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Spanning Trees in Regular Graphs

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Let X be a regular graph with degree $k \geq 3$ and order n. Then the number of spanning trees of X is

$$\kappa(X) < \gamma_k c_k^n \exp\left(- \sum_{i=3}^{nk/2} \left(1 - \frac{2i}{kn}\right) p_i \beta_{i,k}(1/k)\right),$$

where γ_k, c_k and $\beta_{i,k}(1/k)$ are positive constants, and p_i is the number of equivalence classes of certain closed walks of length i in X. The value

$$c_k = \frac{(k-1)^{k-1}}{(k^2-2k)^{(k/2)-1}}$$

is shown to be the best possible in the sense that $\kappa(X_i)^{1/n} \to c_k$ for some increasing sequence X_1, $X_2, \ldots$ of regular graphs of degree k. A sufficient condition for this convergence is established. Finally, for some absolute constant A, $\kappa(X) \leq A c_k^n \log n/(nk \log k)$, a bound which (for fixed k) is high by at most $O(\log n)$.

1. Introduction

In this paper we investigate the number of spanning trees of a regular graph. We succeed in finding a tight upper bound in terms of the numbers of small cycles and other subgraphs. The only previous similar result known to the author was found by Kel'mans [7] and independently by Nosal [13] and Biggs [2]:

Theorem 1.1. *A regular graph of order n and degree k has at most $(nk/(n-1))^{n-1}/n$ spanning trees.*

We will not allow our graphs to have multiple edges, but the same results can easily be extended to that case also.

2. Walks

A *walk* of *length r* in a graph X is a sequence $\mathbf{v} = (v_0, v_1, \ldots, v_r)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq r$. We say that $\mathbf{v}$ *starts* at v_0, *finishes* at v_r, and is *closed* if $v_r = v_0$. Suppose that for some i $(0 < i < r)$ we have $v_{i-1} = v_{i+1}$. Then we can *reduce* $\mathbf{v}$ by deleting the elements v_i and v_{i+1}. The result is clearly a walk of length $r-2$ which is closed if and only if $\mathbf{v}$ is closed. If $\mathbf{v}$ cannot be reduced in this way it is called *irreducible*.

Given any walk $\mathbf{v}$ there is a unique irreducible walk $\bar{\mathbf{v}}$ which can be obtained from $\mathbf{v}$ by a sequence of reductions. The uniqueness of $\bar{\mathbf{v}}$ is proved in [5]. If $\bar{\mathbf{v}}$ has length 0, we will call $\mathbf{v}$ *totally reducible*. Obviously, totally reducible walks are closed.

Our first theorem gives a relationship between the number of walks and the number of irreducible walks between two vertices of X, if X is regular.

Theorem 2.1. *Let X be regular with degree k, and let v and v' be the vertices of X, not necessarily distinct. Define $a(x) = \sum_{i=0}^{\infty} a_i x^i$ and $b(x) = \sum_{i=0}^{\infty} b_i x^i$ where, for each i, a_i is the number of walks of length i in X which start at v and finish at v', and b_i is the*

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number of such walks which are irreducible. Then

$$a(x)=\frac{k-2-k(1-4(k-1)x^2)^{1/2}}{2(k^2x^2-1)}\,b\left(\frac{1-(1-4(k-1)x^2)^{1/2}}{2(k-1)x}\right),$$

and

$$b(x)=\frac{1-x^2}{1+(k-1)x^2}\,a\left(\frac{x}{1+(k-1)x^2}\right).$$

PROOF. For each $r\geq 0$, define $\boldsymbol{W}_r$ to be the $n\times n$ matrix whose (i,j)th entry is the number of irreducible walks in X which start at x_i and finish at x_j. Let $\boldsymbol{W}(x)=\sum_{r=0}^{\infty}\boldsymbol{W}_rx^r$.

Obviously, $\boldsymbol{W}_0=\boldsymbol{I}$, $\boldsymbol{W}_1=\boldsymbol{A}$ and $\boldsymbol{W}_2=\boldsymbol{A}^2-k\boldsymbol{I}$, where $\boldsymbol{A}$ is the 0–1 adjacency matrix of X. From [1] we know that for $r\geq 2$, $\boldsymbol{W}_{r+1}=\boldsymbol{W}_r\boldsymbol{A}-(k-1)\boldsymbol{W}_{r-1}$. Therefore

$$\boldsymbol{W}(x)=(1-x^2)\boldsymbol{I}+x\boldsymbol{W}(x)\boldsymbol{A}-(k-1)x^2\boldsymbol{W}(x).$$

The second equation now follows on solving for $\boldsymbol{W}(x)$, and the first on a simple change of variable.

Let $\boldsymbol{v}=(v_0,v_1,\ldots,v_r)$ be a closed irreducible walk of length $r\geq 3$ in X, such that $v_1\neq v_{r-1}$ and all cyclic permutations of $\boldsymbol{v}$ are distinct. The *primitive circuit* $\mathscr{C}(\boldsymbol{v})$ is the equivalence class containing all cyclic permutations of $\boldsymbol{v}$ and all cyclic permutations of the reverse walk $(v_r,v_{r-1},\ldots,v_0)$. Clearly $\mathscr{C}(\boldsymbol{v})$ contains exactly $2r$ irreducible closed walks and is uniquely defined by any one of its members. The simplest example of a primitive circuit is an ordinary cycle.

We now show that in order to count the closed walks in X it suffices to count the primitive circuits, provided X is regular.

THEOREM 2.2. *Let X be a regular graph of order n and degree k. Let w_i be the number of closed walks of length i in X $(i\geq 0)$, and let p_i be the number of primitive circuits of length i in X $(i\geq 3)$. Define*

$$w(x)=\sum_{i=0}^{\infty}w_ix^i\qquad\text{and}\qquad p(x)=\sum_{i=3}^{\infty}\frac{ix^i}{1-x^i}p_i.$$

Then

$$w(x)=\frac{k-2-k(1-4(k-1)x^2)^{1/2}}{2(k^2x^2-1)}\,n+\frac{2}{(1-4(k-1)x^2)^{1/2}}\,p\left(\frac{1-(1-4(k-1)x^2)^{1/2}}{2(k-1)x}\right).$$

PROOF. Let d_i be the number of irreducible closed walks of length i in X, for $i\geq 0$. Then $d_0=n$ and $d_1=d_2=0$. Define $d(x)=\sum_{i=0}^{\infty}d_ix^i$.

An irreducible closed walk of nonzero length in X is necessarily of the form $(z_0,z_1,\ldots,z_m,\ v_1,\ v_2,\ldots,v_r,\ldots,v_1,v_2,\ldots,v_r,\ z_{m-1},\ z_{m-2},\ldots,z_0)$, where $(z_0,z_1,\ldots,z_m)$ is an irreducible walk (not necessarily closed), $\boldsymbol{v}=(v_0,v_1,\ldots,v_r)$ is an element of a primitive circuit, repeated $s\geq 1$ times, $z_m=v_0=v_r$ and $v_1\neq z_{m-1}\neq v_{r-1}$ (if $m>0$). The length of this walk is $rs+2m$.

Given the primitive circuit $\mathscr{C}(\boldsymbol{v})$ of length r, the choice of $\boldsymbol{v}$ can be made in $2r$ ways. If $m\neq 0$, any of the vertices adjacent to $z_m=v_0$ other than v_1 or v_{r-1} can be chosen as z_{m-1}. Further vertices z_{m-2}, $z_{m-3},\ldots$, if required, can each be chosen in $k-1$ ways. Therefore

$$d(x)=n+\sum_{r=3}^{\infty}2rp_r\sum_{s=1}^{\infty}x^{sr}\left(1+(k-2)\sum_{m=1}^{\infty}(k-1)^{m-1}x^{2m}\right)$$

$$=n+\frac{2(1-x^2)}{1-(k-1)x^2}p(x).$$

By Theorem 2.1 we find

$$\frac{1-x^2}{1+(k-1)x^2} w\left(\frac{x}{1+(k-1)x^2}\right) = n + \frac{2(1-x^2)}{1-(k-1)x^2} p(x),$$

from which the theorem follows on a change of variable.

We note that the term $(k-2-k(1-4(k-1)x^2)^{1/2})/2(k^2x^2-1)$ counts totally reducible walks with a fixed starting vertex. This can be deduced from the proofs above, or can be proved by demonstrating a one–one correspondence between these walks and the closed walks with fixed starting vertex in an infinite regular tree of degree k. We state this result in the next theorem, and at the same time recall some of the results we will need from McKay [11].

For notational convenience, define $\omega = 2(k-1)^{1/2}$.

THEOREM 2.3. *Let X be a regular graph of degree k. Let v be a vertex of X and, for $i \geq 0$, let t_i be the number of totally reducible walks of length i in X which start at v. Define $t(x) = \sum_{i=0}^{\infty} t_i x^i$.*

(a) $t(x) = \dfrac{k-2-k(1-\omega^2x^2)^{1/2}}{2(k^2x^2-1)}.$

(b) *$t_i = 0$ if i is odd, and*

$$t_{2r} = \sum_{j=0}^{r} \binom{2r}{j} \frac{2r-2j+1}{2r-j+1}(k-1)^j \qquad (r \geq 0).$$

(c) *Define*

$$f_k(x) = \begin{cases} \dfrac{k(\omega^2-x^2)^{1/2}}{2\pi(k^2-x^2)}, & \text{for } |x| \leq \omega, \\ 0, & \text{otherwise.} \end{cases}$$

Then $t_i = \int_{-\omega}^{\omega} x^i f_k(x)\,dx$.

(d) *If $k \geq 3$,*

$$t_{2r} \sim \frac{4^r k(k-1)^{r+1}}{r(k-2)^2(\pi r)^{1/2}} \qquad \text{as } r \to \infty.$$

(e) *Define the Chebysheff polynomials $T_0(x)$, $T_1(x), \ldots$ by $T_m(\cos\theta) = \cos m\theta$. Then*

$$\int_{-\omega}^{\omega} f_k(x) T_i(x/\omega)\,dx = \begin{cases} 1, & \text{if } i = 0, \\ -\dfrac{k-2}{2(k-1)^m}, & \text{if } i = 2m > 0, \\ 0, & \text{if } i = 2m+1. \end{cases}$$

3. SPANNING TREES

Let X be a graph with vertices $x_1, x_2, \ldots, x_n$ $(n \geq 2)$, and adjacency matrix $\boldsymbol{A} = \boldsymbol{A}(X)$. Let Λ be the $n \times n$ diagonal matrix whose ith diagonal entry is the degree of x_i, and define $\boldsymbol{K} = \boldsymbol{K}(X) = \Lambda - \boldsymbol{A}$. The first lemma in this section reviews some of the basic properties of the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{K}$.

LEMMA 3.1. *Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\boldsymbol{A}$, and let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be the eigenvalues of $\boldsymbol{K}$. Let X have $\kappa(X)$ spanning trees, maximum degree Δ and* *edge-connectivity* η.

(a) *For* $1 \leqslant i \leqslant n$, $-\Delta \leqslant \lambda_i \leqslant \Delta$ *and* $0 \leqslant \mu_i \leqslant 2\Delta$.

(b) *For* $r \geqslant 0$, $w_r = \sum_{i=1}^{n} \lambda_i^r$ *is the number of closed walks of length r in X.*

(c) $\mu_2\mu_3 \cdots \mu_n = n\kappa(X)$.

(d) *If X has c components,* $\mu_1 = \mu_2 = \cdots = \mu_c = 0$ *and* $\mu_{c+1} > 0$ (*if* $c \neq n$).

(e) $\mu_2 \geqslant 2\eta(1-\cos(\pi/n))$.

(f) *If X is regular with degree k,* $\mu_i = k - \lambda_{n-i+1}$ *for* $1 \leqslant i \leqslant n$. *Also,* $\mu_2, \ldots, \mu_n$ *are not all equal unless X is empty or complete.*

PROOF. Part (a) follows from Gershgorin's Theorem. Part (b) follows from the fact that $\sum_{i=1}^{n} \lambda_i^r$ is the trace of A^r. Part (c) is equivalent to the well-known matrix tree theorem, first proved by Borchardt [3], but closely related to a theorem of Kirchhoff [8]. Part (e) was proved by Fiedler [4]. Part (f) is true because $\Lambda = kI$ in this case.

The next lemma is a standard result (see [6] for example).

LEMMA 3.2. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_m$ *be positive real numbers, not all equal. For* $t > 0$ *define* $M(t) = (\frac{1}{m}\sum_{i=1}^{m} \alpha_i^t)^{1/t}$. *Then* $M(t)$ *is strictly increasing and*

$$\lim_{t \to 0+} M(t) = (\alpha_1\alpha_2 \cdots \alpha_m)^{1/m}.$$

THEOREM 3.3. *Let X be a connected regular graph with n vertices and degree* $k \geqslant 3$. *Define* $\tau(X) = (n\kappa(X))^{1/(n-1)}$. *Then*

$$\tau(X) = k \lim_{t \to 0+} \left(\frac{1}{n-1} \sum_{r=0}^{\infty} \binom{t}{r} (-1)^r w_r k^{-r} \right)^{1/t}.$$

Furthermore, the value of the expression on the right for $t > 0$ *is greater than its limit, unless X is a complete graph.*

PROOF. By Lemma 3.1,

$$\tau(X) = \left(\prod_{i=1}^{n-1} (k - \lambda_i) \right)^{1/(n-1)}$$

$$= k \lim_{t \to 0+} \left(\frac{1}{n-1} \sum_{i=1}^{n} \left(1 - \frac{\lambda_i}{k}\right)^{t} \right)^{1/t}, \qquad \text{by Lemma 3.2.}$$

Since $t > 0$, the binomial expansion of $(1-x)^t$ is convergent for $-1 \leqslant x \leqslant 1$. Therefore,

$$\tau(X) = k \lim_{t \to 0+} \left(\frac{1}{n-1} \sum_{i=1}^{n} \sum_{r=0}^{\infty} \binom{t}{r} (-1)^r \lambda_i^r k^{-r} \right)^{1/t},$$

$$= k \lim_{t \to 0+} \left(\frac{1}{n-1} \sum_{r=0}^{\infty} \binom{t}{r} (-1)^r w_r k^{-r} \right)^{1/t}.$$

The second claim follows from Lemma 3.1(f) and Lemma 3.2.

Now write $w_r = nt_r + u_r$, where t_r is as in Theorem 2.3. We can identify u_r as the number of closed but not totally reducible walks of length r in X. Define

$$I_k(t) = \left(\sum_{r=0}^{\infty} \binom{t}{r} (-1)^r t_r k^{-r} \right)^{1/t}$$

$$= \left(\int_{-\omega}^{\omega} \left(1 - \frac{x}{k}\right)^t f_k(x)\, dx \right)^{1/t}, \qquad \text{by Theorem 2.3(c).}$$

Noting that $u_0 = u_1 = u_2 = 0$, Theorem 3.3 can be restated as

$$\tau(X) = kI_k(t) \lim_{t\to 0+} \left(\frac{n}{n-1} + \frac{1}{n-1} I_k(t)^{-t} \sum_{r=3}^{\infty} \binom{t}{r} (-1)^r u_r k^{-r}\right)^{1/t}.$$

Since $f_k(x)$ is an even function with support $(-\omega, \omega)$, $I_k(t) > 0$. Also, $\binom{t}{r}(-1)^r \leq 0$ for $0 < t \leq 1$ and $r \geq 2$. Therefore a consequence of Theorem 3.3 is as follows.

THEOREM 3.4. *Under the conditions of Theorem* 3.3,

$$\tau(X) \leq kI_k(t) \left(\frac{n}{n-1} + \frac{1}{n-1} I_k(t)^{-t} \sum_{r=3}^{\infty} \binom{t}{r} (-1)^r u'_r k^{-r}\right)^{1/t}$$

for $0 < t \leq 1$ *and* $u'_r \leq u_r$ $(r \geq 3)$.

The case $t = 1$ of Theorem 3.4 is equivalent to Theorem 1.1. In order to estimate $I_k(t)$ we first consider a related integral.

LEMMA 3.5. *For* $|\gamma| < 1/\omega$ *define* $J_k(\gamma) = \int_{-\omega}^{\omega} \log(1 - \gamma x) f_k(x)\, dx$. *Then*

$$J_k(\gamma) = -\log\left(\eta \left(\frac{k-\eta}{k-1}\right)^{(k-2)/2}\right), \qquad \text{where } \eta = \frac{1 - (1 - 4(k-1)\gamma^2)^{1/2}}{2(k-1)\gamma^2}.$$

PROOF. A standard result is that $\log(1 - 2yz + z^2) = -2 \sum_{i=1}^{\infty} z^i T_i(y)/i$, for $-1 \leq y \leq 1$ and $|z| < 1$. Putting $z = (1 - (1 - 4(k-1)\gamma^2)^{1/2})/(2\omega)$ and $y = x/\omega$, we find that $\log(1 - \gamma x) = -\log(1 + z^2) - 2 \sum_{i=1}^{\infty} z^i T_i(x/\omega)/i$, for $|x| \leq \omega$.

Since the series on the right is absolutely convergent, we can perform the integration term by term using Theorem 2.3(e). The result is immediate.

THEOREM 3.6. *For any* $k \geq 3$, $kI_k(t) = c_k(1 + O(t))$ *as* $t \to 0+$, *where*

$$c_k = \frac{(k-1)^{k-1}}{(k^2 - 2k)^{(k/2)-1}}.$$

PROOF.

$$\begin{aligned}
I_k(t) &= \left(\int_{-\omega}^{\omega} \left(1 - \frac{x}{k}\right)^t f_k(x)\, dx\right)^{1/t} \\
&= \exp\left(\frac{1}{t} \log \int_{-\omega}^{\omega} \sum_{i=0}^{\infty} \frac{(t \log(1 - (x/k)))^i}{i!} f_k(x)\, dx\right) \\
&= \exp\left(\frac{1}{t} \log\left(1 + t \int_{-\omega}^{\omega} \log\left(1 - \frac{x}{k}\right) f_k(x)\, dx + O(t^2)\right)\right) \\
&= \frac{c_k}{k}(1 + O(t)), \qquad \text{by Lemma 3.5 with } \gamma = \frac{1}{k}.
\end{aligned}$$

Some sample values of c_k are $c_3 \approx 2{\cdot}3094$, $c_4 = 3{\cdot}375$ and $c_5 \approx 4{\cdot}4066$. Asymptotically,

$$c_k = k - \frac{1}{2} - \frac{3}{8k} - O\left(\frac{1}{k^2}\right).$$

LEMMA 3.7. *For $k \geq 3$ and $0 \leq x < 1/k$, $\sum_{i=3}^{\infty} u_i x^i / i = \sum_{i=3}^{\infty} p_i \beta_{i,k}(x)$,* where

$$\beta_{i,k}(x) = 2^{i+2} i \int_0^{\frac{1}{2}\sin^{-1}(\omega x)} \frac{\mathrm{d}\phi}{\sin 2\phi (\omega^i \cot^i \phi - 2^i)}.$$

PROOF.

$$\sum_{i=3}^{\infty} \frac{u_i}{i} x^i = \int_0^x \frac{u(t)}{t} \mathrm{d}t$$

$$= \int_0^x \frac{zp(y)}{t} \mathrm{d}t, \qquad \text{where } y = \frac{2(1-(1-\omega^2 t^2)^{1/2})}{\omega^2 t} \quad \text{and} \quad z = \frac{2}{(1-\omega^2 t^2)^{1/2}}$$

$$= \sum_{i=3}^{\infty} p_i \beta_{i,k}(x), \qquad \text{where } \beta_{i,k}(x) = \int_0^x \frac{izy^i}{t(1-y^i)} \mathrm{d}t.$$

The expression for $\beta_{i,k}(x)$ in the lemma is now easily obtained via the substitution $\omega t = \sin 2\phi$.

THEOREM 3.8. *For any $k \geq 3$ and $R \geq 3$ there is a constant $D = D(k, R)$ such that*

$$\tau(X) \leq c_k \exp\left(-\frac{1}{n} \sum_{i=3}^{R} p_i \beta_{i,k}(1/k) + \frac{D}{n^{1/2}} \right).$$

PROOF. Firstly, note that $I(t)^{-t} = 1 + \mathrm{O}(t)$ and that $\binom{t}{i}(-1)^{i+1} < t(1-t)^{i-1}/i$ for $0 < t < 1$ and $i \geq 3$. By Theorems 3.4 and 3.6 we have

$$\tau(X) \leq c_k (1+\mathrm{O}(t)) \left(1 + \frac{t}{n-1}\left(\frac{1}{t} - (1+\mathrm{O}(t)) \sum_{i=3}^{\infty} \frac{(1-t)^{i-1}}{ik^i} u_i\right)\right)^{1/t}$$

$$\leq c_k (1+\mathrm{O}(t)) \exp\left(\frac{1}{n-1}\left(\frac{1}{t} - (1+\mathrm{O}(t)) \sum_{i=3}^{R} p_i \beta_{i,k}\left(\frac{1-t}{k}\right)\right)\right).$$

Now $\sum_{i=3}^{R} p_i \beta_{i,k}((1-t)/k) = \sum_{i=3}^{R} p_i \beta_{i,k}(1/k) - \mathrm{O}(tn)$, since $p_i \leq n(k-1)^i/i$. The theorem now follows on putting $t = n^{-1/2}$.

A table of values of $\beta_{i,k}(1/k)$ can be found in [9]. Some example values are $\beta_{3,3}(1/3) \approx 0{\cdot}26706$, $\beta_{3,4}(1/4) \approx 0{\cdot}07548$, $\beta_{4,3}(1/3) \approx 0{\cdot}12908$ and $\beta_{5,5}(1/5) \approx 0{\cdot}00195$. It can be shown that $\beta_{i,k}(1/k) \sim 2/(k-1)^i$ as $i + k \to \infty$.

By refining the techniques above, a reasonably good upper bound on D, and thus one on $\tau(X)$, can be found. However, viewed as an upper bound on $\kappa(X)$, the uncertainty involved in D (a factor of $\mathrm{e}^{\mathrm{O}(n^{1/2})}$ is annoyingly large. Fortunately, there is a technique by which this factor can be reduced to a constant. We begin with a result from [10].

THEOREM 3.9. *Let $2 < K_0 \leq K_1$ be constants. Let $n_1 < n_2 < \cdots$ be a sequence of natural numbers and, for each i, let $\boldsymbol{k}_i = (k_i^{(1)}, k_i^{(2)}, \ldots, k_i^{(n_i)})$ be a graphical degree sequence for which $k_i^{(l)} \leq K_1$ for $1 \leq l \leq n_i$ and with arithmetic mean $\bar{k}_i \geq K_0$. Define $\hat{k}_i$ to be the geometric mean of the entries of $\boldsymbol{k}_i$.*

For each i define $\bar{\kappa}_i$ to be the average number of spanning trees over all labelled graphs with degree sequence $\boldsymbol{k}_i$. Then there are constants $A>0$ and B, independent of i, such that $Ac(\boldsymbol{k}_i)^{n_i}/n_i \leq \bar{\kappa}_i \leq Bc(\boldsymbol{k}_i)^{n_i}/n_i$, where

$$c(\boldsymbol{k}_1)=\frac{\hat{k}_i(\bar{k}_i-1)^{\bar{k}_i-1}}{\bar{k}_i^{\bar{k}_i/2}(\bar{k}_i-2)^{(\bar{k}_i/2)-1}}.$$

Since X has n vertices and degree k, it has $m=nk/2$ edges. Label these $e_1, e_2, \ldots, e_m$ in any order. For $1 \leq j \leq m$ form the $(n+1)$-vertex graph X_j by inserting a new vertex in the middle of the edge e_j. In other words, replace the edge e_j by a path of length two.

LEMMA 3.10. *For $i \geq 3$ define $p_{i,j}$ to be the number of primitive circuits of length i in X_j. Then*

$$\kappa(X_j) \leq \left(\frac{k-1}{k-2}\right)^2 c_k^n \exp\left(-\sum_{i=3}^{\infty} p_{i,j}\beta_{i,k}(1/k)\right).$$

PROOF. Fix $0<\delta<\frac{1}{2}$. By Theorem 3.9 there is an increasing sequence of graphs G_1, $G_2, \ldots$ such that

(a) G_r has degree sequence $\boldsymbol{k}_r=(k_r^{(1)}, k_r^{(2)}, \ldots, k_r^{(m_r)})$, where $k_r^{(l)}=k-2$ for $1 \leq l \leq \delta m_r$ and $k_r^{(l)}=k$ for $\delta m_r < l \leq m_r$.

(b) For some constant A, depending only on δ and k, $\kappa(G_r) \geq Ac(\boldsymbol{k}_r)^{m_r}/m_r$.

For each r form the graph H_r by taking one copy of G_r and $\lfloor \delta m_r \rfloor$ copies of X_j, then identifying each of the vertices of degree $k-2$ in G_r with the vertex of degree 2 in one of the copies of X_j. Clearly, H_r is regular of degree k and order $N=m_r+n\lfloor \delta m_r \rfloor$.

Since the number of primitive circuits of length i in H_r is at least Mp_{ij}, where $M=\lfloor \delta m_r \rfloor$, Theorem 3.8 tells us that for some $D=D(R)$,

$$\tau(H_r) \leq c_k \exp\left(-\frac{M}{N}\sum_{i=3}^{R} p_{i,j}\beta_{i,j}(1/k)+\frac{D}{N^{1/2}}\right).$$

Now $\tau(H_r)=(N\kappa(H_r))^{1/(N-1)}$, by definition, and $\kappa(H_r)=\kappa(G_r)\kappa(X_j)^M$, obviously. Therefore

$$\begin{aligned}\kappa(X_j) &= \left(\frac{\tau(H_r)^{N-1}}{N\kappa(G_r)}\right)^{1/M} \\ &\leq \left(\frac{c_k^{(N-1)}m_r}{NAc(\boldsymbol{k}_r)^{m_r}}\right)^{1/M} \exp\left(-\frac{N-1}{N}\sum_{i=3}^{R} p_{i,j}\beta_{i,k}(1/k)\right)\exp\left(\frac{D(N-1)}{M(N)^{1/2}}\right).\end{aligned}$$

Letting $r \to \infty$ we obtain

$$\kappa(X_j) \leq c_k^n \exp\left(-\sum_{i=3}^{R} p_{i,j}\beta_{i,k}(1/k)\right) \lim_{r\to\infty}\left(\frac{c_k}{c(\boldsymbol{k}_r)}\right)^{1/\delta}.$$

Since D has disappeared, we may replace R by ∞. Finally, from Theorem 3.9, we find that

$$\lim_{\delta\to 0+}\lim_{r\to\infty}\left(\frac{c_k}{c(\boldsymbol{k}_r)}\right)^{1/\delta}=\left(\frac{k-1}{k-2}\right)^2.$$

LEMMA 3.11.

$$\kappa(X) < 2^{(2/k)-1}\left(\prod_{j=1}^{m}\kappa(X_j)\right)^{1/m}.$$

PROOF. For $1 \leqslant j \leqslant m$ define α_j to be the proportion of the spanning trees of X which use edge e_j. Then $\kappa(X_j) = (2-\alpha_j)\kappa(X)$, and so

$$\left(\prod_{j=1}^{m} \kappa(X_j)\right)^{1/m} = \kappa(X)\left(\prod_{j=1}^{m} (2-\alpha_i)\right)^{1/m}$$

$$\geqslant \kappa(X)\exp\left(\frac{\log 2}{m}\sum_{j=1}^{m}(1-\alpha_j)\right), \qquad \text{since } 0 \leqslant \alpha_j \leqslant 1.$$

Since each spanning tree uses $n-1$ edges and we are summing over every edge, $\frac{1}{m}\sum_{j=1}^{m} \alpha_j = 2(n-1)/(nk) < 2/k$. Therefore

$$\left(\prod_{j=1}^{m} \kappa(X_j)\right)^{1/m} > \kappa(X)\exp\left(\frac{k-2}{k}\log 2\right)$$

$$= 2^{1-(2/k)}\kappa(X).$$

THEOREM 3.12.

$$\kappa(X) < 2^{(2/k)-1}\left(\frac{k-1}{k-2}\right)^2 c_k^n \exp\left(-\sum_{i=3}^{nk/2}\left(1-\frac{2i}{nk}\right) p_i \beta_{i,k}(1/k)\right).$$

PROOF. Since a primitive circuit of length i uses at most i edges, it is clear that

$$\frac{1}{m}\sum_{j=1}^{m} p_{ij} \geqslant \left(1-\frac{i}{m}\right) p_i.$$

The theorem now follows by taking the geometric mean over $1 \leqslant j \leqslant m$ of Lemma 3.10 and then applying Lemma 3.11.

EXAMPLE. Let X be the cartesian product $C_{10} \times C_{10}$. Thus $n = 100$ and $k = 4$. Theorem 1.1 gives $\kappa(X) < 1{\cdot}09 \times 10^{58}$. Using the trivial bounds $p_i \geqslant 0$, Theorem 3.12 gives $\kappa(X) < 1{\cdot}07 \times 10^{53}$. With the actual values $p_4 = 100$, $p_6 = 200$ and $p_8 = 1300$, Theorem 3.12 gives $\kappa(X) < 3{\cdot}76 \times 10^{51}$. The correct value of $\kappa(X)$ is approximately $1{\cdot}545 \times 10^{50}$.

4. ASYMPTOTIC RESULTS

In this section we consider a sequence $X_1, X_2, \ldots$ of regular connected graphs of degree $k \geqslant 3$, and investigate the limit points of the sequence $\tau(X_1), \tau(X_2), \ldots$. In particular we will show that the value c_k is best possible in the sense that there are sequences $X_1, X_2, \ldots$ for which $\tau(X_i) \to c_k$ as $i \to \infty$.

Let $X_1, X_2, \ldots$ be a sequence of connected regular graphs of degree $k \geqslant 3$. For each i, define n_i to be the order of X_i and let $\bar{\tau}_i = \log(n_i\kappa(X_i))/n_i$. We will assume throughout that $n_1 < n_2 < \cdots$. Define the function F_i: $\mathbb{R} \to \mathbb{R}$, where $F_i(x)$ is the proportion of the eigenvalues of $A(X_i)$ which are less than or equal to x. Thus $F_i(x)$ is a non-decreasing right-continuous step function with $F_i(x) = 0$ for $x < -k$ and $F_i(x) = 1$ for $x \geqslant k$.

For $i \geqslant 1$ and $m \geqslant 3$ define $C_i(m)$ to be the number of cycles in X_i with length m or less. The sequence $X_1, X_2, \ldots$ will be said to satisfy *Condition* (*A*) if, for each fixed m, $C_i(m)/n_i \to 0$ as $i \to \infty$. The sequence will satisfy *Condition* (*B*) if there are constants $\alpha > (2\log(k/\omega))^{-1}$ and $\varepsilon > 0$ such that $C_i(r_i) = O(n_i(\log n_i)^{-1-\varepsilon})$ as $i \to \infty$, where $r_i = 2\lfloor \alpha \log\log n_i \rfloor$. Clearly Condition (B) implies Condition (A). The existence of a sequence $X_1, X_2, \ldots$ which satisfies Condition (B) follows from the existence of regular graphs of high girth (see [14]). In fact, rather more is true:

THEOREM 4.1. [12] *For $i \geqslant 1$ choose X_i at random from the set of all labelled connected regular graphs of degree k and order n_i. Then Condition (B) is satisfied with probability one.*

Our principal tool in the following is a result from [11]:

LEMMA 4.2. *If $X_1, X_2, \ldots$ satisfies Condition (A) then, for each real x, $F_i(x) \to F(x)$ as $i \to \infty$, where*

$$F(x) = \int_{-\infty}^{x} f_k(t)\,\mathrm{d}t.$$

By Lemma 3.1, $\bar{\tau}_i = \int_{-k}^{k-0} \log(k-x)\,\mathrm{d}F_i(x)$, where the integral excludes the jump in $F_i(x)$ at $x = k$. Unfortunately Lemma 4.2 does not imply that, under Condition (A), $\bar{\tau}_i \to \int_{-k}^{k} \log(k-x)\,\mathrm{d}F(x) = \log c_k$. However, our next theorem shows that Condition (B) is sufficient.

THEOREM 4.3. *If $X_1, X_2, \ldots$ satisfies Condition (B) then $\bar{\tau}_i \to \log c_k$ as $i \to \infty$.*

PROOF. Since $\varepsilon > 0$ and $\alpha > (2\log(k/\omega))^{-1} > 0$, there is a number z such that $\max\{\omega \exp(1/2\alpha), k \exp(-\varepsilon/2\alpha)\} < z < k$. Then $z > \omega$, so that

$$\int_{-k}^{z} \log(k-x)\,\mathrm{d}F_i(x) \to \int_{-k}^{z} \log(k-x)\,\mathrm{d}F(x) \qquad \text{as } i \to \infty,$$

by Lemma 4.1, since $F(x)$ is constant outside $[-\omega, \omega]$, and $\log(k-x)$ is uniformly continuous on $[-\omega, \omega]$. Therefore, it will suffice to prove that

$$\int_{z}^{k-0} \log(k-x)\,\mathrm{d}F_i(x) \to 0 \qquad \text{as } i \to \infty.$$

By Lemma 3.1(e), the second largest eigenvalue of $\mathbf{A}(X_i)$ is less than or equal to $k - 2(1-\cos(\pi/n_i)) < k - 1/n_i^2$. It follows that $\int_z^{k-0} \log(k-x)\,\mathrm{d}F_i(x) \to 0$ if $(1 - F_i(z)) \log n_i \to 0$ as $i \to \infty$.

For sufficiently large i, $r_i \geqslant 0$. Let $r = r_i$ and define $w_r = n_i t_r + u_r$ as in Section 3. By Lemma 3.1,

$$w_r = n_i \int_{-\infty}^{\infty} x^r\,\mathrm{d}F_i(x) \geqslant n_i \int_{z}^{k} x^r\,\mathrm{d}F_i(x) \geqslant n_i z^r (1 - F_i(z)).$$

Therefore,

$$1 - F_i(z) \leqslant \frac{w_r}{n_i z^r} = \frac{t_r}{z^r} + \frac{u_r}{n_i z^r}.$$

We first consider the term t_r/z^r. By Theorem 2.3(d), there is a constant K depending only on k, such that $t_r < K\omega^r$. Therefore,

$$\frac{t_r}{z^r} \log n_i < K \left(\frac{\omega}{z}\right)^r \log n_i.$$

Now $r = 2\delta + 2\alpha \log\log n_i$, where $-1 < \delta \leq 0$. Therefore

$$\frac{t_r}{z^r} \log n_i < K \left(\frac{\omega}{z}\right)^{2\delta} (\log n_i)^{1+2\alpha \log(\omega/z)}$$

$$\to 0 \quad \text{as } i \to \infty, \qquad \text{since } z > \omega\, e^{1/2\alpha}.$$

Now consider the term $u_r/(n_i z^r)$. Each closed walk of length r in X_i which is not totally reducible is a cyclic permutation of one which starts at a vertex on a cycle of length r or less. Since there are at most k^r closed walks of length r starting at a given vertex, we must have $u_r \leq C_i(r) k^r r^2$. Therefore

$$\frac{u_r}{n_i z^r} \log n_i \leq \frac{\log n_i}{n_i} C_i(r) r^2 \left(\frac{k}{z}\right)^{2\delta} (\log n_i)^{2\alpha \log(k/z)}$$

$$\to 0 \quad \text{as } i \to \infty,$$

since

$$z > k\, e^{-\varepsilon/2\alpha}, \qquad C_i(r) = O(n_i (\log n_i)^{-1-\varepsilon}) \qquad \text{and} \qquad r = O(\log\log n_i).$$

We conclude that $(1 - F_i(z)) \log n_i \to 0$ as $i \to \infty$, and so $\bar{\tau}_i \to \log c_k$ as $i \to \infty$.

COROLLARY 4.4. *If $X_1, X_2, \ldots$ satisfies Condition (B) then $\tau(X_i) \to c_k$ as $i \to \infty$.*

We wish to point out that we know of no sequence $X_1, X_2, \ldots$ which satisfies Condition (A) but for which $\tau(X_i) \not\to c_k$. In other words Condition (B) may be too strong. We suspect that a deeper analysis using the techniques of Section 3 might solve this problem. However, we can show that Condition (A) is necessary for $\tau(X_i) \to c_k$.

THEOREM 4.5. *If $X_1, X_2, \ldots$ violates condition (A), then* $\liminf_{i \to \infty} \tau(X_i) < c_k$.

PROOF. If $X_1, X_2, \ldots$ violates Condition (A), there is a subsequence $X_{i_1}, X_{i_2}, \ldots$ and constants $r \geq 3$ and $a > 0$ such that $C_{i_j}(r) \geq a n_{i_j}$ for $j \geq 1$. The claim now follows easily from Theorem 3.12.

The method used in the proof of Theorem 4.3 can be used in conjunction with Theorem 4.4 of McKay [11] to obtain a rudimentary lower bound for $\kappa(X)$ in terms of the order, degree and girth of X. We will leave the details to the reader.

5. UNIFORM BOUNDS

A trivial corollary to Theorem 3.2 is that

$$\kappa(X) < 2^{(2/k)-1} \left(\frac{k-1}{k-2}\right)^2 c_k^n,$$

which Theorem 3.9 shows to be too high by at most $O(n)$. In this section we will sharpen this bound until it is high by at most $O(\log n)$. We begin with a collection of necessary lemmas. All notation is as in Section 3.

LEMMA 5.1.
(a) *Let ρ_j be the largest eigenvalue of $\mathbf{A}(X_j)$. Then $\rho_j \geq k - 2(k-2)/(kn)$.*
(b) *For $i \geq 1$, $t_{2i} \leq 4\omega^{2i} i^{-3/2}$.*
(c) *If $-1 < x < 1$ and $r \geq 0$, then $\sum_{i=2r}^{\infty} x^i/i > 0$.*
(d) *For some j, $\kappa(X) \leq k\kappa(X_j)/(2k-2)$.*

PROOF. To prove part (a), recall that $\rho_j \geq \boldsymbol{x}^{\mathrm{T}} \boldsymbol{A}(X_j)\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ for any non-zero vector $\boldsymbol{x}$. The required bound is obtained on chosing the entries of $\boldsymbol{x}$ thus: $2k$ for the vertex of degree two, k^2-1 for its two neighbours, and k^2 for every other vertex.

Part (b) can be proved from Theorem 2.3(b). Part (c) follows from the identity $\sum_{i=2r}^{\infty} x^i/i = \int_0^x z^{2r-1}\,dz/(1-z)$. Part (d) is implicit in the proof of Lemma 3.11.

THEOREM 5.2. *$\kappa(X) \leq \alpha(k) c_k^n \log n/n$, where $\alpha(k) = \mathrm{O}(1/(k \log k))$.*

PROOF. The proofs of Theorems 3.8 and Lemma 3.10 can be reworked with essentially no change to derive the inequality

$$\kappa(X_j) \leq \left(\frac{k-1}{k-2}\right)^2 c_k^n \exp\left(-\sum_{i=3}^{\infty} \frac{u_{i,j}}{ik^i}\right),$$

where $u_{i,j}$ is the number of closed walks of length i in X_j which are not totally reducible. By Lemma 5.1(a)–(c),

$$\sum_{i=3}^{\infty} \frac{u_{i,j}}{ik^i} \geq \sum_{i=2r}^{\infty} \frac{1}{i}\left(1-\frac{2(k-2)}{k^2 n}\right)^i - 2(n+1)\sum_{i=r}^{\infty} \frac{1}{i^{5/2}}\left(\frac{\omega}{k}\right)^{2i}$$

$$\geq \log\frac{k^2 n}{2(k-2)} - \log(2r) - \gamma - \frac{2(n+1)}{r^{5/2}}\left(\frac{\omega}{k}\right)^{2r}\left(\frac{k}{k-2}\right)^2,$$

where $\gamma \approx 0{\cdot}5772$ is Euler's constant.

Choose $r = \lceil \log n/2 \log(k/\omega) \rceil$. Lemma 5.1(d) and the trivial inequality $n \geq k+1$ then yield the required bound, with

$$\alpha(k) = \frac{k-1}{k(k-2)}\left(\frac{1}{\log(k/\omega)} + \frac{2}{\log(k+1)}\right)\exp\left(\gamma + \frac{2(k+2)k^2}{(k+1)(k-2)^2}\left(\frac{2\log(k/\omega)}{\log(k+1)}\right)^{5/2}\right).$$

It is clear that the bound in Theorem 5.2 can be reduced further by doing the calculations more carefully. However, we are unable to reduce it by an increasing function of n. Indeed, such a reduction may not be possible. The argument used in the proof ignores closed walks of length less than $2r$; the average contribution of the primitive circuits of length less than $2r$ to the bound in Lemma 3.10 is within a constant of $\log n$. Of course, closed walks longer than $2r$ can use primitive circuits shorter than $2r$, so this argument is hardly conclusive. Nevertheless, we are confident enough to conjecture that the bound in Theorem 5.2 is high by at most a function of k.

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