Integrator Backstepping using Barrier Functions for Systems with Multiple State Constraints

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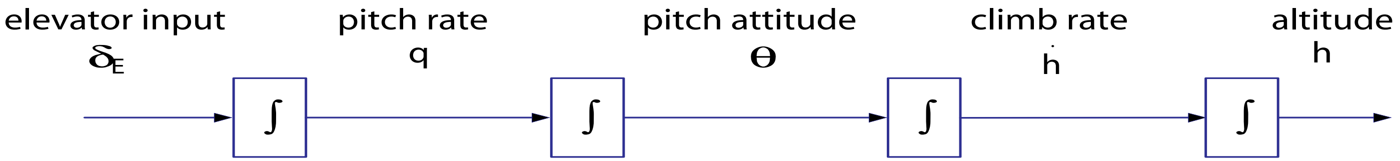
Motivating problem

Consider the idealized model of the longitudinal dynamics of an aircraft:



- Control margin provided by elevators is sufficient for all required manoeuvres.
- Magnitude constraints on:
  - Climb rate due to stall.
  - Pitch attitude and pitch rate due to “passenger comfort” factor.
- Control design must guarantee that the constrained states always remain below their limits.

Some flight control design techniques

Gain scheduled linear control design:

Does not explicitly include constraint limits. Constrained designs tend to be based on low gain arguments.

Feedback linearization:

Demands accurate an dynamic model. Does not directly incorporate constraints.

Model Predictive Control:

Requires a good model and is computationally intensive.

Forwarding:

Incorporates input constraints and can be extended to state constraints. Low gain design.

Backstepping

- Affords control designers great freedom in selecting final control law.
- Able to accommodate large nonlinearities and uncertainties in system's model, ignored dynamics, input/measurement disturbances.
- Aircraft longitudinal dynamics can be transformed into "lower-triangular feedback" form.
- The lack of 'gain-limiting' feedforward interconnection terms means no gain restriction on intermediate control signals.
- High gains are required to impose state constraints.

Motion systems dynamics

$$\dot{\xi}_1 = \xi_2$$ Translation position (unconstrained)

$$\dot{\xi}_2 = \xi_3$$ Translation velocity

$$\dot{\xi}_3 = \xi_4$$ Attitude

$$\dot{\xi}_4 = b(\xi) + a(\xi)u$$ Angular velocity

Magnitude constraints on system states

$$|\xi_i(t)| \leq \Xi_i, \quad i = 2, 3, 4$$

The proposed approach can be extended to:

- Arbitrary lower triangular systems.
- Systems of arbitrary order.

Step I: Growth condition on first state for cLf

Consider the first scalar system

$$\dot{\xi}_1 = \xi_2$$

Since ξ_1 is unbounded the cLf design must incorporate a growth bound on its dependence on ξ_1.

$$V_1(\xi_1) = \mathbf{O}(\xi_1), \quad \text{as} \quad |\xi_1| \to \infty$$

A candidate cLf is

$$V_1(\xi_1) = k_1 \xi_1 \arctan(\xi_1), \quad k_1 > 0$$

The time differential of V_1 is bounded in ξ_1

$$\dot{V}_1 = k_1 \xi_2 \left[\arctan(\xi_1) + \frac{\xi_1}{1 + \xi_1^2} \right]$$

Step I: Bounded reference trajectory

Define the error variable z_1

$$z_1 = \xi_2 - \xi_{2_{ref}}$$

The reference trajectory $\xi_{2_{ref}}$ must be bounded.

$$\xi_{2_{ref}} = -c_1 \arctan(\xi_1), \quad c_1 > 0$$

By choosing bounded growth in V_1 and bounded reference trajectory, $\xi_{2_{ref}}$, this ensures bounded propagation of virtual error in the back-stepping procedure.

Step I: Error dynamics of ξ_1-subsystem

$$
\begin{aligned}
\dot{\xi}_1 &= -c_1 \arctan(\xi_1) + z_1 \\
\dot{V}_1 &= -k_1 c_1 \arctan(\xi_1) \left[\arctan(\xi_1) + \frac{\xi_1}{1+\xi_1^2} \right] \\
&\qquad\qquad\qquad\qquad + k_1 z_1 \left[\arctan(\xi_1) + \frac{\xi_1}{1+\xi_1^2} \right] \\
&\leq -W(\xi_1) + k_1 z_1 \left[\arctan(\xi_1) + \frac{\xi_1}{1+\xi_1^2} \right]
\end{aligned}
$$

where $W_1(\xi_1)$ is positive-definite in ξ_1,

Step II: Barrier function on z_1

Need to choose an augmented cLf V_2 that takes V_1 and adds a term for the stability of the z_1 state. We propose to incorporate a barrier condition on the cLf V_2 with respect to z_1

$$|z_1| \to k_2 \Longrightarrow V_2(z_1) \to \infty$$

A candidate cLf is

$$V_2(\xi_1, z_1) = V_1 + \frac{1}{2} k_3 \log\left(\frac{k_2^2}{k_2^2 - z_1^2}\right)$$

- Local quadratic structure of V_2

$$V_2 \approx k_1 \xi_1^2 + k_3 z_1^2, \qquad \text{for } |\xi_1|, |z_1| \text{ small}$$

- Linear growth in ξ_1.
- Unbounded growth in V_2 as $z_1 \to k_2$.

Step II: Choosing stabilising function $\xi_{3_{ref}}$

The time-derivative of V_2 is

$$\dot{V}_2 = -W_1(\xi_1) + z_1 \left[k_1 \arctan(\xi_1) + \frac{k_1 \xi_1}{1+\xi_1^2} + \frac{k_3 \dot{z}_1}{k_2^2 - z_1^2} \right]$$

The stabilizing function $\xi_{3_{ref}}$ is chosen as

$$\xi_{3_{ref}} = -c_2 z_1 - \frac{c_1 \xi_2}{1+\xi_1^2} - \frac{(k_2^2 - z_1^2)}{k_3} \left[k_1 \arctan(\xi_1) + \frac{k_1 \xi_1}{1+\xi_1^2} \right],$$

Thus,

$$\dot{z}_1 = -c_2 z_1 - \frac{(k_2^2 - z_1^2)}{k_3} \left[k_1 \arctan(\xi_1) + \frac{k_1 \xi_1}{1+\xi_1^2} \right] + z_2$$

$$\dot{V}_2 = -W_1 - \frac{k_3 c_2 z_1^2}{k_2^2 - z_1^2} + \frac{k_3 z_1 z_2}{k_2^2 - z_1^2}$$

Step II: Boundedness of state ξ_2

Assume for a moment that ξ_2 is directly controlled and one can set $z_2 = 0$. Then

$$\dot{V}_2 = -W_1 - \frac{k_3 c_2 z_1^2}{k_2^2 - z_1^2} \quad \Rightarrow \quad V_2(t) < V_2(0)$$

It follows that for all time $t > 0$

$$|z_1| < k_2$$

and consequently that

$$|\xi_2| = |z_1 + \xi_{2_{ref}}| < k_2 + \frac{\pi}{2} c_1$$

Choosing k_2 and c_1 such that

$$k_2 + \frac{\pi}{2} c_1 \leq \Xi_2$$

guarantees the first state bound holds for all time.

Step III: Continue back-stepping process

Define the error signal variable

$$z_3 = \xi_4 - \xi_{4_{ref}}$$

A candidate cLf is

$$V_3 = V_2 + \frac{1}{2} k_5 \log \left(\frac{k_4^2}{k_4^2 - z_2^2} \right)$$

The time derivative of V_3 is

$$\dot{V}_3 = -W_1 - \frac{k_3 c_2 z_1^2}{k_2^2 - z_1^2} + \frac{k_3 z_1 z_2}{k_2^2 - z_1^2} + \frac{k_5 z_2}{k_4^2 - z_2^2} \dot{z}_2$$

Choose stabilising function

$$\xi_{4_{ref}} = -c_3 z_2 + \dot{\xi}_{3_{ref}}, \quad c_3 > 0, \quad \Rightarrow \quad \boxed{\dot{z}_2 = -c_3 z_2 + z_3}$$

Step III: Stability analysis

The cross-term $\frac{k_3 z_1 z_2}{k_2^2 - z_1^2}$ cannot be directly canceled.

The cLf time derivative is given by

$$\dot{V}_3 = -W_1 \underbrace{- \frac{k_3 c_2 z_1^2}{k_2^2 - z_1^2} + \frac{k_3 z_1 z_2}{k_2^2 - z_1^2} - \frac{k_5 c_3}{k_4^2 - z_2^2} z_2^2}_{\text{need to show negative semi-definite}} + \frac{k_5 z_2 z_3}{k_4^2 - z_2^2}$$

Case I: If $(k_2 - |z_1|)$ is large relative to $(k_4 - |z_2|)$ then it is possible to choose gains to ensure that the three terms are negative using a 'completion of squares' argument.

Case II: If $(k_2 - |z_1|)$ is small relative to $(k_4 - |z_2|)$ then $|z_1| >> 0$. Since $|z_4| < k_4$, it is possible to choose the gains such that

$$-k_3 c_2 z_1^2 + k_3 z_1 z_2 < 0$$

Step IV: Crank the handle

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -A_{11} & 1 & 0 & 0 \\ -A_{21} & -c_2 & 1 & 0 \\ 0 & 0 & -c_3 & 1 \\ 0 & 0 & 0 & -c_4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

where

$$A_{11} = c_1 \frac{\arctan(\xi_1)}{\xi_1}$$

$$A_{21} = \frac{\left(k_2^2 - z_1^2\right)}{k_3} \left[k_1 \frac{\arctan(\xi_1)}{\xi_1} + \frac{k_1}{1 + \xi_1^2} \right]$$

For suitable choice of $c_1 \cdots c_4$ it is straightforward to show stability of the error system.

However, to guarantee the constraints are satisfied at all times then it is necessary to show a non-negative decrease property of constructed cLf V_4.

Step V: State bounds and control tuning

Let $Z = (k_1, \ldots, k_{2n}, c_1, \ldots, c_n)$ denote the set of gains used.

At each back-stepping step there were two bounds derived from the stability analysis

$$c_i \geq \frac{k_{2i}^2}{[k_i\alpha_{i-1}]^2} + \frac{1}{2} \qquad \Rightarrow \quad \Phi_i(Z) \geq 0$$

$$\frac{k_{2j+1}c_{j+1}}{k_{2j}^2} \geq \frac{k_{j+1}}{k_j^2 - [k_j\alpha_{j-1}]^2} \qquad \Rightarrow \quad \Psi_j(Z) \geq 0$$

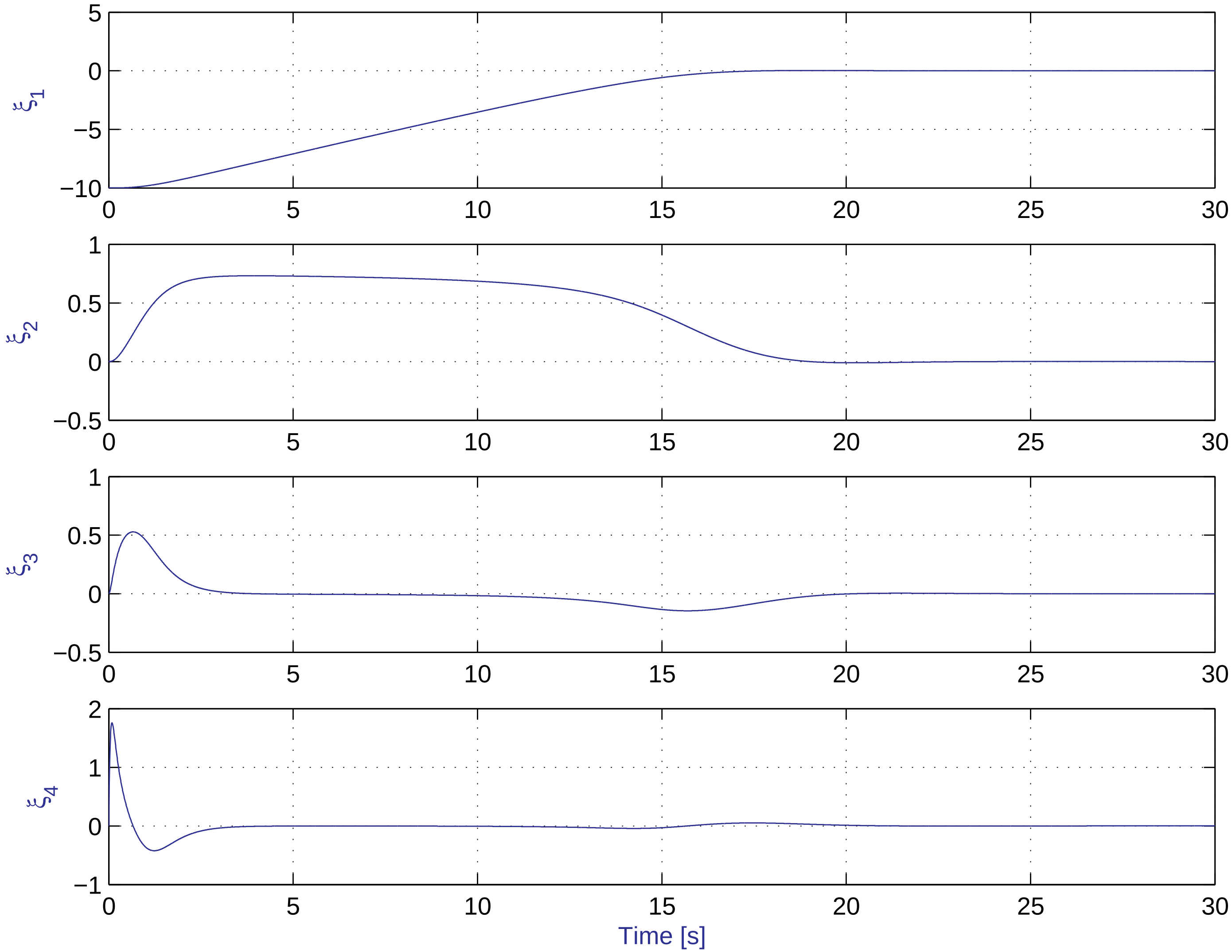
There are also the original bounds on the states that are interpreted as constraints on Z

$$|\xi_k(t)| = |z_{k-1}(t) + \xi_{k_{ref}}(t)| = |X_k(k_1, \ldots, k_{2n}, c_1, \ldots, c_n)| < \Xi_k, \quad k = 2, \ldots, n$$

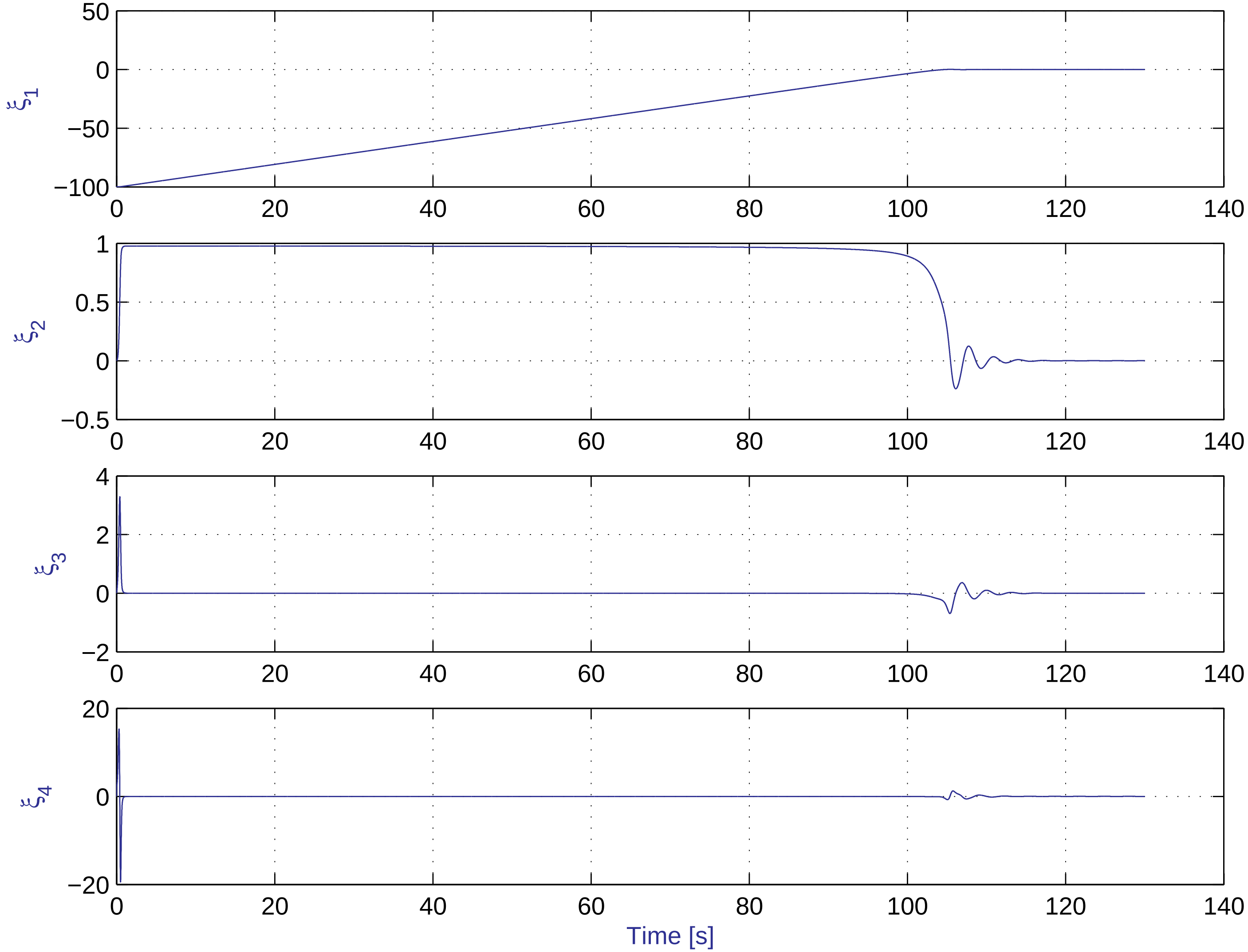
Optimisation problem:

Minimise $F(Z) = \sum(\Xi_i - |X_i(Z)|)/(\Xi_i)$ over Z subject to $\{\Phi_i(Z) \geq 0\}$ and $\{\Psi_j(Z) \geq 0\}$ and $\{|X_k(Z)| < \Xi_k\}$.

Simulation for initial position error of 10m



Simulation for initial position error of 100m



Conclusions

- Use the flexibility of the backstepping procedure to introduce barrier function characteristics in the constructed control Lyapunov function.

  This introduces parameterized soft bounds on the state error variables in the transformed system.

- Derive stability conditions on the gain choices at each iteration of back-stepping procedure to guarantee stability.

- Derive a set of parameter bounds for the control tuning based on the state bounds.

- Pose a constrained optimisation problem in the control parameters whose solution leads to good closed-loop performance while preserving the state constraints.