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NOTE

HADAMARD EQUIVALENCE VIA GRAPH ISOMORPHISM

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Two $m \times n$ matrices with ± 1 entries are *Hadamard equivalent* if one may be obtained from the other by a sequence of operations involving independent row and column permutations and multiplications of rows or columns by -1. We solve the computational problem of recognising Hadamard equivalence by reducing it to the problem of determining an isomorphism between two graphs with $2(m+n)$ vertices. Existing graph isomorphism algorithms permit the practical determination of Hadamard equivalence when m and n are of the order of several hundred.

Let H_1 and H_2 be two $m \times n$ matrices with ± 1 entries. We say that H_1 and H_2 are *Hadamard equivalent* if H_2 can be obtained from H_1 by applying an element of the group G generated by the following operations, where S_k is the symmetric group on k letters.

p_α: permute the rows by permutation α $\quad (\alpha \in S_m)$,

q_β: permute the columns by permutation β $\quad (\beta \in S_n)$,

r_i: multiply row i by -1 $\quad (1 \leq i \leq m)$,

c_j: multiply column j by -1 $\quad (1 \leq j \leq n)$.

Suppose that $H = (h_{ij})$ is any $m \times n$ matrix with ± 1 entries. Define $X = X(H)$ to be the graph with vertices $v_1, v_2, \ldots, v_m$, $v'_1, v'_2, \ldots, v'_m$, $w_1, w_2, \ldots, w_n$, $w'_1, w'_2, \ldots, w'_n$ and edges

$$\begin{cases} (v_i, w_j), (v'_i, w'_j) & \text{if} \quad h_{ij} = 1, \\ (v_i, w'_j), (v'_i, w_j) & \text{if} \quad h_{ij} = -1. \end{cases}$$

In addition, $X(H)$ has loops on the vertices $v_1, v_2, \ldots, v_m$, $v'_1, v'_2, \ldots, v'_m$.

Theorem. *Let $X_1 = X(H_1)$ and $X_2 = X(H_2)$. Then H_1 and H_2 are Hadamard equivalent if and only if X_1 and X_2 are isomorphic.*

Proof. Let G^* be the group of relabelling operations generated by the following permutations. Vertices are not mentioned in each case are fixed.

P_α: For each i, map v_i onto $v_{i\alpha}$ and v_i' onto $v_{i\alpha}'$ $(\alpha \in S_m)$,

Q_β: For each j, map w_j onto $w_{j\beta}$ and w_j' onto $w_{j\beta}'$ $(\beta \in S_n)$,

R_i: transpose v_i and v_i' $(1 \leqslant i \leqslant m)$,

C_j: transpose w_j and w_j' $(1 \leqslant j \leqslant n)$.

Define ϕ to be the homomorphism from G onto G^* which takes p_α onto P_α, q_β onto Q_β, r_i onto R_i and c_j onto C_j, for each $\alpha \in S_m$, $\beta \in S_n$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. It is easily verified that ϕ is a group isomorphism, and that $X(H_1 g) = X(H_1)(g\phi)$ for each $g \in G$. Therefore, the Hadamard equivalence of H_1 and H_2 implies the isomorphism of X_1 and X_2.

Suppose conversely that there is an isomorphism θ from X_1 to X_2. Let e_1 be any edge of X_1 and let $e_2 = e_1\theta$ be its image in X_2. For $k \in \{1, 2\}$, define Y_k to be the subgraph of X_k induced by those vertices adjacent to either end of e_k. The structure of X_k ensures that Y_k has three important properties.

(i) Exactly one of v_i and v_i' is in Y_k $(1 \leqslant i \leqslant m)$.
(ii) Exactly one of w_j and w_i' is in Y_k $(1 \leqslant j \leqslant n)$.
(iii) Y_k completely determines X_k.

To explain (iii), suppose, for example, that (v_i, w_j) is an edge of Y_k. Then (v_i', w_j') is an edge of X_k but (v_i, w_j') and (v_i', w_j) are not edges of X_k.

Since θ is an isomorphism, it maps Y_1 onto Y_2. By properties (i) and (ii), we can find $g^* \in G^*$ whose restriction to Y_1 is the same as that of θ. But then g^* is an isomorphism from X_1 to X_2, by property (iii). Therefore $H_2 = H_1(g^*\phi^{-1})$. □

The graph isomorphism algorithm described in [1] can successfully handle most graphs in the order of 800–1000 vertices. Consequently we can expect Hadamard equivalence testing to be practically feasible whenever $n + m \leqslant 400$, approximately.

If m and n are not equal, the loops on $X(H)$ may be omitted without affecting the validity of the theorem. If the loops are omitted when $m = n$, X_1 and X_2 are isomorphic if and only if H_1 is Hadamard equivalent to either H_2 or its transpose.

Reference

[1] B.D. McKay, Computing automorphisms and canonical labellings of graphs. Combinatorial Mathematics, Lecture Notes in Mathematics Vol. 686, (Springer-Verlag, Berlin, 1978) 223–232.