Correlations in two-dimensional vortex liquids

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We report on a high-temperature perturbation expansion study of the superfluid-density spatial correlation function of a Ginzburg-Landau-model superconducting film in a magnetic field. We have derived a closed form which expresses the contribution to the correlation function from each graph of the perturbation theory in terms of the number of Euler paths around appropriate subgraphs. We have enumerated all graphs appearing out to 12th order in the expansion and have evaluated their contributions to the correlation function. Low-temperature correlation functions, obtained using Padé approximants, are in good agreement with Monte Carlo simulation results and show that the vortex liquid becomes strongly correlated at temperatures well above the vortex solidification temperature. We have also evaluated the high-temperature expansion for the free energy of the Ginzburg-Landau model to 13th order, two orders further than in previous work.

I. INTRODUCTION

Because of the combination of high transition temperatures, strong anisotropy, and short coherence lengths which occurs in high-temperature superconductors, strong thermal fluctuations are present over a wide temperature interval in these materials. Thermal fluctuations are especially important in a magnetic field where they are responsible for the melting of the Abrikosov
vortex lattice at temperatures below the mean-field critical temperature giving rise to a vortex liquid state. In this paper we report on a study of correlations in the vortex liquid in the extreme anisotropy limit of decoupled layers. We have evaluated the leading terms in the high-temperature perturbation expansion of the superfluid-density spatial correlation function for the Ginzburg-Landau model of a superconductor film.

The thermodynamics of this system is unusual because of Landau quantization of the order parameter fluctuations; our calculations are carried out within the lowest Landau level approximation in which only the lowest gradient energy fluctuations are retained. This approximation is valid near the mean-field transition temperature and is ordinarily valid throughout the vortex liquid state, although fluctuation effects may be strong enough in some high-temperature superconductors to drive the solidification transition outside of its range of validity. High-temperature perturbation expansions for the free energy of this model, even when evaluated to high order,
exhibit little evidence of the Abrikosov vortex lattice state which is expected to occur at low temperatures. Recent Monte Carlo simulations, on the other hand, generally obtain results indicative of a weak first-order phase transition between the two states. (See, however, Ref. 9 where a different conclusion is reached.) We find that the perturbation expansion for the superfluid-density correlation function, unlike that for the average superfluid density studied by previous workers,
provides clear evidence of strong correlations in the vortex liquid which presage the appearance of an ordered state at low temperatures.

The paper is organized as following. In Sec. II we introduce the quantity we study, \( \langle |\Delta(q)|^2 \rangle \) which is proportional to the Fourier transform of the superfluid-density spatial correlation function. In Sec. III we outline the high-temperature perturbation expansion for this quantity and present our closed form result for the contribution to it from individual diagrams which appear in the expansion. In Sec. IV we discuss our evaluation of all terms out to twelfth order in this expansion and discuss some comparisons with expansions for the free energy performed by earlier workers which allow us to check our results. Extrapolations of our finite-order results to low temperatures using Padé approximants are presented and compared with Monte Carlo simulations. In Sec. V we discuss our results on the high-temperature expansion for the free energy. We have evaluated the coefficients of this expansion exactly out to 13th order, two orders further than in the most recent prior work. Previous workers speculated on the basis of existing results that the zero-temperature limits of Padé approximants to the free energy series extrapolated to the exact zero-temperature free energy in the limit of infinite-order approximants. Our results allow us to determine an approximant of one higher order than was previously possible and do not support this speculation. Section VI contains a brief summary.

II. SUPERFLUID-DENSITY CORRELATION FUNCTION

The free energy per unit area of the Ginzburg-Landau-model superconducting film in a perpendicular magnetic
field is given by
\[ f[\Psi] = \alpha(T)|\Psi|^2 + \frac{\beta}{2} |\Psi|^4 + \frac{1}{2m} \left[ (-i\hbar \nabla - 2e\vec{A})\Psi \right]^2, \]

(1)

where \( \Psi(\vec{r}) \) is the order parameter. \( \Psi(\vec{r}) \) is proportional to the wave function for the center of mass of the Cooper pairs\(^{10} \) and \( \vec{B} = \nabla \times \vec{A} \). We employ the Landau gauge so that \( \vec{A} = (0, Bz, 0) \). The quadratic terms in Eq. (1) are minimized by order parameters which correspond to a lowest Landau level (LLL) wave function for the Cooper pairs. In using Eq. (1) we are neglecting screening so that \( \vec{B} \) is the external magnetic field. This approximation is valid for type-II superconductors near the upper critical field, the regime of interest in this article. The mean-field-theory superconducting instability occurs when the quadratic terms in Eq. (1) become negative for the LLL, i.e., at \( T_c^{MF} \) where \( \alpha_H(T_c^{MF}) = 0 \) \( (\alpha_H \equiv \alpha + e\hbar B/m^*) \). In the LLL approximation we assume that fluctuations in higher Landau level channels can be neglected or at least absorbed in a renormalization of \( \alpha(T) \).\(^{4,6,11,12} \) Then the order parameter \( \Psi(\vec{r}) \) can be expanded as
\[
\Psi(\vec{r}) = \sum_{p} C_p(L_y)^{-1/2} \left( \frac{2eB}{\pi \hbar} \right)^{1/4} e^{ip\vec{r}} e^{-\frac{\beta}{\hbar} \left( z - \frac{x^2}{2} \right)^{2}},
\]

(2)

where the number of terms in the sum over \( p \) is \( N_\phi = L_x L_y \pi^2 = L_x L_y(eB/\pi \hbar) \). This expansion leads to the following expression for the Ginzburg-Landau model free energy:
\[
\int f[\Psi]d\vec{r} = \left[ \alpha_H \sum_{p} |C_p|^2 + \frac{\beta}{2} \sum_{p_1 p_2 p_3 p_4} (L_y)^{-1}(2\pi^2)^{-1/2} \right. \\
\times \exp \left\{ -\frac{\beta}{2} \left[ \sum_{i=1}^{4} p_i^2 - \frac{1}{4} \left( \sum_{i=1}^{4} p_i \right)^2 \right] \right\} \\
\times \hat{C}_{p_1} \hat{C}_{p_2} \hat{C}_{p_3} \hat{C}_{p_4} \delta_{p_1+p_2, p_3+p_4},
\]

(3)

where \( \alpha_H = \alpha(T)|1 - H/H_c2(T)| \). Fluctuation effects in the model are regulated by the dimensionless parameter \( g = \alpha_H(\pi^2/\beta k_B T)^{1/2} \).

The central quantity in our work is the superfluid-density spatial correlation function which we define by
\[
\chi_{SFD}(\vec{R}) \equiv \langle |\Psi(\vec{r})|^2 |\Psi(\vec{r} + \vec{R})|^2 \rangle \\
- \langle |\Psi(\vec{r})|^2 \rangle \langle |\Psi(\vec{r} + \vec{R})|^2 \rangle.
\]

(4)

Translational invariance of the system guarantees that the right hand side of Eq. (4) is independent of \( \vec{r} \). The modulation of the average superfluid density when the homogeneous system is disturbed by weak pinning can be expressed in terms of \( \chi_{SFD}(\vec{R}) \):
\[
\delta \langle |\Psi(\vec{r})|^2 \rangle = -\frac{1}{k_B T} \int d\vec{r}' \chi_{SFD}(\vec{r}' - \vec{r}) \delta \alpha(\vec{r}'),
\]

(5)

where \( \delta \alpha(\vec{r}') \) reflects the modulation of the mean-field transition temperature by pinning. The Fourier transform of \( \chi_{SFD}(\vec{r}) \) is
\[
\chi_{SFD}(\vec{q}) \equiv \frac{1}{L_x L_y} \int d^2\vec{r} \int d^2\vec{r}' \langle |\Psi(\vec{r})|^2 |\Psi(\vec{r}')|^2 \rangle \\
- \langle |\Psi(\vec{r})|^2 \rangle \langle |\Psi(\vec{r}')|^2 \rangle \exp[i\vec{q} \cdot (\vec{r} - \vec{r}')].
\]

(6)

We evaluate \( \chi_{SFD}(\vec{q}) \) by expressing it in terms of
\[
\Delta(\vec{q}) \equiv \frac{1}{N_\phi} \sum_{p_1 p_2} \hat{C}_{p_1} \hat{C}_{p_2} \delta_{p_1+p_2, q},
\]

(7)

so that
\[
\chi_{SFD}(\vec{q}) = \frac{N_\phi^2}{L_x L_y} \exp(-q^2 \ell^2/2) |\Delta(\vec{q})|^2 \\
- \langle \Delta(\vec{q} = 0) \rangle^2.
\]

(8)

\( \Delta(\vec{q}) \) satisfies the following informative sum rule\(^8 \) for each configuration of the Ginzburg-Landau system,
\[
\frac{1}{N_\phi} \sum_{\vec{q}} \langle |\Delta(\vec{q})|^2 \rangle - 1/N_\phi = 0,
\]

(9)

where \( \Delta(\vec{q}) \equiv \Delta(\vec{q})/\Delta_0 \) and \( \Delta_0 \equiv \Delta(\vec{q} = 0) \) is proportional to the integrated superfluid density.\(^{13} \) [Note that \( \Delta(\vec{q}) \) is invariant when the order parameter is multiplied by an overall constant.] \( |\Delta(\vec{q})|^2 \) is a particularly revealing quantity to examine in studying correlations in the vortex liquid. Equation (9) guarantees that (for large \( N_\phi \) \( \lim_{q \to \infty} N_\phi \Delta(\vec{q})^2 = 1 \) for any vortex liquid configuration. For example, in the high-temperature (vortex gas) limit \( \langle |\Delta(\vec{q})|^2 \rangle = 1/N_\phi \) for all \( \vec{q} \neq \vec{0} \). On the other hand it is easy to show that in the low-temperature limit the mean-field Abrikosov lattice configuration of the order parameter gives
\[
\langle |\Delta(\vec{q})|^2 \rangle = \begin{cases} 1 & \text{at } \vec{q} = \vec{G}, \\
0 & \text{otherwise}, \end{cases}
\]

(10)

where \( \vec{G} \) is any reciprocal lattice vector. Thus \( N_\phi |\Delta(\vec{q})|^2 \equiv s_V(q) \) shows exactly the behavior which would be expected for the static structure function of a classical fluid with \( N_\phi \) particles in both low-temperature (solid) and high-temperature (gas) limits. It seems clear that \( s_V(q) \) must be closely related to the static structure factor of the zeros of the order parameter, the vortices, although we do not believe that they are identical at all temperatures. For the purposes of the present study it is sufficient to observe that \( h_V(q) \equiv s_V(q) - 1 \), which we call the vortex correlation function, is a convenient measure of the degree of correlation in this system.
III. HIGH-TEMPERATURE PERTURBATION EXPANSION

$$|\Delta(q)|^2$$ can be expressed in the form

$$|\Delta(q)|^2 = N_\phi^{-2} \sum_{p_1p_2p_3p_4} \langle \overline{C}_{p_1} C_{p_2} C_{p_3} C_{p_4} \delta_{p_1+p_2+p_3+p_4} \rangle \times (\delta_{p_1,q_1} + \delta_{p_2,q_2} + e^{i\ell_2 q_2 (p_4 - p_3)}),$$

(11)

At high temperatures we can evaluate its thermal average

$$\langle |\Delta(q)|^2 \rangle = \frac{1}{Z} \int \prod_p dC_p dC_{\overline{p}} |\Delta(q)|^2 e^{-\frac{\alpha_H}{k_B T} \sum \delta q_i^2} \times e^{-\frac{\beta}{k_B T} \sum \delta q_i^2} e^{\frac{i}{2} \sum \delta q_i^2},$$

(12)

where $Z$ is the partition function

$$Z = \int \prod_p dC_p dC_{\overline{p}} e^{-\frac{\alpha_H}{k_B T} \sum \delta q_i^2} \times e^{-\frac{\beta}{k_B T} \sum \delta q_i^2} e^{\frac{i}{2} \sum \delta q_i^2},$$

(13)

by expanding the contribution to thermal weighting factors from the quartic contribution to the Landau-Ginzburg free energy. The perturbation series can most easily be handled in terms of Feynman diagrams. At nth order, there are contributions from diagrams with $(n+1)$ vertices and $2(n+1)$ edges in which the edges represent the Gaussian approximation correlation functions, $n$ vertices represent $|\Psi|^4$ terms proportional to $\beta$, and the additional "external" vertex corresponds to $|\Delta(q)|^2$. An important consequence of the Landau quantization of order parameter fluctuations is the fact that the Gaussian approximation correlation functions,

$$\langle \overline{C}_{p'} C_p \rangle = \delta_{p',p} k_B T / \alpha_H,$$

(14)

are independent of the momentum $p$. Each vertex has two directed outlying lines to represent the factors of $C_p$ associated with it and two incoming lines to represent the factors of $\overline{C}_p$. All diagrams containing single-loop dressings of edges or vertices (except the external vertex) can be eliminated by following Ruggeri and Thouless and expanding in terms of "Hartree-Fock" approximation correlation functions. This renormalization replaces $\alpha_H$ in Eq. (14) by $\alpha \equiv -2 \alpha_H / (g^4 + 4g^2)^{1/2}$. The $+$ and $-$ sign apply for $g < 0$ ($g > 0$). $\alpha$ remains positive for all temperatures so that the renormalized expansion parameter of the high-temperature perturbation expansion, $x = (\beta k_B T) / (4 \alpha^2 q^2)$, remains finite at all temperatures. $\alpha \approx \alpha_H$ for $g \gg 0$, $\alpha \approx |\alpha_H| / g^2$ for $g \ll 0$; $\alpha \approx |\alpha_H| (1 - 4x)^{-1}$, $g^2 = (4x - 1)^2 / 4x$ and $x = 1 / 4$ at $T_2$ [MP]. When the expansion is performed in terms of the Hartree-Fock correlation functions, $\langle |\Delta(q)|^2 \rangle$ is the sum of all the connected diagrams without single-loop dressings except at the external vertex.

It is convenient to label the external vertex as vertex 1 and to label the incoming momenta at vertex $i$ as $p_{qi}$ and $p_{zi-1}$. Then the contribution to $\langle |\Delta(q)|^2 \rangle$ from an nth-order diagram is given by $(k_B T / \alpha)^n N_\phi^{-1} I(q) / n!$ where

$$I(q) = \int d\rho_2 \cdots \int d\rho_{2n+2} \prod_{\mu=2}^{n+1} \delta \left( \sum_{i=1}^{n+1} (M_{\mu i} - N_{\mu i}) p_i \right) \times (\delta_{p_1} + \delta_{p_2} + e^{i\ell_2 (q_2 + i\ell_2 p_2)} \exp \left\{ \frac{-\ell^2}{2} \sum_{\mu=2}^{n+1} (p_2 - p_{2\mu-1})^2 \right\} ,$$

(15)

$$I(q) = \left\{ \begin{array}{l} \frac{1}{TA} \exp \left( -\frac{T^B}{2TA} q^2 \ell^2 \right) \\ + \frac{1}{TB} \exp \left( -\frac{T^A}{2TB} q^2 \ell^2 \right) \end{array} \right\}$$

(16)

if $T_A \times T_B \neq 0$, and

$$I(q) = \left( N_\phi \delta_{q,0} + 1 \right) / (T_A + T_B)$$

(17)

if $T_A \times T_B = 0$.

Using the above result for $I(q)$ and writing the number of appearances of a given graph as $4^{n+1} n! / G_{n+1,2}$ we obtain the following formally exact expression for $\langle |\Delta(q)|^2 \rangle$:

$$\langle |\Delta(q)|^2 \rangle = \frac{1}{N_\phi} \left( \frac{k_B T}{\alpha} \right)^2 \sum_{n=0}^{\infty} \sum_g a_{n,g}(q)(-4x)^{n}$$

$$= \frac{1}{N_\phi} \left( \frac{k_B T}{\alpha} \right)^2 \left\{ 1 + N_\phi \delta_{q,0} - 4x \exp \left[ -\frac{q^2 \ell^2}{2} \right] + 4x^2 \exp \left[ -\frac{q^2 \ell^2}{2} \right] \right.$$ 

$$+ 8x^2 (1 + N_\phi \delta_{q,0}) + 16x^2 \left( 0.5 \exp \left[ -\frac{q^2 \ell^2}{4} \right] + \exp[-q^2 \ell^2] \right) + \ldots \right\} ,$$

(18)
where $g$ labels graphs and the sum at $n$th order is over $(n + 1)$-vertex graphs. $a_{n,g}(q)$ is given by

$$a_{n,g}(q) = \begin{cases} 
\frac{2}{G_{n+1,g}} \left( T_{n,g}^{A} \exp \left[ -\frac{T_{n,g}^{A}}{2T_{n+1,g}} q^2 \ell^2 \right] + T_{n,g}^{B} \exp \left[ -\frac{T_{n,g}^{B}}{2T_{n+1,g}} q^2 \ell^2 \right] \right), \\
\left( 1 + N\delta q_{0} \right)
\end{cases}$$

if $T_{n,g}^{A} \times T_{n,g}^{B} \neq 0$, \hspace{1cm}

if $T_{n,g}^{A} \times T_{n,g}^{B} = 0$, \hspace{1cm}

where $T_{n,g}^{A}$ and $T_{n,g}^{B}$ are the number of the Euler paths of the two contracted $n$-vertex graphs and $G_{n+1,g}$ is the number of automorphisms of the $(n + 1)$-vertex graph with one external vertex. The diagrams which appear up to second order in the series and their associated properties are listed in Table I. The explicit expression in Eq. (18) can be confirmed from the entries in this table.

We observe in Eq. (19) that contributions which survive to the large-$|q|$ limit come only from graphs where $T_{n,g}^{A}$ equals zero or $T_{n,g}^{B}$ equals zero and that the only terms in $\langle |\Delta(q)|^2 \rangle$ which are independent of $N\phi$ come from the same set of diagrams. The contribution independent of $N\phi$ is $\langle \Delta^{2}_{g} \rangle q_{0}$. The remaining contributions to $\langle |\Delta(q)|^2 \rangle$, which are proportional to $N\phi^{-1}$, contribute to XSFD($q$) and are due to correlations in the thermally fluctuating superfluid density. We thus obtain explicitly from the perturbation expansion that

$$\lim_{|q| \to \infty} \langle |\Delta(q)|^2 \rangle = \langle \Delta^{2}_{g} \rangle / N\phi. \hspace{1cm} (20)$$

This result was claimed earlier on the basis of the sum rule [Eq. (9)].

The real space correlation function is given by

$$\langle |\Psi(\vec{r})|^2 |\Psi(\vec{r} + \vec{R})|^2 \rangle = \frac{1}{(2\pi)^2} \int dq |\Delta(q)|^2 \times \exp[-q^2 \ell^2 / 2] \exp[-iq \cdot \vec{R}]$$

$$\times \left( \frac{N\phi}{L_{x}L_{y}} \right)^2 \left( \frac{k_{B} T}{\alpha} \right)^2 \times \sum_{n=0} \sum_{g} a_{n,g}(R)(-4\pi)^n; \hspace{1cm} (21)$$

where

$$a_{n,g}(R) = \frac{2}{G_{n+1,g} T_{n+1,g}} \left\{ \exp \left[ -\frac{T_{n,g}^{A}}{2T_{n+1,g}} \left( \frac{R}{\ell} \right)^2 \right] \right. \right.$$ 

$$\left. + \exp \left[ -\frac{T_{n,g}^{B}}{2T_{n+1,g}} \left( \frac{R}{\ell} \right)^2 \right] \right\}. \hspace{1cm} (23)$$

In Eq. (23) $T_{n+1,g}^{A} = T_{n,g}^{A} + T_{n,g}^{B}$ is the number of Euler paths in the uncontracted $(n + 1)$-vertex graph. We see again here that only graphs with $T_{n,g}^{A} = 0$ or $T_{n,g}^{B} = 0$ remain finite for $R \to \infty$ where correlations vanish. The contribution of this subset of graphs is $\langle |\Psi(\vec{r})|^2 \rangle^2$.

### IV. FINITE-ORDER RESULTS AND EXTRAPOLATION TO LOW TEMPERATURES

We have developed a fast graph-generation computer program which generates all relevant graphs represented by their adjacency matrices.\(^{(15)}\) (The description of the algorithm is given in Appendix B.) From the adjacency matrices we calculate the number of Euler paths for the subgraphs ($T_{n,g}^{A}$ and $T_{n,g}^{B}$) and from the graph-generating algorithm we calculate the symmetry factor $G_{n+1,g}$. In Fig. 1, we show one of the graphs which appears at second order in the expansion, the two graphs which result from the deletion of its external vertex, and the adjacency matrices of all three graphs. (The number of Euler paths equals the determinant of the minor of the matrix appropriately formed from the corresponding adjacency matrix.\(^{(16)}\)) In this way, we have evaluated the series exactly up to 12th order.\(^{(18)}\) We have checked our results for $\langle |\Delta(q)|^2 \rangle$ by confirming that the sum rule Eq. (9) is satisfied and that the results for both $\langle |\Delta(q)|^2 \rangle$ and $\sum_{g} |\Delta(q)|^2 \exp[-q^2 \ell^2 / 2]$ are correct order by order in perturbation theory. The latter two quantities can be

![Graph](image)
related to derivatives of the free energy. The free energy is given by

\[ F = -k_B T \ln Z = k_B T N_\phi [\ln(\tilde{\alpha}/\pi k_B T) + f_{2D}(x)] \]  

(24)

where the perturbation expansion for \( f_{2D}(x) \) was first evaluated to order \( x^0 \) by Ruggeri and Thouless.\(^4\) The expansion was extended to order \( x^{11} \) by Brézin, Fujita, and Hikami\(^5\) and is extended to order \( x^{13} \) in the present work

\[
\langle |\Delta_0|^2 \rangle - \langle \Delta_0 \rangle^2 = \frac{1}{N_\phi} \left( \frac{k_B T}{\tilde{\alpha}} \right)^2 \left( \frac{1 - 4x}{1 + x} \right)^2 \left( 1 - 2x f_{2D}(x) \right)
\]

(26)

where \( \langle \Delta_0 \rangle = (k_B T/\tilde{\alpha})[1 - 2x f'_{2D}(x)]/(1 + 4x) \).

The following two expressions give the explicit expansions for \( \langle |\Delta(\tilde{G})|^2 \rangle \) and \( \langle |\Delta_0|^2 \rangle \), where \( |\tilde{G}| = 2.6935476^{-1} \) is the smallest reciprocal vector for the triangular lattice:

\[
\langle |\Delta(\tilde{G})|^2 \rangle = \frac{1}{N_\phi} \left( \frac{k_B T}{\tilde{\alpha}} \right)^2 \left( 1 - 0.106 319 841 061 557 4x + 9.421 892 685 500 202 1x^2
\]

\[
-98.684 577 215 755 340 x^3 + 1222.173 413 973 232 8x^4 - 17 509.235 550 184 578 x^5
\]

\[
+284.197.095 063 130 492 x^6 - 5 152 806.531 339 371 6x^7 + 103 259 468.748 141 05x^8
\]

\[
-2 267 805 862.815 430 6x^9 + 54 203 017 441.510 056 x^{10} - 1 401 439.349 032 462 2x^{11}
\]

\[
+38 992 399 883 388.438 x^{12}
\]

and

\[
\langle |\Delta_0|^2 \rangle = \left( \frac{k_B T}{\tilde{\alpha}} \right)^2 \left( 1 + 8x^2 - \frac{248}{3} x^3 + \frac{14792}{15} x^4 - 13 639.131 890 331 891 x^5
\]

\[
+214.140.340 471 362 08x^{6} - 3 763 359.335 800 620 3x^7 + 73 235 246.076 008 394 x^8
\]

\[
-1 565 815 099.394 278 5x^9 + 36 491 644 165.736 839 x^{10} - 921 480 933 812.921 26x^{11}
\]

\[
+25 075 334 330 238.250 x^{12}
\]

(27)

The asymptotic high-temperature expansion of \( \langle |\Delta(q)|^2 \rangle \) can be extrapolated to low temperatures using Padé approximants to describe the \( x \) dependence at each value of \( q \). Comparisons between Padé approximants of the series and our Monte Carlo simulation results\(^6\) are given in Fig. 2 for different temperatures. \( s_V(q) \) shows a well-defined peak for \( q \) near \( |\tilde{G}| \), where \( \tilde{G} \) is a reciprocal lattice vector of the vortex solid. Such a peak is characteristic of a strongly correlated liquid. Quantitative agreement with Monte Carlo simulation results is obtained for temperatures above \( T_c^{MF} \) and for temperatures below \( T_c^{MF} \) with \( g^2 < 10 \). A continuous phase transition to a vortex solid state at low temperatures would be indicated by the divergence of \( s_V(q) \) at \( q \) near \( |\tilde{G}| \) and hence by a pole in the Padé approximant to \( \langle |\Delta(q)|^2 \rangle \) along the positive real \( x \) axis. We find that poles do occur in some Padé approximants to \( \langle |\Delta(q)|^2 \rangle \) at values of \( x \) near those at which Monte Carlo simulations find a weak first-order phase transition. However, the poles do not appear consistently in approximants of different order so the high-temperature perturbation expansion does not provide any reliable information about the temperature and the nature of the phase transition. Nevertheless, we believe that the structure seen in \( s_V(q) \) is a precursor of the solidification of the vortex lattice at lower temperatures. The proximity of the vortex lattice state is apparent in the perturbation theory for \( s_V(q) \) but is hidden in the perturbation theory for the free energy because of the weakness of the thermodynamic singularity associated with the solidification phase transition.

V. ANALYSES OF THE FREE ENERGY EXPANSION

By developing a fast graph-generation algorithm, we were able to extend the expansion for \( f_{2D}(x) \) up to 13th order which is given by

\[
f_{2D}(x) = -2x - x^2 + \frac{38}{9} x^3 - \frac{1199}{30} x^4 + 471.396 594 516 594 46 x^5 - 6471.562 574 955 144 6x^6
\]

\[
+101 279.327 845 970 63x^7 - 1 779 798.787 594 752 2x^8 + 34 709 019.614 363 678 x^9
\]

\[
-744 093 435.668 222 31x^{10} + 17 399 454 123.559 521 x^{11}
\]

\[
-440 863 989 257.285 10x^{12} + 12 035 432 945 204.531 x^{13}
\]

(29)
The Abrikosov ratio $\beta_A$, which measures the average smoothness of the superfluid density, can be obtained in term of $f_{2D}(x)$ as

$$\beta_A = \frac{4 + (1 - 4x)f_{2D}(x)(1 + 4x)}{[1 - 2x f_{2D}(x)]^2}.$$

In the low-temperature (large-$x$) limit, the $f_{2D}(x)$ is linear in $x$, $f_{2D}(x) = -ax$, and the Abrikosov ratio $\beta_A$ is given by $4/a$. The high-temperature expansion results for $f_{2D}(x)$ can be extrapolated to the large-$x$ limit by using the $[n, n-1]$ Padé approximants for the expansion series. The resulting low-temperature limits for the Abrikosov ratio $\beta_A$ obtained by [4,3, [5,4, [6,5, and [7,6] Padé approximants are $1.39894, 1.29872, 1.25667, \text{and } 1.24057$, respectively. The [7,6] approximant requires the high-temperature expansion to be known to order $x^{13}$ and so was not previously available. Figure 3 shows the low-temperature limits of the approximants. The dashed line shows an extrapolation of the low-temperature limits of the $[n, n-1]$ approximants for $n = 4, n = 5, \text{and } n = 6$ to $n = \infty$, which suggests that the high-temperature expansion if carried out to sufficiently high order could provide an accurate estimate of the zero-temperature $\beta_A$ and hence the zero-temperature free energy as has been argued by Hikami, Fujita, and Larkin.\footnote{At zero temperature $\beta_A \sim 1.16$.} However the newly available $n = 7 [n, n-1]$ approximant clearly deviates from this extrapolation.

VI. SUMMARY AND CONCLUSIONS

We have studied the superfluid-density spatial correlation function $\langle |\Delta(q)|^2 \rangle$ using high-temperature perturbation expansions and extrapolated our results to temperatures below the mean-field transition temperature by means of Padé approximants. Good agreement with Monte Carlo simulation data is obtained for $g > -\sqrt{10}$. Our results demonstrate that the vortex liquid is strongly correlated below $T_c^\text{MF}$. A result argued for previously on the basis of a sum rule which related the large-wave-vector limit of the correlation function to the average superfluid density was obtained explicitly from the perturbation expansion. We argue that these perturbation expansion studies give clear indications of the phase transition to a two-dimensional vortex solid that is believed to occur at lower temperatures, although they cannot at present provide reliable information about the nature of the phase transition or the temperature at which it occurs. We have also evaluated two more orders in the high-temperature expansion of the free energy. These results demonstrate that caution needs to be exercised when extrapolating low-temperature results because of the slow convergence of the series.

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APPENDIX A

The integral to be evaluated has the form

$$I(q) \equiv e^{-i\xi q} \left( \frac{2\pi}{\ell^2} \right)^{-n/2} \sum_{p_2} \int dp_3 \cdots \int dp_{2n+2} \prod_{\mu=2}^{n+1} \delta \left( \sum_i (M_{\mu i} - N_{\mu i}) p_i \right) \times \left( \delta_{p'_2+q} + \delta_{p'_3+q} \right) e^{-i\xi' (q_2+iq_3)p_2} \exp \left\{ -\frac{\ell^2}{2} \sum_{\mu=2}^{n+1} (p_{2\mu} - p_{2\mu-1})^2 \right\}.$$  \hspace{1cm} (A1)
Of the \((2n + 1)\) variables \(p_i\) only \(n\) are independent because of the \((n + 1)\) \(\delta\) functions. It is convenient to choose \(s_\mu = p_{2\mu} - p_{2\mu - 1}\) \((\mu = 2, \ldots, n + 1)\) as the \(n\) independent variables. We label the contributions to \(I(q)\) from the two choices for the \(\delta\) function at the external vertex as \(I^A(q)\) and \(I^B(q)\) respectively. We use this \(\delta\) function to eliminate the sum over \(p_2 = s_1\) which can then be expressed as a linear function of the independent variables. For \(X = A\) or \(B\)

\[
s_1^X = -\alpha X q_b + \sum_{\mu=2}^{n+1} t_\mu^X s_\mu. \tag{A2}
\]

The integral over the independent variables can then be expressed in terms of these coefficients. The Jacobian for changing variables from \(p_b\) to \(s_{\mu-1}\) is independent of \(q_b\); at \(q_b = 0\) the change of variables is identical to that required for the diagram obtained by deleting the external vertex and contracting the two outgoing edges at that vertex with the two incoming edges. [The two ways of doing the contraction correspond to \(I^A(q)\) and \(I^B(q)\) respectively.] Following the work of McCauley and Thouless, we note that the inverses of the required Jacobians equal the number of Euler paths \(^{11}\) for the \(n\)-vertex graphs which results from the deletion of the external vertex and the two possible contractions. We denote the Euler path numbers by \(T^X\). The integral over the \(s_\mu\) is then elementary and we obtain

\[
I^X(q) = (e^{t^2 q_b q_v})^n_X (e^{-t^2 |q_v|^2})^n_2 (e^{-t^2 |q_v|^2})^n_3 \frac{1}{T^X}, \tag{A3}
\]

where \(n_1^X = -1 + \alpha X - \frac{1}{2} \sum_{\mu=2}^{n+1} (t_\mu^X)^2\), \(n_2^X = -1 + \alpha X - \frac{1}{2} \sum_{\mu=2}^{n+1} (t_\mu^X)^2\), and \(n_3^X = \frac{1}{2} \sum_{\mu=2}^{n+1} (t_\mu^X)^2\). However, we know that \(I^X(q)\) is real, which leads to the requirement that \(n_1^X = 0\) and implies that \(I^X(q) = \exp[-q^2 t^2 (\alpha X - 1)/2]/T^X\). The value of \(\alpha X\) can be inferred by noting that \(\sum_q \exp(-q^2 t^2/2) I^X(q)\) is proportional to an integral which appears in the expansion of the free energy and equals \(N_\phi/T\) where \(T\) is the number of Euler paths in the original graph before deletion. It follows that \(\alpha X = T/T^X\) and hence that

\[
I(q) = \left\{ \frac{1}{T_A} \exp \left( - \frac{T_B}{2T_A} q^2 t^2 \right) \right. \\
\left. + \frac{1}{T_B} \exp \left( - \frac{T_A}{2T_B} q^2 t^2 \right) \right\}. \tag{A4}
\]

(Note that \(T_A + T_B = T\).) This analysis fails in the special case where \(T_A \times T_B = 0\). For that case we have \(I^X(q) = n_X^B q_0\) or \(I^X(q) = n_X^C\) and a similar analysis gives the corresponding \(n_X^0 = N_\phi/T\) or \(n_X = 1/T\), and so

\[
I(q) = (N_\phi q_0 + 1)/(T_A + T_B). \tag{A5}
\]
morphs only arise from the symmetries of \( g \), but we chose not to use that fact.) After this local isomorph rejection is complete, we have exactly one representative from each equivalence class of basic digraphs of order \( n + 1 \), except for the primitive digraphs. The latter are easily added separately. The number of connected basic digraphs of each order up to 13 are shown in Table II.

The diagrams required for the expansion are easily made from the basic digraphs. Recall that the diagrams are connected, have a distinguished “external” vertex labeled separately (and allowed to carry a loop), and have an ordering specified for the two incoming edges at each vertex and for the two outgoing edges. Suppose \( g \) is a connected basic digraph with automorphism group having order \( G \) (including interchange of parallel edges) and vertex orbits \( O_1, \ldots, O_k \). We can make diagrams with \( n + 1 \) vertices in two distinct ways.

To make diagrams with no loops, choose any \( g \) with \( n + 1 \) vertices. Then choose any orbit \( O_i \) and designate an arbitrary vertex of that orbit to be the external vertex. The number of labeled diagrams in this class is \( n!|O_i|^{|G^{n+1}/G|} \).

To make diagrams with the external vertex carrying a loop, choose \( g \) to have \( n - j + 1 \) vertices for some \( j \geq 1 \) and delete some edge \( e = (v, w) \). Append new vertices \( u_1, \ldots, u_j \) and new edges \( (v, u_1), (u_1, u), (u, u_{j+1}), (u_{j+1}, u_j) \) (\( 1 \leq i \leq j - 1 \)), and \( (u_j, u) \). Vertex \( u_j \) becomes the external vertex. The number of corresponding labeled diagrams depends on the action of the automorphism group of \( g \) on the edge \( e \). However, since the number of Euler paths in the resulting diagram does not depend on \( e \), we can group together all the diagrams derivable from \( g \) in this manner. The total number of labeled diagrams in this class is \( 2(n+1)!G^{n+1}/G \).

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11. The limits of validity of this approximation are discussed by Ryuske Ikeda (unpublished).
12. We also neglect fluctuations in the vector potential. This approximation is permitted as long as the temperature is not too far below the mean-field transition temperature or the films are thin and widely separated.
13. We remark that Eq. (9) is exact even for finite-size systems with quasiperiodic boundary conditions applied to the order parameter. In that case the number of \( g \) values in the sum over wave vectors is \( N_0^2 \). Finite-size effects have to be considered carefully when terms of order \( N_0^{-1} \) are retained on the left hand side of Eq. (9).
18. The perturbation expansion of the Fourier transform of this function has previously been evaluated to sixth order using a different approach by Ryuske Ikeda, Tetsuo Ohmi, and Toshihiko Tsuneto [J. Phys. Soc. Jpn. 59, 1397 (1990)].

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**TABLE II. Number of connected basic digraphs for each order up to 13.**

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