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Combinatorial Mathematics VI, Lecture Notes in Mathematics
No. 748, Springer 1979. pp 21-31

THE CURRENT STATUS OF THE GENERALISED MOORE GRAPH PROBLEM

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1. INTRODUCTION AND BACKGROUND

The concept of a generalised Moore graph has been developed by Cerf, Cowan, Mullin, Stanton, in a series of paper [2]-[7]. Basically, the problem arose in connection with the topological design of computer communication networks; in Figure 1, we reproduce from [3] the graph of P(N,V) against N, where N is the number of nodes in a regular graph of valence V, P(N,V) is the average path length in the graph, and P(N,V) is minimal. It is clear from the figure that P(N,V) grows logarithmically for $V \geq 3$, and we have restricted our published discussions to the case V = 3, which is typical, economical and convenient.

2. REPHRASING THE PROBLEM

To rephrase the problem in purely graph-theoretic terms, we look at one particular node, which we call the root node R. Recall that we are dealing with valence 3; then there are to be 3 nodes at unit distance from R, 6 at distance 2 from R, 12 at distance 3 from R, etc. We determine k such that

$$1 + 3 + 6 + 12 + \dots + 3.2^{k-2} < N,$$
$$1 + 3 + 6 + 12 + \dots + 3.2^{k-1} \geq N.$$

This ensures that all nodes are packed as close to R as possible; there are 3.2^{j-1} nodes at distance j from R for j = 1,2,3,...,k-1.

If the kth level contains 3.2^{k-1} nodes, we speak of a complete Moore graph; otherwise, the graph is incomplete. We employ the notation M(N,3) for a generalised Moore graph on N nodes (complete if and only if $N = 3.2^k-2$).

Furthermore, we require that the distance pattern be the same no matter what node is chosen as root node R. This is the essential graph-theoretic condition.

3. COMPLETE MOORE GRAPHS

In a complete Moore graph, the maximal distance from R is k, and the smallest circuit is a (2k+1)-gon. If we count all these circuits, their number is easily determined.

Number of nodes in graph = $N = 3.2^k-2$.

Number of edges in graph = $\frac{3N}{2} = 3(3.2^{k-1}-1)$.

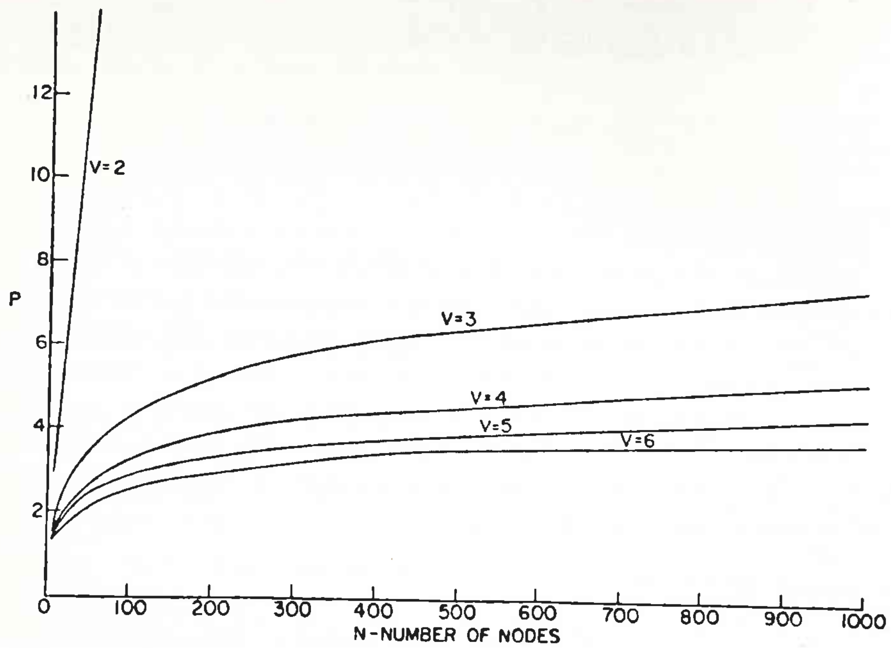


FIGURE 1 - Plot of P(N,V)

Number of edges in joins at levels lower than $k = 3 + 6 + 12 + \ldots + 3.2^{k-1}$
$= 3(2^k-1)$.

Number of edges at level $k = 3(3.2^{k-1}-2^k) = 3.2^{k-1}$.

Each edge at level k determines one (2k+1)-gon through R. Hence:
Number of (2k+1)-gons through $R = 3.2^{k-1}$.

Total number of (2k+1)-gons $= 3.2^{k-1}(3.2^k-2)$.

Total number of distinct (2k+1)-gons $= \dfrac{3.2^k(3.2^{k-1}-1)}{2k+1}$.

We thus have the

THEOREM. *A complete Moore graph can only exist when* 2k + 1 *divides* (3.2^{k-1}).
We note that this occurs (for $k \leq 125{,}000$*) only for* k = 1,2,6425, *and* 116983.

The first 2 cases lead to the tetrahedral graph and the Petersen graph. None

of the others can exist, since they would be minimal graphs of girth 2k + 1 and valence 3; it is known from various results, summarised in Biggs [1], that graphs satisfying this girth condition can only exist (for k = 3) when the girth does not exceed 5.

4. MANUAL RESULTS

Papers [2]-[7] give a thorough discussion of possible graphs on N vertices for N = 4(2)24. As a summary of results the number of distinct graphs on N vertices is given as f(N) in the following table.

N	4	6	8	10	12	14	16	18	20	22	24
f(N)	1	2	2	1	2	~~8~~ 7	6	1	1	0	1

In addition to these published results, D.D. Cowan has determined that f(26) = 2. Also, examples of graphs on 28, 30, 32, 34 nodes have been given; these were obtained in several cases by a type of inductive heuristic approach.

By the time one reaches the case N = 26, the case approach has become quite complicated. The main problem is that the "same" graph occurs many times, and so isomorphism rejection is needed again and again.

5. COMPUTERISATION OF THE PROCEDURE

The type of argument used in [2]-[7] can be computerised readily, and employed in conjunction with the fast isomorphism algorithm described by McKay [8]. That algorithm has handled isomorphism of graphs with up to 3000 vertices, and also has the advantage of producing the automorphism group aut (G) of the graph G.

When the graphs were determined by computer, previous results for N ≥ 20 were verified, and a number of new ones obtained, as summarised in the following table. Here N denotes the number of vertices in G, s is the number of orbits in aut (G), g is the order of aut (G).

N	s	g	N	s	g	N	s	g
4	1	24	14	1	336	16	4	6
6	1	72	14	4	4	16	6	4
6	1	12	14	3	8	16	4	6
8	3	12	14	2	14	16	6	4
8	1	16	14	3	8	16	9	2
10	1	120	14	8	2	18	4	8
12	2	18	14	5	4	20	3	20
12	2	16	14	6	4	24	2	32
			16	9	2			

The new results are as follows.

N	s	g
26	3	48
26	1	52
28	16	2
28	15	2
28	16	2
28	15	2
28	5	12
28	10	4
28	15	2
28	1	336
28	3	14

we thus see that f(28) = 9. A subsidiary program was used to produce the largest cycle in the graph. It turns out that the Petersen graph and the Coxeter graph are the only non-Hamiltonian graphs for N ≤ 30. This gives the

THEOREM. *For* N ≤ 30, *all graphs* M(N,3) *are Hamiltonian save the Petersen graph and the Coxeter graph.*

6. THE CASE N = 30

The case N = 30 is particularly interesting since there are 40 distinct generalised Moore graphs with aut (g) ranging from 1440, in the case of the 8-cage, down to the identity. The N, s, g table follows (N is omitted, since N = 30 in all cases).

s	g	s	g	s	g	s	g
1	1440	16	2	16	2	30	1
5	16	16	2	30	1	15	2
4	16	10	4	30	1	8	6
9	4	17	2	15	2	12	3
5	8	30	1	18	2	30	1
4	16	16	2	30	1	17	2
15	2	10	4	30	1	15	2
9	4	16	2	30	1	17	2
18	2	16	2	17	2	17	2
10	4	16	2	5	8	30	1

The case N = 30 is perhaps of sufficient complexity to merit a further table. The algorithm looks at the possible top layers; for each top layer, A generalised Moore graphs are determined, B of these being non-isomorphic; the time employed is given as t. The total for the B column is 355, but each graph may occur in several rows,

and only 40 of the graphs are non-isomorphic. The total time is 13,238 seconds (about 3.7 hours); this time was on a CDC Cyber 73. For running times on the 370/168, divide by 5, roughly.

Possible top layers	A	B	t
1) o o o o o o o o	2	1	1
2) o o o o o o o—o	6	1	3
3) o o o o o o—o—o	79	2	96
4) o o o o o—o o—o	123	5	194
5) o o o o o—o—o—o	59	6	199
6) o o o o—o o—o—o	59	10	413
7) o o o o—o—o—o—o	14	9	228
8) o o o—o o—o o—o	139	17	1207
9) o o o—o—o o—o—o	199	18	991
10) o o o—o o—o—o—o	281	32	1807
11) o o o—o—o—o—o—o	81	21	824
12) o o—o o—o o—o—o	163	28	1199
13) o o—o o—o—o—o—o	70	26	732
14) o o—o—o o—o—o—o	134	31	566
15) o o—o—o—o—o—o—o	52	16	754
16) o o—o—o—o—o—o—o	2	1	89
17) o—o o—o o—o o—o	219	15	561
18) o—o o—o—o o—o—o	83	16	582
19) o—o o—o o—o—o—o	124	20	920
20) o—o o—o—o—o—o—o	107	25	748
21) o—o—o o—o—o—o—o	25	14	263
22) o—o—o—o o—o—o—o	153	26	534
23) o—o—o—o—o—o—o—o	34	11	303
24) o—o—o—o—o—o—o—o	7	4	24
	2215	355 (40)	13238

All Hamiltonian

TABLE FOR N = 30

7. CASES WITH N > 30

For N = 32, we hope to obtain results by rewriting the algorithms to speed up the search. However, there are some interesting results in the neighbourhood of N = 46 (no graph) which differ from the results in the neighbourhood of N = 22. We first prove

<u>LEMMA 1</u>. *In the case of* M(3.2^k-4,3), *we may always assume that there are no joins at level* k - 1, *provided that* $3 < k < 6426$.

<u>Proof</u>. Let there be a joins at level k, c at level k - 1, b between levels k and k - 1. Then

$$2a + b = 3(3.2^{k-1}-2),$$
$$2c + b = 3.2^{k-1}.$$

However, $b \geq 3.2^{k-1} - 2$; hence we have two possibilities, namely,

$$b = 3.2^{k-1} - 2, \quad c = 1, \quad a = 3.2^{k-1} - 2;$$
$$b = 3.2^{k-1}, \quad c = 0, \quad a = 3.2^{k-1} - 3.$$

Suppose, if possible, that we always have the first case, no matter what our choice of root node R. Then we can count the number of (2k-1)-gons in the graph; it is simply

$$\frac{3.2^k - 4}{2k - 1},$$

since we get one such circuit for each choice of root node and each circuit is counted 2k - 1 times. Thus it is clear that 2k - 1 must divide $3.2^{k-2} - 1$; for k = 2,3, this does occur (cf. M(8,3),M(20,3)). This cannot occur for $3 < k < 6426$.

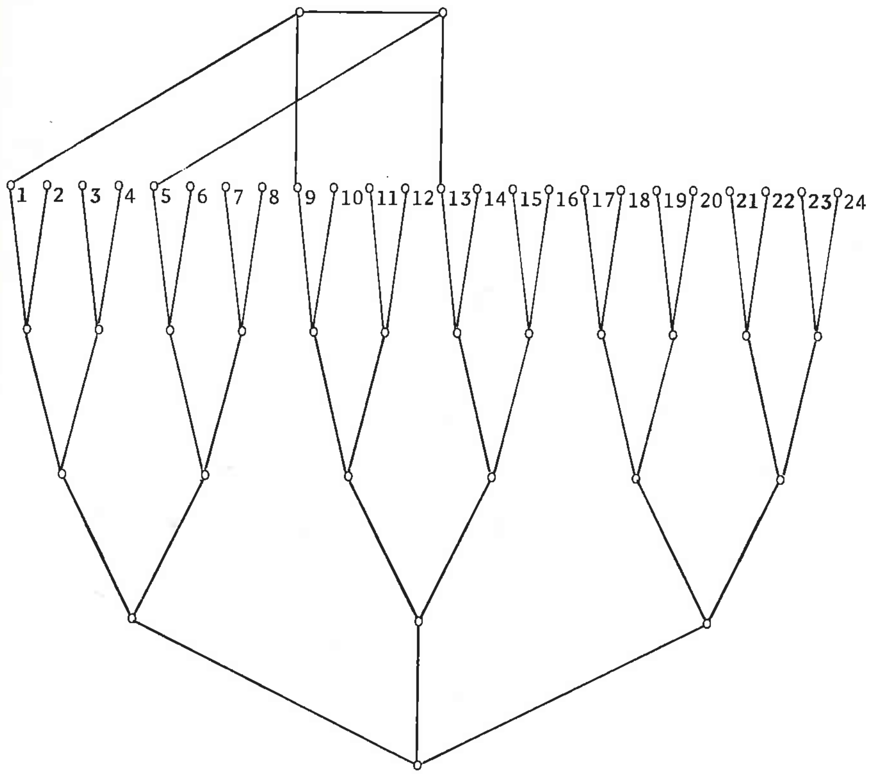
Thus, we may assume that there exists a vertex R such that there are no joins at level k - 1.

We have not completed a discussion of M(44,3), but this lemma shows that we may consider the case of a root node R for which the 22 vertices at level 4 have 21 lines at level 4, 24 lines joining level-4 vertices to level-3 vertices.

We now pass to the casee of 3.2^k vertices, that is the case just above a complete Moore graph. We restrict ourselves to m(48,3), that is k = 4.

First, we deal with the case a = 1, b = 4, c = 22. Then there are 2 distinct ways of joining the vertices at level r.

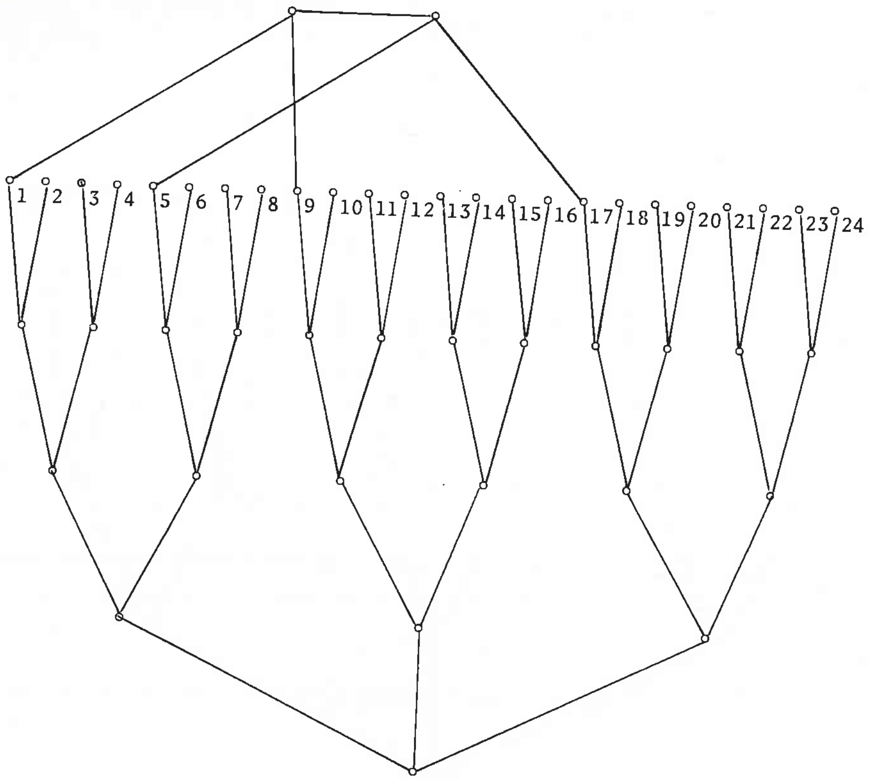
<u>Case A</u>



Vertex 1 must join to {17,...,24}; so join 1-17, be equivalence.

Then 9 must join to {21,...,24}; so join 9 to 21. Vertex 17 can then go to either 15 or 16; so join 17 to 15; similarly, 21 can go to either 7 or 8; so join 21 to 7. Now 15 is forced to join to 8, and 7 cannot be joined to any vertex.

Case B.



In this case, the subtrees containing {1,...,8}, {9,...,16}, {17,...,24}, are referred to as the *flowers* F_1,F_2,F_3, respectively. Any vertex not joined to level 5 must be adjacent to a vertex of each of the flowers not containing it.

As in Case A, we find (using equivalence) a forced sequence of edges. We join 9 to 21. Then, with no loss of generality, 10 is joined to 19 in F_3. We can then join 21 to 7, 17 to 13, 13 to 3, 1 to 15, 5 to 23, 10 to 4. Again, without loss of generality, 18 may be joined to F_2 at 11; then we join 18 to 8, 3 to 22; and 4 can not be joined to any vertex.

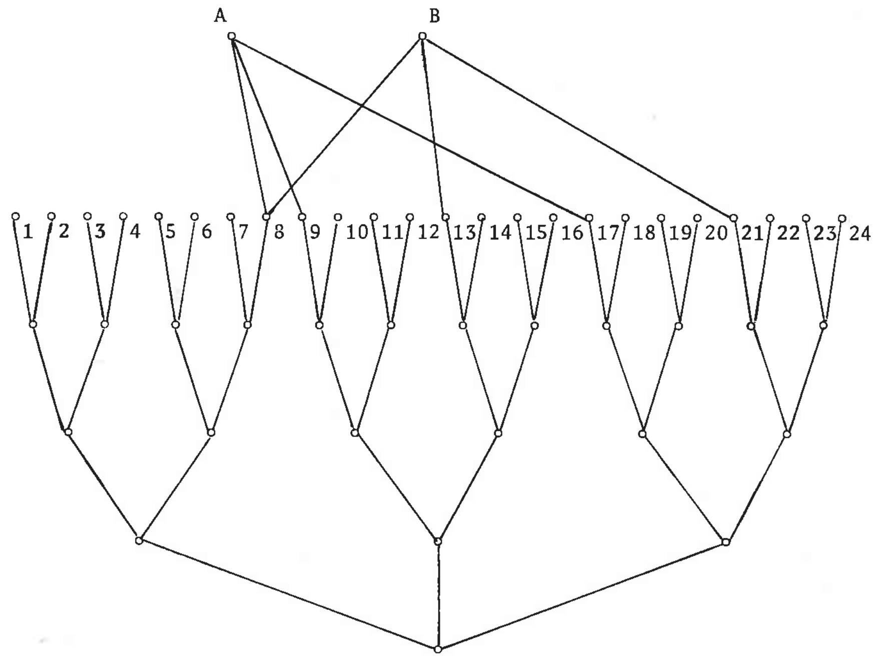
An alternative forced sequence is 3-17; 4-21; 2-19; 1-23; 23-15; 24-11; 15-7; 24-6. Then 6 has no possible join.

Now suppose that, for all R, all the 6 lines from A and B, the two vertices at level 5, go to distinct level-4 vertices. This leaves 18 vertices at level 4 which have two level-4 edges emanating from them. We count decagons.

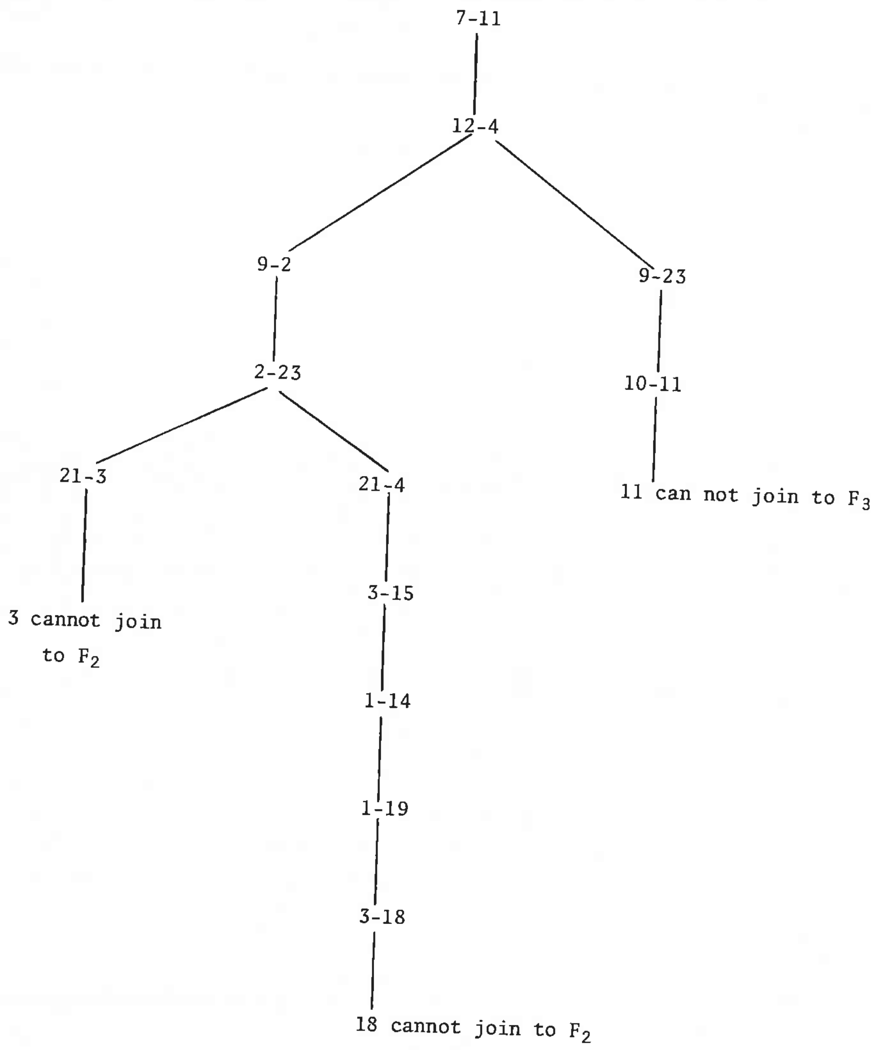
Decagons through R either go to A (three) or B (three) or have two edges at level 4 (eighteen, one for each of the level-4 vertices which has two level-4 lines from it). Thus the total number of decagons is 24 (48)/10, and this is impossible. Thus we have

LEMMA 3. *There is a root* R *for which (exactly) one level-4 vertex is joined to both* A *and* B.

We now discuss this case, illustrated below.



We represent forced joins by a tree (using equivalence).



We thus have the

<u>THEOREM</u>. *A generalised Moore graph* M(48,3) *on* 48 *vertices does not exist.*

The result of this theorem has been verified by running a programme analogous to that used to produce results on 30 vertices; the same programme also established the

<u>THEOREM</u>. *The generalised Moore graph* M(50,3) *does not exist.*

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