_j - \bar{\Delta})}\right)\Big/\left(1 + \frac{g_{\max}^2}{2(E_g - E_h - \Delta - (e-1)E_j/e)}\right)\right]^{E_j}.$$

Proof: Part (a) comes immediately from Theorem 2·5 on making the observation that

$$\frac{N(g, L, L)}{N(g, H, H)} \leq \frac{N(g, L, L)}{N(g, L, H)}.$$

For part (b), notice that $\mathcal{G}(g, H, H) = \bigcup_{S \subseteq J} \mathcal{G}(g, L, H \cup S)$. Therefore

$$\frac{N(g, L, L)}{N(g, H, H)} = \frac{N(g, L, L)}{N(g, L, H)}\Big/\sum_{S \subseteq J}\frac{N(g, L, H \cup S)}{N(g, L, H)}.$$

A lower bound on $N(\boldsymbol{g}, L, L)/N(\boldsymbol{g}, L, H)$ can be found in Theorem 2·9. To handle the sum we have, by Theorem 2·5,

$$\sum_{S\subseteq J} \frac{N(\boldsymbol{g}, L, H\cup S)}{N(\boldsymbol{g}, L, H)} \le \sum_{S\subseteq J} \frac{\prod_{i=1}^{n}(g_i - h_i)^{[s_i]}}{2^{E_s}(E_g - E_h - \Delta)^{[E_s]}},$$

where $(s_1, s_2, \ldots, s_n) = \boldsymbol{d}(S)$ and $E_s = \frac{1}{2}\sum_{i=1}^{n} s_i$. Therefore

$$\sum_{S\subseteq J} \frac{N(\boldsymbol{g}, L, H\cup S)}{N(\boldsymbol{g}, L, H)} \le \sum_{m=0}^{E_j} \binom{E_j}{m} \frac{g_{\max}^{2m}}{2^m(E_g - E_h - \Delta)^{[m]}}.$$

By Stirling's approximation we find that $x^{[m]} \ge (x - (e-1)m/e)^m$ for $0 \le m \le x$. Therefore

$$\sum_{S\subseteq J} \frac{N(\boldsymbol{g}, L, H\cup S)}{N(\boldsymbol{g}, L, H)} \le \left(1 + \frac{g_{\max}^2}{2(E_g - E_h - \Delta - (e-1)E_j/e)}\right)^{E_j},$$

from which (b) follows immediately. □

It is interesting to consider the ratio between the two bounds given in Theorem 2·10 for large n. Let $\phi(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{j})$ be the value of the upper bound in (a), divided by the lower bound in (b). Suppose that the requirements for each part are met. If $\Delta E_j = o(E_g - E_h - E_j - \Delta)$ as $n \to \infty$ then $\phi(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{j}) \to 1$. The asymptotic value of $N(\boldsymbol{g}, L, L)/N(\boldsymbol{g}, H, H)$ given is the same as that which can be obtained by two applications of a theorem of Bender and Canfield [2], when their conditions overlap ours (essentially when Δ is bounded).

If $\Delta E_j = O(E_g - E_h - E_j - \Delta)$ as $n \to \infty$ then $\phi(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{j})$ is bounded. In some cases this is less accurate than the result obtainable from [2], although the latter only gives asymptotic values without error bounds. Wormald [12] and Bollabás [3] have also obtained results which partially overlap ours.

All of the methods used in this chapter can be repeated, using Theorems 2·5 and 2·9 to achieve much closer bounds. In order to do this with substantial success it is necessary to consider in more detail the precise structure of the subgraphs we are considering. Indeed, for the case $\Delta = O(1)$ and $E_g - E_h - E_j \to \infty$, the results of [2] prove that upper and lower bounds of the form of Theorem 2·10 cannot be found which are asymptotically equal and only depend on $\boldsymbol{g}$, $\boldsymbol{h}$ and $\boldsymbol{j}$.

3. BICOLOURED GRAPHS

The techniques of Section 2 can be applied equally as easily to the investigation of subgraphs of bicoloured graphs. The only essential difference is that we must restrict our switchings to maintain the bipartition.

Let $m, n > 0$. Define $V_1 = \{1, 2, \ldots, m\}$, $V_2 = \{1', 2', \ldots, n'\}$ and $K_{m,n} = V_1 \times V_2$. A *bicoloured graph of order* (m, n) is a subset $G \subseteq K_{m,n}$. As in Section 2, $(u, v) \in K_{m,n}$ will be abbreviated to uv. The *degree sequence* of G is the $(n+m)$-tuple $\boldsymbol{d}(G) = \boldsymbol{g} = (g_1, g_2, \ldots, g_m; g_{1'}, g_{2'}, \ldots, g_{n'})$ where $g_u = |\{ v \mid uv \in G\}|$ for $u \in V_1$ and $g_v = |\{ u \mid uv \in G\}|$ for $v \in V_2$. If $H \subseteq L \subseteq K_{m,n}$, the set of all $G \subseteq K_{m,n}$ such that $G \cap L = H$ will be denoted by $\mathcal{B}(\boldsymbol{g}, L, H)$. The cardinality of $\mathcal{B}(\boldsymbol{g}, L, H)$ will be denoted by $M(\boldsymbol{g}, L, H)$.

We will now list the results which correspond to Corollary 2·4, Lemma 2·8 and Theorems 2·5, 2·9 and 2·10. We will omit the proofs, as they are very similar to those in Section 2.

Let $H, J \subseteq L \subseteq K_{m,n}$ and $H \cap J = \emptyset$. Define $\boldsymbol{h} = \boldsymbol{d}(H)$, $\boldsymbol{j} = \boldsymbol{d}(J)$, $\boldsymbol{l} = \boldsymbol{d}(L)$, $E_g = \sum_{i=1}^{m} g_i$, E_j and E_h similarly, $g_{\max} = \max\{ g_i \mid i \in V_1 \cup V_2 \}$ and $l_{\max}$ similarly. Let $uv \in L \setminus H$.

3·1 Theorem *Let $\Gamma = 2g_{\max}(g_{\max} + l_{\max} - 2) + 1$. Then*

$$(E_g - E_h - \Gamma) M(\boldsymbol{g}, L, H \cup \{uv\}) \leq (g_u - h_u)(g_v - h_v) M(\boldsymbol{g}, L, H).$$

□

3·2 Theorem *Let $\bar{\Gamma} = 2g_{\max}(g_{\max} + l_{\max} - 1) + 2$. Then, if $E_g - E_h > \bar{\Gamma}$,*

$$(E_g - E_h - 1) M(\boldsymbol{g}, L, H \cup \{uv\}) \leq (g_u - h_u)(g_v - h_v)\left(1 - \frac{g_{\max} l_{\max}}{E_g - E_h - \bar{\Gamma}}\right) M(\boldsymbol{g}, L, H).$$

□

3·3 Theorem *If $E_g - E_h - \Gamma \leq E_j$, then*

$$(E_g - E_h - \Gamma)^{[E_j]} M(\boldsymbol{g}, L, H \cup J) \leq \prod (g_i - h_i)^{[j_i]} M(\boldsymbol{g}, L, H).$$

□

3·4 **Theorem** *If $E_g - E_h - \bar{\Gamma} - E_j > g_{\max} l_{\max}$, then*

$$(E_g - E_h - 1)^{[E_j]} M(g, L, H \cup J) \geq \prod_{i=1}^{n} (g_i - h_i)^{[j_i]} \left(1 - \frac{g_{\max} l_{\max}}{E_g - E_h - \bar{\Gamma} - E_j + 1}\right)^{E_j} M(g, L, H).$$

□

3·5 **Theorem** *Suppose $J = L \setminus H$ and $B(g, L, H) \neq \emptyset$.*

(a) *If $E_g - E_h - \Gamma \geq E_j$, then*

$$\frac{M(g, L, L)}{M(g, H, H)} \leq \frac{\prod (g_i - h_i)^{[j_i]}}{(E_g - E_h - \Gamma)^{[E_j]}}.$$

(b) *If $E_g - E_h - \bar{\Gamma} - E_j + 1 \geq g_{\max} l_{\max}$, then*

$$\frac{M(g, L, L)}{M(g, H, H)} \geq \frac{\prod (g_i - h_i)^{[j_i]}}{(E_g - E_h - 1)^{[E_j]}} \times \left[\left(1 - \frac{g_{\max} l_{\max}}{E_g - E_h - E_j - \bar{\Gamma} + 1}\right) \Big/ \left(1 + \frac{g_{\max}^2}{E_g - E_h - \Gamma - (e-1)E_j/e}\right)\right]^{E_j}.$$

□

Some asymptotic results partially overlapping Theorem 3·5 can be derived from O'Neil [9], Bender [1] and Wormald [12].

It is clear that the same method we have used for graphs and bicoloured graphs can be also applied to multigraphs, pseudographs and even to hypergraphs (via the vertex–edge incidence graph). Variances can also be investigated by using Theorem 2·10 or 3·5 to bound pair-wise covariances. The analysis in this case is usually quite complicated.

4. SPANNING TREES

For a graph G, let $\kappa(G)$ denote the number of spanning trees of G. We say that G is *k-regular* if $d(G) = (k, k, \ldots, k)$. The number of spanning trees of a k-regular graph has been investigated in McKay [7], where a generalisation of the following theorem is proved.

4·1 Theorem *Let G be a connected k-regular graph of order n, where $k \geq 3$. Then $\kappa(G) \leq \gamma_k c_k^n$, where*

$$\gamma_k = \frac{(k-1)^{2k-2}}{(k^2-2k)^{k-2}(k-2)^{\frac{k}{k-1}}(k+1)^{\frac{k-3}{k-1}}2^{\frac{k-2}{k}}}, \quad \textit{and}$$
$$c_k = \frac{(k-1)^{k-1}}{(k^2-2k)^{k/2-1}}.$$

□

We will apply Theorem 2·10 to investigate the sharpness of the bound $\gamma_k c_k^n$. For functions $f_1(n)$ and $f_2(n)$ we will write $f_1(n) \stackrel{c}{\sim} f_2(n)$ if there are constants n_0 and $0 < A \leq B$ such that $Af_1(n) \leq f_2(n) \leq Bf_1(n)$ for $n \geq n_0$. The next theorem demonstrates that the bound of Theorem 4·1 is high by at most $O(n)$.

4·2 Theorem *For an infinity of values of n, let $\boldsymbol{g}(n) = \big(g_1(n), g_2(n), \ldots, g_n(n)\big)$ be such that $\mathcal{G}(\boldsymbol{g}(n), \emptyset, \emptyset) \neq \emptyset$. Define $g_{\max}(n) = \max\{\, g_i(n) \mid 1 \leq i \leq n\}$, $\bar{g} = \bar{g}(n) = \big(\sum g_i(n)\big)/n$ and $\hat{g} = \hat{g}(n) = \big(\prod g_i(n)\big)^{1/n}$. Suppose that $\bar{g} \geq 2 + \epsilon$ for some $\epsilon > 0$ independent of n and that $g_{\max} \leq K$ for some constant K. Then the average number of spanning trees over $G \in \mathcal{G}\big(\boldsymbol{g}(n), \emptyset, \emptyset\big)$ is*

$$\kappa_n \stackrel{c}{\sim} \frac{1}{n}\left(\frac{\hat{g}(\bar{g}-1)^{\bar{g}-1}}{\bar{g}^{\bar{g}/2}(\bar{g}-2)^{\bar{g}/2-1}}\right)^n.$$

Proof: According to [8], the number of trees T with $\boldsymbol{d}(T) = \boldsymbol{l} = (l_1, l_2, \ldots, l_n)$ for $n \geq 2$ is

$$\binom{n-2}{l_1-1, l_2-1, \ldots, l_n-1}.$$

Therefore, by Theorem 2·10,

$$\begin{aligned}\kappa_n &\stackrel{c}{\sim} \sum_{\boldsymbol{l}} \binom{n-2}{l_1-1, l_2-2, \ldots, l_n-1} \frac{\prod_{i=1}^n g_i^{[l_i]}}{2^n (n\bar{g}/2)^{[n-1]}} \\ &= \frac{(n-2)!\,\hat{g}^n}{(n\bar{g}/2)^{[n-1]}2^n} \sum_{\boldsymbol{l}} \prod_{i=1}^n \binom{g_i-1}{l_i-1} \\ &= \frac{(n-2)!\,\hat{g}^n}{(n\bar{g}/2)^{[n-1]}2^n} \binom{n(\bar{g}-1)}{n-2},\end{aligned}$$

from which the desired result follows on application of Stirling's formula. □

A similar application of Theorem 3·5 leads easily to the next result.

4·3 Theorem *Let $N = N(r) \to \infty$ as $r \to \infty$, and define $m = \alpha(r)N$ and $n = (1-\alpha(r))N$, where $0 < \alpha(r) < 1$. Suppose that $B(\mathbf{g}(r), \emptyset, \emptyset) \neq \emptyset$, where $\mathbf{g}(r) = \big(g_1(r), \ldots, g_m(r); g_{1'}(r), \ldots, g_{n'}(r)\big)$. Define $g_{\max} = \max\{\, g_i(r) \mid i \in V_1 \cup V_2 \,\}$, $\bar{g} = \big(\sum g_i(r)\big)/N$ and $\hat{g} = \big(\prod g_i(r)\big)^{1/N}$. If $\bar{g} \geq 2 + \epsilon$ and $g_{\max} \leq K$ for constants $\epsilon > 0$ and K, then the average number of spanning trees over $G \in B(\mathbf{g}(r), \emptyset, \emptyset)$ is*

$$\tau_r \overset{c}{\sim} \frac{1}{N}\left(\frac{\hat{g}(\bar{g}-2\alpha)^{\bar{g}/2-\alpha}(\bar{g}+2\alpha-2)^{\bar{g}/2+\alpha-1}}{\bar{g}^{\bar{g}/2}(\bar{g}-2)^{\bar{g}/2-1}}\right)^{N}. \qquad \square$$

REFERENCES

1. E. A. Bender, The asymptotic number of non-negative integer matrices with given row and column sums, *Discrete Math.* **10** (1974) 217–223.

2. E. A. Bender and E. R. Canfield, The asymptotic number of labeled graphs with given degree sequences, *J. Combinatorial Theory (A)* **24** (1978) 296–307.

3. B. Bollobás, Private Communication.

4. R. B. Eggleton, Graphic sequences and graphic polynomials: A report, *in* Infinite and Finite Sets I (ed. A. Hajnal *et al.*), *Colloq. Math. Soc. J. Bolyai 10* (North-Holland, 1975) 385–392.

5. S. L. Hakimi, On realizability of a set of integers as degrees of the vertices of a linear graph I, *J. Soc. Indust. Appl. Math.*, **10** (1962) 496–506.

6. V. Havel, A remark on the existence of finite graphs, *Časopis Pěst. Math.* **80** (1955) 477–480.

7. B. D. McKay, Spanning trees in regular graphs, *Vanderbilt University, Computer Science Tech. Rep.* CS-81-01 (1980–1981).

8. J. W. Moon, The second moment of the complexity of a graph, *Mathematika* **11** (1964) 95–98.

9. P. E. O'Neil, Asymptotics and random matrices with row-sum and column-sum restrictions, *Bull. Amer. Math. Soc.* **75** (1969) 1276–1282.

10. H. J. Ryser, Combinatorial properties of matrices of zeros and ones, *Canad. J. Math.* **9** (1957) 371–377.

11. J. K. Senior, Partitions and their representative graphs, *Amer. J. Math.* **75** (1951) 663–689.

12. N. C. Wormald, Some problems in the enumeration of labelled graphs, *Ph. D. Thesis, Dept. of Math., University of Newcastle* (1978).