Proc. 3rd Car. Conf. Comb.
& Comp. pp. 139-143

SPANNING TREES IN RANDOM REGULAR GRAPHS

Brendan D. McKay

Computer Science Dept., Vanderbilt University, Nashville, Tennessee 37235

Let $n_1 < n_2 < \cdots$ be possible orders of connected regular graphs of fixed degree $k \geq 3$. For each i, choose a graph X_i at random from the set of all connected regular graphs of order n_i and degree k. Let $\kappa(X_i)$ be the number of spanning trees of X_i. Then, with probability one,

$$\kappa(X_i)^{1/n_i} \to \frac{(k-1)^{k-1}}{(k^2-2k)^{k/2-1}} \quad \text{as } i \to \infty.$$

§1 INTRODUCTION

Let $X = X_1, X_2, \ldots$ be a sequence of connected regular graphs of degree $k \geq 3$ and orders $n_1 < n_2 < \cdots$. Define κ_i to be the number of spanning trees of X_i. For $i \geq 1$ and $m \geq 3$, let $C(m, X_i)$ be the number of cycles of length at most m in X_i. We will say that X *satisfies Condition (B)* if there are constants m_0 (arbitrary), $m_1 > 1/\log(k^2/4(k-1))$ and $\epsilon > 0$ such that $C(2\lfloor m_0 + m_1 \log\log n_i \rfloor, X_i) = O(n_i(\log n_i)^{-1-\epsilon})$ as $i \to \infty$. The ordinal B is used for consistency with [2]. Our first theorem was proved in [2].

1·1 · **Theorem** *If X satisfies Condition (B), then*

$$\kappa(X_i)^{1/n_i} \to \frac{(k-1)^{k-1}}{(k^2-2k)^{k/2-1}} \quad \text{as } i \to \infty.$$

□

In this paper we will prove that, if X is constructed by choosing each X_i at random from the set of all connected labelled regular graphs of order n_i and degree k, then X satisfies Condition (B) with probability one.

§2 SUBGRAPHS OF RANDOM REGULAR GRAPHS

In this section we will be concerned exclusively with the set $R(n,k)$ of all connected labelled regular graphs of degree k and order n. Two such graphs, X_1 and X_2, are *closely related* if $|E(X_1)\setminus E(X_2)| = 2$. Since we are only considering simple graphs, it is clear that each of $E(X_1)\setminus E(X_2)$ and $E(X_2)\setminus E(X_1)$ consists of two independent edges and that these four edges together form a square. Another way of defining this relationship is to say that X_1, $X_2 \in R(n,k)$ are closely related if there are distinct vertices v_1, v_2, v_3, v_4 of X_1 such that $v_1v_2, v_3v_4 \in E(X_1)$, $v_1v_3, v_2v_4 \notin E(X_1)$ and $E(X_2) = E(X_1)\setminus\{v_1v_2, v_3v_4\}\cup\{v_1v_3, v_2v_4\}$.

Let F be any non-empty subset of $E(K_n)$, and define $R(n,k,F)$ to be the subset of $R(n,k)$ consisting of those graphs X for which $F \subseteq E(X)$. Let $v_1v_2 \in F$.

2·1 Lemma $|R(n,k,F\setminus\{v_1v_2\})| \geq \dfrac{nk-2|F|-2(k^2-1)}{2k^2}|R(n,k,F)|$.

Proof: Let M be the number of pairs (X_1, X_2) of closely related graphs, where $X_1 \in R(n,k,F)$ and $X_2 \in R(n,k,F\setminus\{v_1v_2\})$. We will prove the lemma by estimating M in two different ways.

Firstly, consider an arbitrary $X_1 \in R(n,k,F)$. We can construct any closely related graph in $R(n,k,F\setminus\{v_1v_2\})$ by finding adjacent vertices v_3 and v_4 of X_1 and then replacing the edges v_1v_2 and v_3v_4 by v_1v_3 and v_2v_4. For this to be valid, it is necessary that $\{v_3,v_4\}\cap\{v_1,v_2\} = \emptyset$, $v_3v_4 \notin F$, and $v_1v_3, v_2v_4 \notin E(X_1)$. This leaves us with at least $nk-2|F|-2(k^2-1)$ choices for (v_3, v_4), except that we are also requiring X_2 to be connected. Furthermore, it is easy to see that, since X_1 is connected, at least one of X_2 and X_2' is connected, where $E(X_2) = E(X_1)\setminus\{v_1v_2, v_3v_4\}\cup\{v_1v_3, v_2v_4\}$ and $E(X_2') = E(X_1)\setminus\{v_1v_2, v_3v_4\}\cup\{v_1v_4, v_2v_3\}$. Therefore,

$$M \geq (nk/2 - |F| - k^2 + 1)|R(n,k,F)|. \tag{1}$$

Now consider an arbitrary $X_2 \in R(n,k,F\setminus\{v_1v_2\})$. Since X_2 has degree k, the number of closely related graphs in $R(n,k,F)$ is at most k^2. Therefore

$$M \leq k^2|R(n,k,F\setminus\{v_1v_2\})|. \tag{2}$$

The lemma now follows easily from (1) and (2). □

2·2 Corollary *If $R(n,k) \neq \emptyset$ and $2|F| < nk - 2(k^2-1)$, then*

$$\frac{|R(n,k,F)|}{|R(n,k)|} \leq \left(\frac{2k^2}{nk - 2(k^2-1) - 2|F|}\right)^{|F|}.$$

§3 CYCLES

Our primary aim in this section is to prove that X satisfies Condition (B) with probability one. In doing so we will prove rather more.

Since a direct application of Corollary 2·2 to the estimation of $C(m, X_i)$ does not appear to be sufficient, we will instead bound the expectation of $C(m, X_i)^2$. The following lemma will prove very useful.

3·1 Lemma *Let $1 \leq l < r \leq s \leq n$. The number of ways of choosing an r-cycle C_1 and s-cycle C_2 from K_n such that $|E(C_1) \cap E(C_2)| = l$ is*

$$w(n,s,r,l) \leq \frac{n^{r+s-l-1}}{2}\left(1 + \sqrt{\frac{2(l-1)}{n}}\right)^{r+s-2l-2}.$$

Proof: Since $l < r \leq s$, $E(C_1) \cap E(C_2)$ consists of a collection of disjoint paths. Suppose that the number of paths is t.

The number of ways of choosing t paths, containing l edges altogether, is

$$\binom{l-1}{t-1}\frac{n!}{2^t t!\,(n-l-t)!} < \frac{n^{l+t}(l-1)^{t-1}}{2^t t!\,(t-1)!}.$$

Given $E(C_1) \cap E(C_2)$, we can construct C_1 by arranging the t paths in a cycle and then inserting $r-l-t$ new vertices. The number of ways this can be done is

$$2^{t-1}(r-l-1)!\binom{n-l-t}{r-l-t} < 2^{t-1}n^{r-l-t}\frac{(r-l-1)!}{(r-l-t)!}.$$

Constructing C_2 similarly, we find that the pair (C_1, C_2) can be chosen in less than

$$\frac{n^{r+s-l-t}2^{t-2}}{t}\binom{r-l-1}{t-1}\binom{s-l-1}{t-1}$$

ways.

The sum of this expression over t is bounded by the function given in the statement of the lemma. □

We will use $\mathcal{E}(Z)$ to denote the expectation of a random variable Z.

3·2 Theorem *Let $m = m(n) = O(n^{\frac{1}{2}-\epsilon})$, for some $\epsilon > 0$. Then*

$$\mathcal{E}\big(C(m(n_i), X_i)^2\big) = O\Big(\frac{(2k)^{2m}}{m^2}\Big).$$

Proof: First note that, by Corollary 2·2, the probability that X_i contains any specified set of $t \leq m(n_i)$ edges is bounded above by $A(2k/n_i)^t$, for some constant A.

Now define $F_1, F_2, \ldots, F_N$ to be the set of all cycle (read sets of edges forming cycles) of length at most m in K_n. For $1 \leq j \leq N$, define

$$Z_j = \begin{cases} 0, & \text{if } F_j \not\subseteq E(X_i), \\ 1, & \text{if } F_j \subseteq E(X_i). \end{cases}$$

Then clearly

$$\mathcal{E}\big(C(m, X_i)^2\big) = \sum_{1 \leq u,v \leq N} \mathcal{E}(Z_u Z_v). \tag{3}$$

We will break the sum above into three parts, according to $|F_u|$, $|F_v|$ and $|F_u \cap F_v|$. Put $n = n_i$.

(a) First consider the contribution to (3) with $u = v$. This is clearly equal to

$$\begin{aligned} \sum_{1 \leq u \leq N} \mathcal{E}(Z_u) &\leq A \sum_{s=3}^{m} \frac{n!}{2s(n-s)!}\Big(\frac{2k}{n}\Big)^s \\ &< A \sum_{s=3}^{m} \frac{(2k)^s}{2s} \\ &= O\Big(\frac{(2k)^m}{m}\Big). \end{aligned}$$

(b) Next, consider the terms of (3) for which $|F_u \cap F_v| = 0$. The contribution here is

$$\begin{aligned} A \sum_{r=3}^{m} \sum_{s=3}^{m} \frac{n!}{2r(n-r)!}\,\frac{n!}{2s(n-s)!}\Big(\frac{2k}{n}\Big)^{r+s} &< A \sum_{r=3}^{m} \sum_{s=3}^{m} \frac{(2k)^{r+s}}{4rs} \\ &= O\Big(\frac{(2k)^{2m}}{m^2}\Big). \end{aligned}$$

(c) Finally, by Lemma 3·1, the contribution to (3) from the terms not included in (a) or (b) is bounded above by

$$A \sum_{r=3}^{m} \sum_{s=r}^{m} \sum_{l=1}^{r-1} n^{r+s-l-1}\Big(\frac{2k}{n}\Big)^{r+s-l}\Bigg(1 + \sqrt{\frac{2(l-1)}{n}}\Bigg)^{r+s-2l-2}.$$

This expression can be shown to be $O\big((2k)^{2m}/m^2\big)$, but we will leave the gory details to the reader. □

3·3 **Theorem** *Let* $m(n) = O(n^{\frac{1}{2}-\epsilon})$ *for some* $\epsilon > 0$, *and suppose that* $T_1, T_2, \ldots$ *is a sequence of positive numbers such that*

$$\sum_{i=1}^{\infty} \frac{(2k)^{2m(n_i)}}{m(n_i)^2 T_i^2} < \infty.$$

Then, with probability one, $C(m(n_i), X_i) = O(T_i)$ *as* $i \to \infty$.

Proof: This is a simple application of the Borel-Cantelli Lemma and Chebychev's Inequality (see [1] for both) to Theorem 3·2. □

As a simple application we have:

3·4 **Corollary** *Let* $\alpha > 0$ *be constant. Then, with probability one,* $C(\alpha \log_{2k} n_i, X_i) = O(n^{\frac{1}{2}+\alpha})$ *as* $i \to \infty$. □

3·5 **Corollary** *X satisfies Condition (B) with probability one.* □

§4 NOTES

The technique used to prove of Lemma 2·1 can be strengthened considerably, and also used to prove a corresponding upper bound. This will be demonstrated at length in [3], but without the connectivity restriction. For random regular graphs, not necessarily connected, the bound of Theorem 3·2 can be lowered to $O\big((k-1)^{2m}/m^2\big)$, which is best possible. See Wormald [4] for many related results.

REFERENCES

1. W. Feller, An introduction to probability theory and its applications, Vol. I, Third Edition (Wiley, NY 1968).

2. B. D. McKay, Spanning trees in regular graphs, *Vanderbilt University, Computer Science Technical Report CS–81–01* (1981).

3. B. D. McKay, Subgraphs of random graphs with specified degrees, to appear.

4. N. C. Wormald, The asymptotic distribution of short cycles in random regular graphs, to appear.