SOME COMPUTATIONAL RESULTS ON THE SPECTRA OF GRAPHS

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The polynomial *of a graph is the characteristic polynomial of its 0-1 adjacency matrix. Two graphs are* cospectral *if their polynomials are the same.*

In this paper some of the results from a numerical study of the polynomials of graphs are presented. The study has encompassed 9 point graphs, 9 point bipartite graphs, 14 point trees and 13 point forests. Also given are several theoretical results which were prompted by the numerical data. These include two characterizations of those cospectral graphs which have cospectral complements, and a proof that, in the sense of Schwenk [20] "almost no" trees are characterized by their polynomials together with the polynomials of their complements. In addition, mention is made of those co-spectral graphs which have cospectral linegraphs, and those which are cospectral to their own complements.

1. INTRODUCTION

Graphs referred to in this paper have a finite, non-zero number of vertices and no loops or multiple edges. For such a graph G, $\overline{G}$ refers to the complement of G, and L(G) to the linegraph of G. For brevity, a graph on n vertices will be called an n-graph.

Suppose G is an n-graph. The *adjacency matrix* of G, also denoted G, is the $n \times n$ matrix whose (i,j)th entry is the number of edges from vertex i to vertex j. The *polynomial* of G, denoted $G(\lambda)$, is the characteristic polynomial of the adjacency matrix of G. An *eigenvalue* of G is a root of $G(\lambda)$. The eigenvalues of G, together with their multiplicities, constitute the *spectrum* of G. Two graphs which have the same polynomial, and hence the same spectrum are called *cospectral.*

Other graph theoretic concepts not defined here can be found in Harary [9] or in Behzad and Chartrand [2]. For any square matrix A, the trace of A is denoted tr A. J will always refer to a square matrix with each element one and I to an identity matrix.

The main purpose of this paper is to give the preliminary results of a computational study of the spectra of graphs. Previous studies of this kind have been made by Collatz and Singowitz [4] (5 point graphs and 8 point trees), King [13] (7 point graphs) and Mowshowitz [16] (10 point trees). In this study, the polynomials of 9 point graphs, 9 point bipartite graphs, 14 point trees and 13 point forests have been computed. Theoretical results which have been motivated by the numerical data include several characterisations of those cospectral graphs which have cospectral complements, and a proof that the proportion of trees of a given size which are characterized by their spectra plus the spectra of their complements goes to zero as the size increases - strengthening a result of Schwenk [20]. Also considered are cospectral graphs which have cospectral linegraphs, and graphs which are cospectral to their own complements.

2. COSPECTRAL GRAPHS

It has been known for some time [4] that the polynomial of a graph does not always determine the graph uniquely. The smallest example of non-isomorphic cospectral graphs is the pair of 5-graphs shown in Figure 1. The smallest such connected pair is shown in Figure 2.

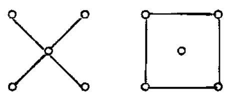


FIGURE 1

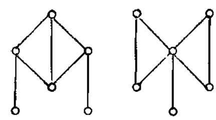


FIGURE 2

Despite some success in constructing large families of cospectral graphs [8], the problem of finding them all seems almost hopelessly difficult. Accordingly several computer searches for cospectral graphs have been made. The polynomials of the 7-graphs were first computed by King and Read [11], [13], [17]. In this work the graphs with 8 or 9 vertices have been included. The source for these graphs was the tape distributed by Baker, Dewdney and Szilard [1]. The polynomials of the 274,668 nine-graphs were computed in about 50 minutes, using Danilevsky's method [3] programmed in assembly-language on a CDC Cyber 73.

In Table 1 (Appendix), the following data are given:

- e : number of edges,
- g_e : number of graphs with e edges,
- ϕ_i : number of families of exactly i cospectral graphs with e edges, $i = 2,3,\ldots,$
- c_e : total number of graphs with e edges not determined by their spectra.

Values of e where $c_e = 0$ have been omitted. Missing values of g_e can be found in [9] or [10].

It is seen that about 18.6% of all the 9-graphs are not determined by their spectra and that this percentage appears to be increasing with the number of vertices. Unfortunately, the number of cospectral families is too large for them to be all listed here. However those with 7 or fewer vertices are given in Table 2. Graphs in the same row of the Table are cospectral.

In Table 3 the extraordinary set of ten cospectral 9-graphs with 16 edges is listed. The equally large set with 20 edges is the set of complements of these graphs.

3. COSPECTRAL GRAPHS WITH COSPECTRAL COMPLEMENTS

In this section we consider the question of those cospectral graphs whose complements are also cospectral. The smallest of these are on 7 vertices, and are indicated by asterisks in Table 2. In Table 6 the statistical distribution of such families is presented in the same format as in Table 1. Note that complementarity is a symmetric relationship, and so the lower half of each table has been omitted. It is perhaps surprising that about 86% of those 9-graphs not determined by their polynomials are still not determined when the polynomials of their complements are also considered. The following loose relationship probably provides only a partial explanation for this phenomenon.

Theorem 3.1 [5]. *If the spectrum of a graph* G *contains an eigenvalue* λ *with multiplicity* p $(p > 1)$ *then the spectrum of* $\overline{G}$ *contains an eigenvalue* $-\lambda-1$ *with multiplicity* $\overline{p}$ *satisfying* $p-1 \leq \overline{p} \leq p+1$.

We now proceed to characterize those cospectral graphs whose complements are also cospectral.

Consider each edge of a graph G as two oppositely oriented directed edges. A *walk of length* k $(k \geq 1)$ on G is a sequence of directed edges $e_1, e_2, \ldots, e_k$ of G such that consecutive edges in the sequence are adjacent. If c_k is the number of such walks, for each k, then the formal power series

$$W_G(x) = c_1x + c_2x^2 + c_3x^3 + \ldots$$

is called the *walk generating function* for G. Note that we do not recognize walks of zero length.

Let A and B be arbitrary $n \times n$ matrices, and let G be an n-graph. The following lemmas are stated without proof:

Lemma 3.2. A *and* B *are cospectral iff*

$$\operatorname{tr} A^r = \operatorname{tr} B^r \qquad r = 1,2,\ldots,n-1.$$

Note that by the Cayley-Hamilton theorem [14], the upper bound on r can be removed.

Lemma 3.3. $\operatorname{tr} AB = \operatorname{tr} BA$.

Lemma 3.4. *(a)* $J^k = n^{k-1}J \qquad k = 1,2,\ldots$

where J *is the* $n \times n$ *matrix with every element one.*

(b) $\operatorname{tr} JAJB = \operatorname{tr} JA \operatorname{tr} JB$.

Lemma 3.5. $W_G(x) = c_1x + c_2x^2 + \dots$ *where*

$$c_k = \sum_{i,j=1}^{n} (G^k)_{ij} = \mathrm{tr}\ JG^k = z^TG^kz, \qquad k = 1,2,\dots$$

where z *is an* n*-vector of all ones, and* $(G^k)_{ij}$ *is the* (i,j)*th entry of* G^k.

Theorem 3.6. *Let* G *and* H *be cospectral* n*-graphs. Then* $\overline{G}$ *and* $\overline{H}$ *are cospectral iff* $W_G(x) = W_H(x)$.

Proof. Clearly $\overline{G}(\lambda) = (J-G)(\lambda+1)$, so that by 3.2, $\overline{G}$ and $\overline{H}$ are cospectral iff $\mathrm{tr}\ (J-G)^k = \mathrm{tr}\ (J-H)^k$, $k = 1,2,\dots,n-1$. Expanding $(J-G)^k$ and using Lemmas 3.3 and 3.4 we have

3.7 $$\mathrm{tr}\ (J-G)^k = (-1)^k[\mathrm{tr}\ G^k - k\ \mathrm{tr}\ JG^{k-1}] + \phi_k(G)$$

where $\phi_k(G)$ is a polynomial in $\{\mathrm{tr}\ JG^r\}_{r=1}^{k-2}$, with a corresponding expression for $\mathrm{tr}\ (J-H)^k$.

Since $\mathrm{tr}\ G^k = \mathrm{tr}\ H^k$, the condition $\mathrm{tr}\ JG^r = \mathrm{tr}\ JH^r$, $r = 1,2,\dots,k-1$ gives us $\mathrm{tr}(J-G)^k = \mathrm{tr}\ (J-H)^k$, so that $\overline{G}$ and $\overline{H}$ are cospectral.

Conversely, the equation 3.7 can be solved uniquely for $\mathrm{tr}\ JG^{k-1}$, and by easy induction on k we find that the condition $\mathrm{tr}\ (J-G)^r = \mathrm{tr}\ (J-H)^r$ $r = 1,2,\dots$ gives us $\mathrm{tr}\ JG^{k-1} = \mathrm{tr}\ JH^{k-1}$ $k = 2,3,\dots$ //

Corollary 3.8 [6]. *If* G *and* H *are regular cospectral graphs, then* $\overline{G}$ *and* $\overline{H}$ *are cospectral.*

Proof. If G and H are regular of degree k, then from [5] we have

$$W_G(x) = W_H(x) = \frac{nkx}{1-kx} \quad .//$$

Let G_1 and G_2 be two graphs. The *join* $G_1 : G_2$ of G_1 and G_2 is formed by taking the disjoint union of G_1 and G_2 and joining each vertex of G_1 to each vertex of G_2.

Theorem 3.9. *Let* G *and* H *be cospectral* n*-graphs. Let* L *be any graph. Then* $\overline{G}$ *and* $\overline{H}$ *are cospectral iff* $G : L$ *and* $H : L$ *are cospectral.*

Proof. Our original proof of this theorem was made obsolete by the following result of Cvetković [5], from which the result is immediate.

$$(G : L)(\lambda) = (-1)^{\ell}G(\lambda)\overline{L}(-\lambda-1) + (-1)^{n}L(\lambda)\overline{G}(-\lambda-1) - (-1)^{n+\ell}\overline{G}(-\lambda-1)\overline{L}(-\lambda-1)$$

and similarly for $H : L$, where ℓ is the number of vertices of L.//

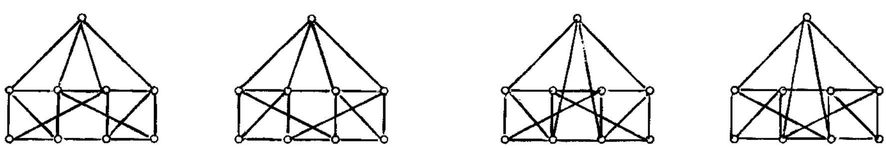
4. GRAPHS COSPECTRAL TO THEIR OWN COMPLEMENTS

A graph is called *self-complementary* if it is isomorphic to its own complement. Such graphs are obviously cospectral to their own complements but it has been discovered, probably for the first time, that there are non-self-complementary graphs cospectral to their own complements. The smallest such graphs are shown in Figure 3. Of course, the complements of these graphs have the same property.

FIGURE 3

A list of all those graphs on 8 or 9 vertices which are cospectral to their own complements is given in Table 5. Those that are actually self-complementary are indicated by an asterisk. Otherwise, the complement of the graph must be added to the Table.

As indicated in the table by brackets, some cospectral families occur amongst graphs in this class. A particularly interesting family is that drawn in Figure 4.



(a) (b) (c) (d)

FIGURE 4

Graphs (a) and (b) are complements of each other, whereas graphs (c) and (d) are both self-complementary. All of these graphs have the polynomial $\lambda^9-18\lambda^7-14\lambda^6+67\lambda^5+60\lambda^4-62\lambda^3-46\lambda^2+12\lambda+8$. Graphs (a) and (b) have the additional property that their linegraphs are cospectral and have cospectral complements.

5. COSPECTRAL GRAPHS WITH COSPECTRAL LINEGRAPHS

In a similar fashion, one can ask for those cospectral graphs which have cospectral linegraphs. Such graphs are surprisingly uncommon, at least on a small number of vertices. The only such families on 9 or fewer vertices are the four pairs of 9-graphs listed in Table 4. Note that the fourth pair is that drawn in Figure 4(a) and (b). Except for the third pair, these pairs also have cospectral complements. However, their complements do not have cospectral linegraphs except for the fourth pair.

No simple characterization of such graphs seems to be known. However small amounts of information may be deduced, as for example in the following theorem.

Theorem 5.1. *Let* G *and* H *be cospectral* n-*graphs with cospectral linegraphs. Then if the degrees of the vertices of* G *and* H *are* $\{g_i\}_1^n$ *and* $\{h_i\}_1^n$ *respectively,*

$$\sum_{i=1}^{n} g_i^k = \sum_{i=1}^{n} h_i^k, \qquad k = 1,2 \text{ and } 3.$$

Proof. Let B be the incidence matrix of G, [9],[2]. Then $B^TB = L(G)+2I$, $BB^T = G+\Gamma_G$ where I is the identity matrix and $\Gamma_G = \text{diag}(g_1,g_2,\ldots,g_n)$. Since B^TB and BB^T have the same non-zero eigenvalues [14], the polynomial of L(G) is determined by the polynomial of $G+\Gamma_G$. Similarly for H.

From 3.2 the conditions of the theorem now become

$$\text{tr } G^k = \text{tr } H^k; \quad \text{tr } (G+\Gamma_G)^k = \text{tr } (H+\Gamma_H)^k \quad k = 1,2,\ldots .$$

The results now follow by expanding the second equation for the cases k = 1,2 and 3.//

Unfortunately, these conditions do not ensure that G and H have the same degree sequences, the third pair of graphs in Table 4 providing a counter-example.

6. COSPECTRAL BIPARTITE GRAPHS

A graph G is said to be *bipartite* (also *bicolorable*) if the vertices of G can be divided into two classes in such a way that no edge of G joins two members of the same class. Such graphs are characterized by their spectrum, as shown in the following theorem ([7], quoted incorrectly in [21]).

Theorem 6.1. *Let* G *be a graph. Then*

(a) G *is bipartite iff to every eigenvalue* λ_i *of* G *there corresponds an eigenvalue* $-\lambda_i$ *of the same multiplicity. (Thus the polynomial of* G *has all odd or all even powers of* λ.)

(b) If G *is connected and* r *is the largest positive eigenvalue of* G, *then* G *is bipartite iff* -r *is an eigenvalue of* G.//

The distribution of cospectral families amongst the bipartite graphs is given in Table 10, where b_e is the number of bipartite graphs with e edges. Since the actual values of b_e seem to have only been published as far as 6 vertices [12], all values of e where $b_e \neq 0$ are included.

7. COSPECTRAL TREES

In the case of trees, the work of the preceding sections has been extended through 14 vertices. For this purpose, the trees themselves were machine generated, using a method similar to that described by Read [18].

The spectrum of a tree bears the following simple relationship to the structure of the tree. A k-*matching* of a tree T is a set of k mutually non-incident edges of T.

Theorem 7.1. *Let* T *be a tree on* n *vertices. Then* T *has the polynomial*

$$T(\lambda) = \sum_{k=0}^{p} m_k \lambda^{n-2k}, \quad \textit{where } m_k \quad (k > 0)$$

is the number of k*-matchings of* T*,* $m_0 = 1$*, and* p *is the integral part of* $\frac{n}{2}$.

Proof. First given by Sachs [19], and later independently by Mowshowitz [16].//

The smallest pair of cospectral trees, and the smallest foursome are shown in Figure 5.

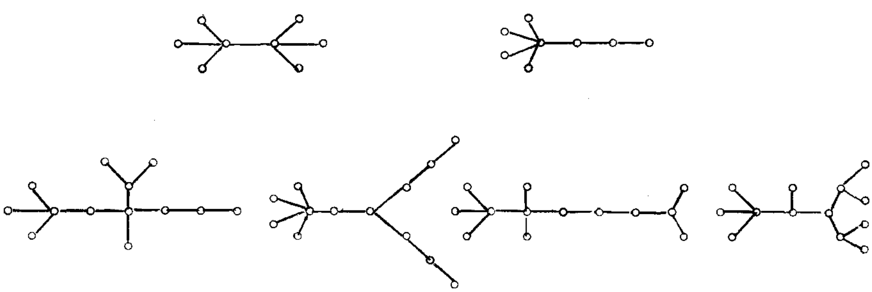


FIGURE 5

The distribution of cospectral families of trees from 8 through 14 vertices is given in Table 7, where t_n is the number of trees on n vertices, and the other symbols are as in Table 1. It is seen that the proportion of trees identified by their spectrum appears to be dropping rapidly, and may even fall below 50% between 20 and 25 vertices.

The asymptotic value of this proportion has been discovered by Schwenk [20]:

Theorem 7.2. *Let* p_n *be the proportion of trees on* n *vertices which are identified (within the class of trees) by their spectrum.*

Then $p_n \to 0$ *as* $n \to \infty$.//

8. TREES WITH A 1-FACTOR

A *1-factor* of a graph G is a regular spanning subgraph of G of degree one. Clearly a 1-factor of a tree on 2n vertices corresponds to an n-matching. From 7.1 those trees with a 1-factor are just those which have no zero eigenvalues.

The smallest examples of cospectral trees in this class are the two pairs shown in Figure 6.

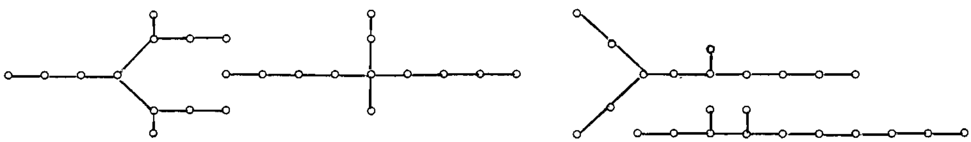


FIGURE 6

The frequency of occurrence of such families is given in Table 8. As no enumeration of trees with a 1-factor is known to the authors, all sizes from 2 through 14 vertices are included.

9. COSPECTRAL TREES WITH COSPECTRAL COMPLEMENTS

Let G and H be n vertex and m vertex rooted graphs, respectively. Define G_1 and H_1 to be the (unrooted) induced subgraphs of G and H, respectively, formed by removing the root. The *merge* $G \cdot H$ of G and H is defined as the n+m-1 vertex graph formed from the disjoint union of G and H by identifying the two roots.

Lemma 9.1.

$$(G \cdot H)(\lambda) = G(\lambda)H_1(\lambda) + G_1(\lambda)H(\lambda) - \lambda G_1(\lambda)H_1(\lambda),$$

where $G(\lambda)$ *and* $H(\lambda)$ *are the polynomials of* G *and* H *considered as unrooted.*

Proof. The determinant $|\lambda I - G \cdot H|$ can be written as

$$\left|\begin{array}{c|c|c} \lambda & a^T & b^T \\ \hline a & \lambda I - G_1 & 0 \\ \hline b & 0 & \lambda I - H_1 \end{array}\right| \qquad \text{where a and b are vectors.}$$

Since the determinant of a matrix is a linear function of its first row we have

$$|\lambda I-G\cdot H| = \begin{vmatrix} \lambda & a^T & 0 \\ a & \lambda I-G_1 & 0 \\ b & 0 & \lambda I-H_1 \end{vmatrix} + \begin{vmatrix} \lambda & 0 & b^T \\ a & \lambda I-G_1 & 0 \\ b & 0 & \lambda I-H_1 \end{vmatrix} - \begin{vmatrix} \lambda & 0 & 0 \\ a & \lambda I-G_1 & 0 \\ b & 0 & \lambda I-H_1 \end{vmatrix}$$

$$= G(\lambda)H_1(\lambda) + G_1(\lambda)H(\lambda) - \lambda G_1(\lambda)H_1(\lambda) \quad //.$$

This lemma has been previously given by Schwenk [20] for trees, in which case it is easily proved from Theorem 7.1.

Lemma 9.2. *Define generating functions (see §3) as follows:*

W_G : *all walks on* G,

W_G': *all walks on* G *starting at the root,*

W_G'': *all walks on* G *starting and finishing at the root,*

and similarly for H *and* G•H. *The coefficient of unity in all these functions is zero.*

Then

$$W_{G\cdot H} = W_G + W_H + (W_G'^2 W_H'' + 2W_G'W_H' + W_G''W_H'^2) \sum_{r=0}^{\infty} (W_G''W_H'')^r$$
$$= W_G + W_H + \frac{(W_G'^2 W_H'' + 2W_G'W_H' + W_G''W_H'^2)}{1 - W_G''W_H''}$$

Proof. For walks on G•H of the form $e_1, e_2, \ldots, e_k$ where e_1 and e_k are edges of G we have the function

$$W_G + W_G'W_H''W_G' + W_G'W_H''W_G''W_H''W_G' + \ldots \quad ,$$

and similarly when e_1 and e_k are edges of H.

If e_1 is an edge of G and e_k is an edge of H, or vice-versa, we find (in each case) the function

$$W_G'W_H' + W_G'W_H''W_G''W_H' + W_G'W_H''W_G''W_H''W_G''W_H' + \ldots \quad .$$

Adding the four functions thus obtained gives the required result//.

A rooted tree L is called a *limb* of a tree T if for some rooted tree R we have T = R•L.

Theorem 9.3. *Let* G *be an arbitrary rooted graph. Let* S *and* T *be the rooted trees shown in* Figure 7.

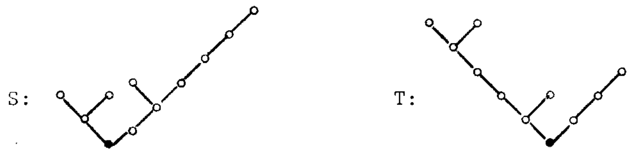


FIGURE 7

Then G•S *and* G•T *are not isomorphic (if* G *does not have an isolated root) but are cospectral and have cospectral complements.*

Proof. (a) Since G•S has P_5 rooted at an end-vertex once more as a limb than has G•T, it is not isomorphic to G•T.

(b) T and S are isomorphic as unrooted trees, and T_1 and S_1 are isomorphic. So G•S and G•T are cospectral by Lemma 9.1.

(c) Direct computation from the adjacency matrices of T and S shows that

$$W_T'(x) = W_S'(x) \quad \text{and} \quad W_T''(x) = W_S''(x).$$

Hence $W_{G\cdot S} = W_{G\cdot T}$ by Lemma 9.2.
Hence G•S and G•T have cospectral complements, by 3.6//.

The following result was first proved by Schwenk [20], who enumerated trees not containing a given limb. This enumeration has since been carried out more directly by McAvaney [15].

Lemma 9.4. *Let* L_1 *and* L_2 *be rooted trees on the same number of vertices. Let* $p_1(n)$ *and* $p_2(n)$ *be respectively the number of (unrooted) trees on* n *vertices not containing* L_1 *or* L_2 *as a limb. Then* $p_1(n) = p_2(n)$ *for all* n.

Furthermore, if t_n *is the number of trees on* n *vertices,*

$$\frac{p_1(n)}{t_n} \to 0 \quad as \quad n \to \infty. \ //$$

Theorem 9.5. *Let* t_n *be the number of trees on* n *vertices. Let* s_n *be the number of trees on* n *vertices which are identified by their spectra together with the spectra of their complements.*

Then $\frac{s_n}{t_n} \to 0$ *as* $n \to \infty$.

Proof. By 9.3, any tree which can be written as R•S for some rooted tree R is cospectral with cospectral complements to the (different) tree R•T, where S and T are as in 9.3. By 9.4, the proportion of all trees which cannot be written in this form goes to zero as the size increases.//

Note 9.6. Theorem 9.3 can be generalized to show that for any rooted graphs G and H the two graphs shown schematically in Figure 8 are cospectral and have cospectral complements. In fact every pair of trees on 14 or fewer vertices which have this property fall into the class of graphs illustrated.

FIGURE 8

10. COSPECTRAL TREES WITH COSPECTRAL LINEGRAPHS

A search of the cospectral trees on up to 14 vertices has produced a small number of pairs which also have cospectral linegraphs. Specifically, there is 1 pair on 11, 1 pair on 12, 1 pair on 13 and 5 pairs on 14 vertices. Those on 11 vertices are shown in Figure 9.

FIGURE 9

11. COSPECTRAL FORESTS

The smallest pair of cospectral forests are those shown in Figure 10.

FIGURE 10

The distribution of such families for forests on up to 13 vertices is given in Table 9. It is seen that the proportion of forests determined by their spectra drops more rapidly with increasing size than it does for trees. Extrapolation by hand suggests that the figure may drop below 50% by 16 vertices.

APPENDIX

In order to reduce space requirements, a compact notation for graphs will be used. This is best illustrated by an example. Take the eight-vertex graph shown in Figure 11, together with its adjacency matrix.

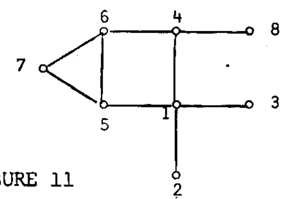


FIGURE 11

0	1	1	1	1	0	0	0
1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
1	0	0	0	0	1	0	1
1	0	0	0	0	1	1	0
0	0	0	1	1	0	1	0
0	0	0	0	1	1	0	0
0	0	0	1	0	0	0	0

The lower triangle of the adjacency matrix, excluding the diagonal, is written down row by row as a binary integer:

1 1 0 1 0 0 1 0 0 0 0 0 0 1 1 0 0 0 0 1 1 0 0 0 1 0 0 0

Zeros are added to the right-hand end if necessary to make the number of digits up to a multiple of three, and then the whole is written as an octal integer: 6 4 4 0 3 0 3 0 4 0. This notation is essentially the same as that employed by Baker, Dewdney and Szilard [1].

The following tables have been computer generated and checked carefully. Where overlap occurs with other existing tables [11], [13], [16], no discrepancies have been found.

5 VERTICES:

e	g_e	ϕ_2	c_e	c_e/g_e
4	6	1	2	.333

TOTALS:

	g_e	ϕ_2	c_e	c_e/g_e
	34	1	2	.059

6 VERTICES:

e	g_e	ϕ_2	c_e	c_e/g_e
4	9	2	2	.444
5	15	1	2	.133
6	21	1	2	.095
7	24	1	2	.083

TOTALS:

	g_e	ϕ_2	c_e	c_e/g_e
	156	5	10	.064

7 VERTICES:

e	g_e	ϕ_2	ϕ_3	c_e	c_e/g_e
4	10	2		4	.400
5	21	0	1	3	.143
6	41	7		14	.341
7	65	4		8	.123
8	97	9		18	.186
9	131	6		12	.092
10	148	8		16	.108
11	148	5	1	13	.088
12	131	6		12	.092
13	97	2		4	.041
14	65	1		2	.031
15	41	1		2	.049
16	21	1		2	.095

TOTALS:

	g_e	ϕ_2	ϕ_3	c_e	c_e/g_e
	1044	52	2	110	.105

8 VERTICES:

e	g_e	ϕ_2	ϕ_3	ϕ_4	c_e	c_e/g_e
4	11	2			4	.364
5	24	1	1		5	.208
6	56	7	1	1	21	.375
7	115	13			26	.226
8	221	25	3	1	63	.285
9	402	28	4		68	.169
10	663	67	10		164	.247
11	980	66	4	1	148	.151
12	1312	97	7	1	219	.167
13	1557	105	3	1	223	.143
14	1646	99	7		219	.133
15	1557	102	2		210	.135
16	1312	68	5		151	.115
17	980	42	1	1	91	.093
18	663	29	2		64	.097
19	402	15			30	.075
20	221	2	2		10	.045
21	115	2			4	.035
22	56	1			2	.036

TOTALS:

	g_e	ϕ_2	ϕ_3	ϕ_4	c_e	c_e/g_e
	12346	771	52	6	1722	.139

TABLE 1. COSPECTRAL GRAPHS

9 VERTICES:

e	g_e	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	c_e	c_e/g_e
4	11	2									4	.364
5	25	1	1								5	.200
6	63	8	1	0	1						24	.381
7	148	19	3	1							51	.345
8	345	50	5	4	1						136	.394
9	771	75	11	4	1						204	.265
10	1637	169	41	10	1	1					512	.313
11	3252	305	35	5	1						740	.228
12	5995	494	109	19	2	3					1419	.237
13	10120	840	89	22	6						2065	.204
14	15615	1273	167	37	10	5	1				3282	.210
15	21933	1833	149	49	2	2					4331	.197
16	27987	2173	261	57	18	8	0	1	0	1	5513	.197
17	32403	2742	179	75	1	2					6338	.196
18	34040	2615	257	72	17	5					6404	.188
19	32403	2626	154	66	0	2					5990	.185
20	27987	1979	205	54	20	4	0	1	0	1	4931	.176
21	21933	1600	110	36	3	1					3695	.168
22	15615	983	118	20	8	3					2458	.157
23	10120	652	41	17	1						1500	.148
24	5995	296	58	3	1	1					789	.132
25	3252	186	6	0	1						395	.121
26	1637	63	12								162	.099
27	771	30	1								63	.082
28	345	6	2								18	.052
29	148	4									8	.054
30	63	1									2	.032
TOTALS:												
	274668	21025	2015	551	95	37	1	2	0	2	51039	.186

TABLE 1. COSPECTRAL GRAPHS (cont.)

6440	3600		7654001	7741100	
			7651400	7741010 *	
64400	36000		7652001	7651004	
24410	34200		7445210	7641003	
64210	36001		7610302	7640403 *	
74440	76001		7445011	3651202 *	
74460	76102		7760402	7762001	
			7742140	7760101	
6440000	3600000		7663040	6771400	
2441000	3420000		7656020	7662440	
6421000	2442020	3600100	7655001	7662041	
3660000	6442040		7660405	7650301 *	
6442020	3642000		7445602	7651005 *	
3640100	6441020		6362444	6651430 *	
6441010	3640001		7762500	7772040	
3441020	3600102		7762140	7762402	
6420404	3420300 *		7655440	7762021	
7444000	7600100		7655401 *	7656041	7760405 *
3660001	3642040		7650341	6655430 *	
7446000	7610200		6636430	7640742 *	
7424400	7441020		7762444	7762430 *	
7420300	7420404 *		7772021	7762504	
7642040	7616000		7672604	7763042	
7642020	7641100		7652243	7760603	
7614001	7446040		7465614	7662407 *	
7541020	7642002		3673214	7652223 *	
6651400	7640404 *		7762544	7762530 *	
7446020	7612002		7762407	7655603 *	
3651200	6660410		7762546	7672634 *	
7445020	7640101		7655770	7762547 *	
7440203	4651410 *		7777603	7777470	

TABLE 2. COSPECTRAL GRAPHS (5,6,7 VERTICES)

* Indicates complements also cospectral

761644106510
746262406450
363646144011
756222120451
743704411430
743234426042
761614130430
365166244011
365233124012
761613006604

TABLE 3. 10 COSP. 9-GRAPHS

714624020500
724622010600
765466121011
765466023006
744725236006
770315226070
753222342524
753212545112

TABLE 4. COSP. GRAPHS WITH COSP. LINEGRAPHS

7773240200*	677546161040	765546144015
7773060010	771746442110*	376252650121*
7763014140*	767642550120*	770315226070*
7762424140*	776252650120*	367645025122*
3773240204*	[776315244044	765164660015
7651350240*	[775613160120	[765131344540
7651344300*	[767642544140*	[376252644141*
7651407440	776252644140*	367644430524*
3663114144*	775714120502	765062362026
6351246310*	777320340502*	753212544514*
3652426250*	377326340101*	377320340503*
3652231260*	766263344110*	353726241103*
	[756711544140	[753212545112
777746440100*	[766263342120*	[754152643122*
777326340100*	765163443210	[753222344514*
777346460001	577325211201	765130350522*
777706014060*	765234631140	353726240505*
777705024060*	376252750120	754542343122*
776657041100	766263242121	765130344542*
757562560200	367706114061*	227645025526*
773364242030	770316216070*	354542343123*
777135140201	637642445142*	376212211427*
377746440101*	766263412031*	365223126017*

TABLE 5. GRAPHS COSPECTRAL TO THEIR OWN COMPLEMENTS

*Indicates self-complementary [Indicates cospectrality

7 VERTICES:

e	g_e	ϕ_2	c_e	c_e/g_e
6	41	1	2	.049
7	65	1	2	.031
8	97	2	4	.041
9	131	3	6	.046
10	148	3	6	.041
TOTALS:				
	1044	20	40	.038

8 VERTICES:

e	g_e	ϕ_2	ϕ_3	ϕ_4	c_e	c_e/g_e
6	56	1			2	.036
7	115	2			4	.035
8	221	2	2		10	.045
9	402	15			30	.075
10	663	26	1		55	.083
11	980	40	1	1	87	.089
12	1312	59	5		133	.101
13	1557	85	1		173	.111
14	1646	86	2		178	.108
TOTALS:						
	12346	546	22	2	1166	.094

9 VERTICES:

e	g_e	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	c_e	c_e/g_e
6	63	1									2	.032
7	148	3									6	.041
8	345	6	2								18	.052
9	771	29	1								61	.079
10	1637	61	12								158	.097
11	3252	180	6	0	1						383	.118
12	5995	284	58	3	2						764	.127
13	10120	646	36	16	1						1469	.145
14	15615	935	113	20	7	3					2342	.150
15	21933	1550	99	36	2	1					3557	.162
16	27987	1861	203	44	15	4	0	1	0	1	4624	.165
17	32403	2482	133	61	0	2					5619	.173
18	34040	2401	224	63	11	4					5805	.171
TOTALS:												
	274668	18477	1550	423	67	24	0	2	0	2	43811	.160

TABLE 6. COSPECTRAL GRAPHS WITH COSPECTRAL COMPLEMENTS

n	t_n	ϕ_2	ϕ_3	ϕ_4	c_n	c_n/t_n
8	23	1			2	.087
9	47	5			10	.213
10	106	4			8	.075
11	235	27	2		60	.255
12	551	49	7		119	.216
13	1301	162	29	1	415	.319
14	3159	349	36	5	826	.261

TABLE 7. COSPECTRAL TREES

n	t_n	ϕ_2	c_n	c_n/t_n
2	1	0	0	..
4	1	0	0	..
6	2	0	0	..
8	5	0	0	..
10	15	0	0	..
12	49	2	4	.082
14	180	12	24	.133

TABLE 8. TREES WITH 1-FACTORS

n	f_n	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	c_n	c_n/f_n
5	10	0					0	••
6	20	1					2	.100
7	37	2					4	.108
8	76	7					14	.184
9	153	16	2				38	.248
10	329	35	5	1			89	.271
11	709	94	10	4			234	.330
12	1598	209	41	7	2		579	.362
13	3650	505	107	23	5	3	1466	.402

TABLE 9. COSPECTRAL FORESTS

5 VERTICES:

e	b_e	ϕ_2	c_e	c_e/b_e
1	1	0	0	••
2	2	0	0	••
3	3	0	0	••
4	4	1	2	.500
5	1	0	0	••
6	1	0	0	••
TOTALS:				
	12	1	2	.167

6 VERTICES:

e	b_e	ϕ_2	c_e	c_e/b_e
1	1	0	0	••
2	2	0	0	••
3	4	0	0	••
4	7	2	4	.571
5	8	1	2	.250
6	6	0	0	••
7	3	0	0	••
8	2	0	0	••
9	1	0	0	••
TOTALS:				
	34	3	6	.176

7 VERTICES:

e	b_e	ϕ_2	ϕ_3	c_e	c_e/b_e
1	1	0		0	••
2	2	0		0	••
3	4	0		0	••
4	8	2		4	.500
5	13	0	1	3	.231
6	19	6		12	.632
7	14	1		2	.143
8	13	0		0	••
9	7	0		0	••
10	4	0		0	••
11	1	0		0	••
12	1	0		0	••
TOTALS:					
	87	9	1	21	.241

8 VERTICES:

e	b_e	ϕ_2	ϕ_3	ϕ_4	c_e	c_e/b_e
1	1	0			0	••
2	2	0			0	••
3	4	0			0	••
4	9	2			4	.444
5	16	1	1		5	.313
6	32	6	1	1	19	.594
7	45	7			14	.311
8	52	5	2		16	.308
9	48	1	1		5	.104
10	40	5			10	.250
11	24	0			0	••
12	16	1			2	.125
13	7	0			0	••
14	3	0			0	••
15	2	0			0	••
16	1	0			0	••
TOTALS:						
	302	28	5	1	75	.248

TABLE 10 - COSPECTRAL BIPARTITE GRAPHS

9 VERTICES:

e	b_e	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	c_e	c_e/b_e
1	1	0					0	••
2	2	0					0	••
3	4	0					0	••
4	9	2					4	.444
5	17	1	1				5	.294
6	38	7	1	0	1		22	.579
7	70	12	3	1			37	.529
8	120	23	4	3	1		75	.625
9	150	23	2	2	1		65	.433
10	179	22	6	3	0	1	80	.447
11	164	23	2				52	.317
12	143	17	4				46	.322
13	94	7					14	.149
14	63	3	1				9	.143
15	32	1					2	.063
16	19	0					0	••
17	7	0					0	••
18	4	0					0	••
19	1	0					0	••
20	1	0					0	••
TOTALS:								
	1118	141	24	9	3	1	411	.368

TABLE 10. COSPECTRAL BIPARTITE GRAPHS (cont.)

Added in proof:

Recent computational work has extended the results of Table 7 to include trees on up to 18 vertices. The extended version of the table is presented below.

n	t_n	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	c_n	c_n/t_n
8	23	1							2	.087
9	47	5							10	.213
10	106	4							8	.075
11	235	27	2						60	.255
12	551	49	7						119	.216
13	1301	162	29	1					415	.319
14	3159	349	36	5					826	.261
15	7741	960	145	20	7				2470	.319
16	19320	2028	326	44	6	1			5246	.272
17	48629	5343	985	221	58	18	3		14944	.307
18	123867	12163	1935	405	88	16	4	4	32347	.261

EXTENDED TABLE 7. COSPECTRAL TREES

It is interesting to note that the proportion c_n/t_n tends to be larger for odd n than for even n, and that the last few values appear to be dropping, despite the fact that c_n/t_n tends to 1 for large n (by 7.2). We have no explanation for these phenomena.

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