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Green's function for thermopiezoelectric plates with holes of various shapes

Q.-H. Qin

Summary Thermoelectroelastic problems for holes of various shapes embedded in an infinite matrix are considered in this paper. Based on Lekhnitskii's formalism, the technique of conformal mapping and the exact electric boundary conditions on the hole boundary, the thermoelectroelastic Green's function has been obtained analytically in terms of a complex potential. As an application of the proposed function, the problem of an infinite plate containing a crack and a hole is analysed. A system of singular integral equations for the unknown temperature discontinuity and the discontinuity of elastic displacement and electric potential (EDEP) defined on crack faces is developed and solved numerically. Numerical results for stress and electric displacement (SED) intensity factors of the crack-hole system are presented to illustrate the application of the proposed formulation.

Key words Piezoelectricity Green's function, thermal stress, crack, integral equation, plate

1 Introduction

The widespread use of piezoelectric materials in structural applications has generated renewed interest in the thermoelectroelastic behaviour. In particular, information on thermal stress concentrations around material or geometrical defects in piezoelectric solids may have wide applications in composite structures. For orthotropic elastic plates with rectangular openings, some research has been done in [1, 2]. The results were based on the Lekhnitskii's formalism [3], and represent only approximate solutions due to the mathematical difficulties involved. The stress field for an anisotropic elastic plate with holes of various shapes subjected to remote uniform mechanical loading was investigated in [4] on the basis of the Stroh formalism and complex conformal mapping. For plane piezoelectric material subjected to a uniform remote load, the electroelastic solutions for a piezoelectric plate with an elliptic hole or an inclusion have been investigated in [5, 6]. As for Green's functions in piezoelectric materials, a contour integral representation of the piezoelectric Green's functions was developed using transform techniques [7]. More recent studies regarding Green's functions in piezoelectric solids can be found in [8–12]. In the literature, however, there is very little work concerning thermoelastic Green's function. Thermoelastic Green's function for a two-dimensional problem of an infinite anisotropic plate subjected to a temperature discontinuity along the axis $x_2 = 0$ has been investigated in [13]. Later on, thermoelectroelastic Green's functions for bimaterial problems as well as for an elliptic hole embedded in an infinite plate were presented in [14, 15], respectively. Recently, a study in [16] on Green's functions for a polygonal hole in an infinite piezoelectric plate has shown that the corresponding transformation is not single-valued, and a simple approach was presented to treat this problem.

The present paper is a continuation of our previous work [16]. A unified form of thermoelectroelastic Green's functions is presented for an infinite thermopiezoelectric plate with various openings subjected to thermal loading. The loading may be a point heat source and a

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temperature discontinuity. The Green's functions developed are used to derive the thermo-electroelastic solution for interactions between a crack and a hole embedded in an infinite thermopiezoelectric plate. Numerical results for SED intensity factors are presented to verify the effectiveness of the proposed formulation.

2 Preliminary formulations

2.1 Basic equations

As in [16, 17], we effect a plane strain analysis, where the material is transversely isotropic and coupling between in-plane stresses and in-plane electric fields takes place. For a cartesian coordinate system $x_1x_2x_3$, choose the x_3-axis as the poling direction. The plane strain constitutive equations are expressed in the matrix form as

$$h_i = k_{ij}H_j \ , \tag{1}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ D_1 \\ D_2 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 & e_{21} \\ c_{12} & c_{22} & 0 & 0 & e_{22} \\ 0 & 0 & c_{33} & e_{13} & 0 \\ 0 & 0 & e_{13} & -\kappa_{11} & 0 \\ e_{21} & e_{22} & 0 & 0 & -\kappa_{22} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ -E_1 \\ -E_2 \end{Bmatrix} - \begin{Bmatrix} \gamma_{11} \\ \gamma_{22} \\ 0 \\ 0 \\ g_2 \end{Bmatrix} \theta \ , \tag{2}$$

or, inversely, as

$$H_i = \rho_{ij}h_j \ , \tag{3}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ -E_1 \\ -E_2 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} & 0 & 0 & p_{21} \\ f_{12} & f_{22} & 0 & 0 & p_{22} \\ 0 & 0 & f_{33} & p_{13} & 0 \\ 0 & 0 & p_{13} & -\beta_{11} & 0 \\ p_{21} & p_{22} & 0 & 0 & -\beta_{22} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ D_1 \\ D_2 \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \\ 0 \\ \lambda_2 \end{Bmatrix} \theta \ , \tag{4}$$

where $\sigma_{ij}, \varepsilon_{ij}, D_j$ and E_j are stress, strain, electric displacement and electric fields respectively, c_{ij} is elastic stiffness, f_{ij} elastic compliance, e_{ij} and p_{ij} are piezoelectric constants, κ_{ij} and β_{ij} dielectric permittivity, g_2 and λ_2 pyroelectric constants, γ_{ij} and α_{ij} piezothermal and thermal expansion constants, H_j and h_j are heat intensity and heat flux, k_{ij} and ρ_{ij} the coefficients of heat conductivity and heat resistivity, θ the temperature change.

Equations (3) and (4) constitute a system of six equations in thirteen unknowns. Additional equations are provided by the elastic equilibrium and Gauss' law

$$h_{1,1} + h_{2,2} = 0, \quad \sigma_{11,1} + \sigma_{12,2} = 0, \quad \sigma_{12,1} + \sigma_{22,2} = 0, \quad D_{1,1} + D_{2,2} = 0 \ , \tag{5}$$

where commas indicate partial differentiation, and the absence of heat sources, body forces and free electric volume charges has been assumed. The following relations are also valid:

$$h_{,i} = -k_{ij}\theta_{,j}, \quad \varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0, \quad E_{1,2} - E_{2,1} = 0 \ . \tag{6}$$

Having formulated the thermoelectroelastic problem, we look for a solution to (3)–(6) subjected to given loading and boundary conditions. To this end, the well-known Lekhnitskii stress functions U and induction function Ψ satisfying the foregoing equilibrium equations are introduced as follows:

$$\sigma_{11} = U_{,22}, \quad \sigma_{22} = U_{,11}, \quad \sigma_{12} = -U_{,12}, \quad D_1 = \Psi_{,2}, \quad D_2 = -\Psi_{,1} \ . \tag{7}$$

Inserting Eq. (7) into Eq. (4), and later into Eq. (6), we have

$$L_4U - L_3\Psi = -\alpha_{11}\theta_{,22} - \alpha_{22}\theta_{,11}, \quad L_3U + L_2\Psi = -\lambda_2\theta_{,1} \ , \tag{8}$$

where

$$L_4 = f_{22}\frac{\partial^4}{\partial x_1^4} + f_{11}\frac{\partial^4}{\partial x_2^4} + (2f_{12}+f_{33})\frac{\partial^4}{\partial x_1^2 \partial x_2^2} \quad ,$$
$$L_3 = p_{22}\frac{\partial^3}{\partial x_1^3} + (p_{21}+p_{13})\frac{\partial^3}{\partial x_1 \partial x_2^2}, \quad L_2 = \beta_{22}\frac{\partial^2}{\partial x_1^2} + \beta_{11}\frac{\partial^2}{\partial x_2^2} \quad . \tag{9}$$

Since this is a linear problem, solutions to Eqs. (8) are assumed to consist of the sum of particular solutions, U_p and Ψ_p, and homogeneous part, U_h and Ψ_h, as

$$U = U_p + U_h, \quad \Psi = \Psi_p + \Psi_h \quad , \tag{10}$$

where U_h and Ψ_h will satisfy

$$L_4 U_h - L_3 \Psi_h = 0, \quad L_3 U_h + L_2 \Psi_h = 0 \quad . \tag{11}$$

A general solution to Eqs. (11) has been well-documented elsewhere [5, 17]. The results are as follows:

$$U_h = 2\mathrm{Re}\sum_{k=1}^{3} U_{hk}(z_k), \quad \Psi_h = 2\mathrm{Re}\sum_{k=1}^{3} \chi_k \Phi_k(z_k) \quad , \tag{12}$$

where 'Re' stands for the real part of the complex function, and

$$z_k = x_1 + \mu_k x_2, \quad \Phi_k(z_k) = U'_{hk}(z_k), \quad \chi_k = -\frac{(p_{21}+p_{13})\mu_k^2 + p_{22}}{\beta_{11}\mu_k^2 + \beta_{22}} \quad , \tag{13}$$

while μ_k are three complex roots (with positive imaginary parts) satisfying the following characteristic equations:

$$f_{11}\beta_{11}\mu^6 + (f_{11}\beta_{22} + f_{33}\beta_{11} + 2f_{12}\beta_{11} + p_{21}^2 + p_{13}^2 + 2p_{21}p_{13})\mu^4$$
$$+ (f_{22}\beta_{11} + 2f_{12}\beta_{22} + f_{33}\beta_{22} + 2p_{21}p_{22} + 2p_{13}p_{22})\mu^2 + f_{22}\beta_{22} + p_{22}^2 = 0 \quad . \tag{14}$$

Thus, through the use of the following relations:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad E_j = -\varphi_{,j} \quad , \tag{15}$$

where u_i and φ are elastic displacement and electric potential, respectively, the EDEP and SED can be expressed as

$$\mathbf{u}_h = 2\,\mathrm{Re}[\mathbf{A}\Phi(\mathbf{z})], \quad \Pi_{1h} = -2\,\mathrm{Re}[\mathbf{BP}\Phi'(\mathbf{z})], \quad \Pi_{2h} = 2\,\mathrm{Re}[\mathbf{B}\Phi'(\mathbf{z})] \quad , \tag{16}$$

where

$$\mathbf{u}_h = \{u_1\, u_2\, \varphi\}_h^{\mathrm{T}}, \quad \Pi_{1h} = \{\sigma_{11}\, \sigma_{12}\, D_1\}_h^{\mathrm{T}}, \quad \Pi_{2h} = \{\sigma_{12}\, \sigma_{22}\, D_2\}_h^{\mathrm{T}} \quad , \tag{17}$$

$$\Phi(\mathbf{z}) = \{\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\}^{\mathrm{T}}, \quad \mathbf{P} = \mathrm{diag}[\mu_1\, \mu_2\, \mu_3] \quad , \tag{18}$$

$$\mathbf{A} = \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ t_1 & t_2 & t_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -\mu_1 & -\mu_2 & -\mu_3 \\ 1 & 1 & 1 \\ -\chi_1 & -\chi_2 & -\chi_3 \end{bmatrix} \quad , \tag{19}$$

with

$$p_k = f_{11}\mu_k^2 + f_{12} - p_{21}\chi_k, \quad q_k = f_{12}\mu_k + \frac{f_{22}}{\mu_k} - \frac{p_{22}\chi_k}{\mu_k},$$
$$t_k = -(p_{13} + \beta_{11}\chi_k)\mu_k \quad , \tag{20}$$

As for the particular solutions, they depend on the form of the known function θ and can be expressed in terms a complex potential $\Phi_t(z_t)$ as

$$\begin{aligned} &\theta = 2\,\mathrm{Re}[\Phi_t'(z_t)], \quad \mathbf{u}_p = 2\,\mathrm{Re}[\mathbf{c}\Phi_t(z_t)], \\ &\Pi_{1p} = -2\,\mathrm{Re}[\tau\mathbf{d}\Phi_t'(z_t)], \quad \Pi_{2p} = 2\,\mathrm{Re}[\mathbf{d}\Phi_t'(z_t)] \ , \end{aligned} \tag{21}$$

where $\mathbf{c} = \{c_1\, c_2\, c_3\}^{\mathrm{T}}$ and $\mathbf{d} = \{d_1\, d_2\, d_3\}^{\mathrm{T}}$ are two vectors associated with the materials constants and well-documented in [17, 18]. For the sake of reference we list them below

$$\mathbf{c} = \begin{bmatrix} c_{11} + \tau^2 c_{33} & \tau(c_{12} + c_{33}) & \tau(e_{21} + e_{13}) \\ & c_{33} + \tau^2 c_{22} & e_{13} + \tau^2 e_{22} \\ \text{symmetric} & & -(\kappa_{11} + \tau^2 \kappa_{22}) \end{bmatrix}^{-1} \begin{Bmatrix} \gamma_{11} \\ \tau\gamma_{22} \\ \tau g_2 \end{Bmatrix} \ , \tag{22}$$

$$\mathbf{d} = \begin{bmatrix} \tau c_{33} & c_{33} & e_{13} \\ c_{12} & \tau c_{22} & \tau e_{22} \\ e_{21} & \tau e_{22} & -\tau\kappa_{22} \end{bmatrix} \mathbf{c} - \begin{Bmatrix} 0 \\ \gamma_{22} \\ g_2 \end{Bmatrix} \ . \tag{23}$$

2.2
Conformal mapping

Since conformal mapping is a fundamental tool used to find complex potentials, the transformation [16]

$$z_k = a\left(a_{1k}\zeta_k + a_{2k}\zeta_k^{-1} + e_{j1}a_{3k}\zeta_k^j + e_{j1}a_{4k}\zeta_k^{-j}\right) \quad (k = 1, 2, 3, t) \ , \tag{24}$$

in which

$$a_{1k} = \frac{1}{2}(1 - \mathrm{i}\mu_k e), \quad a_{2k} = \frac{1}{2}(1 + \mathrm{i}\mu_k e), \quad a_{3k} = \frac{\gamma}{2}(1 + \mathrm{i}\mu_k e), \quad a_{4k} = \frac{\gamma}{2}(1 - \mathrm{i}\mu_k e) \ , \tag{25}$$

with the understanding that $\mu_4 = \tau$, will be used to map the contour of the hole on to a unit circle in the ζ-plane, where $e_{ji} = 0$, if $i = j$; $e_{ji} = 1$, if $i \neq j$, $0 < e \leq 1$, γ and a are real parameters and j is an integer. For a particular value of z_k, it is shown that there exist $2j$ roots for ζ_k in Eq. (24); half of the roots are located outside the unit circle, the remaining are inside the unit circle [16]. This indicates that the transformation (24) will be single-valued for $j = 1$ (ellipse) since only one root is located outside the unit circle. For $j > 1$, however, the transformation (24) is multi-valued, as there are j roots located outside the unit circle. The question is, which transformation should be chosen. This problem has been discussed in [16], and we will omit those details here. As it was done in [16], we choose the root whose magnitude has a minimum value among the j-roots. In a way similar to that in [16], the mapping of the hole region, Ω_0 (s. Fig. 1), can be written as

$$z = a(a_{1c}\zeta + a_{2c}\zeta^{-1} + e_{j1}a_{3c}\zeta^j + e_{j1}a_{4c}\zeta^{-j}) \ , \tag{26}$$

where $z = x_1 + \mathrm{i}x_2$, and

$$a_{1c} = \frac{1}{2}(1 + e), \quad a_{2c} = \frac{1}{2}(1 - e), \quad a_{3c} = \frac{\gamma}{2}(1 - e), \quad a_{4c} = \frac{\gamma}{2}(1 + e) \ , \tag{27}$$

2.3
Boundary conditions

Consider an infinite thermopiezoelectric plate containing a hole subjected to a line temperature discontinuity θ_0 and a line heat source h^*, both located at $\hat{z}_t = x_{10} + \tau x_{20}$. The contour of the hole is represented by (s. Fig. 1)

$$x_1 = a(\cos\psi + \gamma e_{j1} \cos j\psi), \quad x_2 = a(e \sin\psi - \gamma e_{j1} \sin j\psi) \ , \tag{28}$$

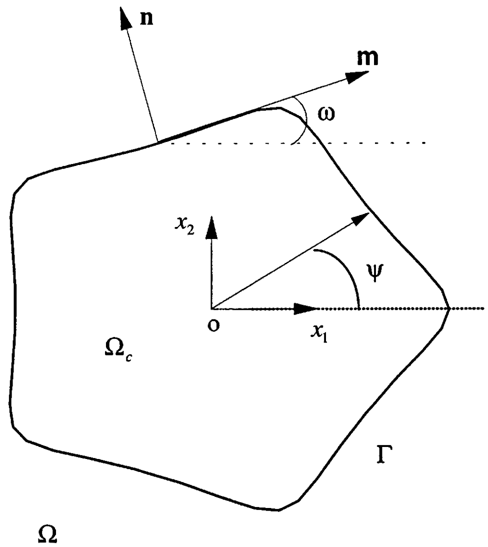

Fig. 1. Geometry of a particular hole ($a = 1,\ e = 1,\ j = 4,\ \gamma = 0.1$)

where ψ is a real parameter. By an appropriate selection of the parameters e, j and γ, we can obtain various special kinds of holes, such as an ellipse ($j = 1$), a circle ($j = 1$ and $e = 1$), a triangle ($j = 2$), a square ($j = 3$) and a pentagon ($j = 4$).

The hole surface is assumed to be free of tractions and also kept at zero heat flux. Besides, on the boundary of the hole the normal component of electric displacement and the electric potential are continuous. Therefore, we have the following boundary conditions:

$$\mathbf{h} \to 0, \quad \Pi \to 0 \ , \tag{29}$$

at infinity

$$\begin{aligned} &h_n = -h_1 \sin\omega + h_2 \cos\omega = 0, \\ &\sigma_1 = 0, \quad \sigma_2 = 0, \quad D_n = -\varepsilon_0 \frac{\partial \varphi^c}{\partial n}, \quad \varphi = \varphi^c \ , \end{aligned} \tag{30}$$

on the hole boundary Γ (s. Fig. 1), and the conditions

$$\int_C \mathrm{d}\theta = \theta_0, \quad \int_C \mathrm{d}\vartheta = -h^* \ , \tag{31}$$

for any closed curve C enclosing the point ζ_t^* where n stands for the normal to the hole boundary, ω is an angle shown in Fig. 1, σ_1 and σ_2 are the rectangular cartesian components of the surface traction, D_n is the normal component of the surface electric displacement, ε_0 is the dielectric constant of vacuum, superscript "c" indicates the quantity associated with the hole medium, and ϑ is a heat-flux function.

The analysis to follow requires the boundary conditions (30) to be expressed in terms of the complex potentials Φ_i ($i = 1, 2, 3, t$). It is easy to show that

$$\frac{\partial F}{\partial x_1} = 2\,\mathrm{Re}\left[\sum_{i=1}^{3} \Phi_i(z_i) + d_2 \Phi_t(z_t)\right] = -\int_0^s \sigma_1\, \mathrm{d}s \ , \tag{32}$$

$$\frac{\partial F}{\partial x_2} = -2\,\mathrm{Re}\left[\sum_{i=1}^{3} \mu_i \Phi_i(z_i) + \tau d_2 \Phi_t(z_t)\right] = -\int_0^s \sigma_2\, \mathrm{d}s \ , \tag{33}$$

$$\psi = 2\,\mathrm{Re}\left[\sum_{i=1}^{3} \chi_i \Phi_i(z_i) + d_3 \Phi_t(z_t)\right] = -\int_0^s D_n\, \mathrm{d}s \ . \tag{34}$$

The substitution of Eqs. (32)–(34) into the conditions (30) yields

$$
\begin{aligned}
&2\,\mathrm{Re}\left\{\sum_{k=1}^{3}\Phi_k + d_2\Phi_t\right\} = 0, \quad 2\,\mathrm{Re}\left\{\sum_{k=1}^{3}\mu_k\Phi_k + \tau d_2\Phi_t\right\} = 0, \\
&2\,\mathrm{Re}\left\{\sum_{k=1}^{3}\chi_k\Phi_k + d_3\Phi_t\right\} = -\varepsilon_0\int_0^s \frac{\partial\varphi^c}{\partial n}\,\mathrm{d}s, \quad 2\,\mathrm{Re}\left\{\sum_{k=1}^{3}t_k\Phi_k + c_3\Phi_t\right\} = \varphi^c \ ,
\end{aligned}
\tag{35}
$$

where s is arc length of Γ. Noting that for an analytic function

$$
\frac{\partial f(x_1 + \mathrm{i}x_2)}{\partial n} = -\mathrm{i}\frac{\partial f(x_1 + \mathrm{i}x_2)}{\partial m} \ , \tag{36}
$$

where m stands for the tangent to the hole boundary (s. Fig. 1), Eq. $(35)_3$ can be further simplified as

$$
2\,\mathrm{Re}\left\{\sum_{k=1}^{3}\chi_k\Phi_k + d_3\Phi_t\right\} = 2\,\mathrm{Re}\{\mathrm{i}\varepsilon_0 F(z)\} \ , \tag{37}
$$

where

$$
\varphi^c = F(z) + \overline{F(z)} \ . \tag{38}
$$

Thus, the key task is to find an appropriate form of Φ_k $(k = 1, 2, 3, t)$ satisfying the conditions (31) and (35), which will be treated in the coming section.

3
Green's function for thermoelectroelastic hole problems

3.1
The thermal potential $\Phi_t(z_t)$

Based on the conformal mapping described above and on the concept of perturbation presented in [19], the general solution for temperature and heat-flux function can be assumed in the form

$$
\theta = 2\,\mathrm{Re}[\Phi_t'(z_t)] = 2\,\mathrm{Re}[f_0(\zeta_t) + f_1(\zeta_t)] \ , \tag{39}
$$

$$
\vartheta = -2\,\mathrm{Re}[\mathrm{i}k\Phi_t'(z_t)] = -2\,\mathrm{Re}[\mathrm{i}kf_0(\zeta_t) + \mathrm{i}kf_1(\zeta_t)] \ , \tag{40}
$$

where $k = \sqrt{k_{11}k_{22} - k_{12}^2}$, while f_0 represents the solution associated with the unperturbed thermal field, and f_1 is the function corresponding to the perturbed field of the plate. Using the heat-flux function ϑ, the heat flux h_i can be expressed as

$$
h_1 = -\vartheta_{,2}, \quad h_2 = \vartheta_{,1} \ . \tag{41}
$$

For a given loading condition, the function f_0 can be easily obtained since it is related to the solution of a homogeneous medium. When an infinite plate is subjected to a line temperature discontinuity θ_0 and a line heat source h^*, both located at $\hat{z}_t = x_{10} + \tau x_{20}$, the function f_0 can be chosen in the form

$$
f_0(\zeta_t) = q_0 \ln(\zeta_t - \zeta_t^*) \ , \tag{42}
$$

where ζ_t and ζ_t^* are related to the complex arguments z_t and $\hat{z}_t$ through the transformation function (24), and q_0 is a complex constant which can be determined from the condition (31). The substitution of Eq. (42) into Eqs. (39) and (40), and later into Eq. (31), yields

$$
q_0 = \frac{\theta_0}{4\pi\mathrm{i}} - \frac{h^*}{4\pi k} \ . \tag{43}
$$

Thus, the boundary condition $(30)_1$ requires that

$$f_1(\zeta_t) = \bar{q}_0 \ln\left(\zeta_t^{-1} - \bar{\zeta}_t^*\right) \quad . \tag{44}$$

The function Φ_t in Eq. (35) can, thus, be obtained by integrating the functions f_0 and f_1 with respect to z_t, which leads to

$$\begin{aligned} \Phi_t(z_t) = {} & a_{1t}\left[q_0 F_1\left(\zeta_t, \zeta_t^*\right) + \bar{q}_0 F_2\left(\zeta_t^{-1}, \bar{\zeta}_t^*\right)\right] + a_{2t}\left[q_0 F_2\left(\zeta_t, \zeta_t^*\right) + \bar{q}_0 F_1\left(\zeta_t^{-1}, \bar{\zeta}_t^*\right)\right] \\ & + e_{j1} a_{3t}\left[q_0 F_3\left(\zeta_t, \zeta_t^*\right) + \bar{q}_0 F_4\left(\zeta_t^{-1}, \bar{\zeta}_t^*\right)\right] + a_{4t}\left[q_0 F_4\left(\zeta_t, \zeta_t^*\right) + \bar{q}_0 F_3\left(\zeta_t^{-1}, \bar{\zeta}_t^*\right)\right] \quad , \end{aligned} \tag{45}$$

where

$$F_1\left(\zeta_t, \zeta_t^*\right) = \left(\zeta_t - \zeta_t^*\right)\left[\ln\left(\zeta_t - \zeta_t^*\right) - 1\right] \quad , \tag{46}$$

$$F_2(\zeta_t, \zeta_t^*) = (\zeta_t^{-1} - \zeta_t^{*-1}) \ln(\zeta_t - \zeta_t^*) + \zeta_t^{*-1} \ln \zeta_t \quad , \tag{47}$$

$$F_3(\zeta_t, \zeta_t^*) = (\zeta_t^j - \zeta_t^{*j})[\ln(\zeta_t - \zeta_t^*) - 1] - \zeta_t^{*j} \sum_{n=1}^{j} \frac{1}{n}\left(\frac{\zeta_t}{\zeta_t^*}\right)^n \quad , \tag{48}$$

$$F_4(\zeta_t, \zeta_t^*) = (\zeta_t^{-j} - \zeta_t^{*-j})[\ln(\zeta_t - \zeta_t^*) + \zeta_t^{*-j} \ln \zeta_t - \zeta_t^{*-j} \sum_{n=1}^{j-1} \frac{1}{n}\left(\frac{\zeta_t^*}{\zeta_t}\right)^n \quad . \tag{49}$$

3.2
The electroelastic potentials $\Phi_k(z_k)$

It can be seen from Eq. (35) that the electroelastic potentials $\Phi_k(z_k)$ should have the same order as that of $\Phi_t(z_t)$ in order to make the equality (35) be valid. Thus, possible forms of $\Phi_k(z_k)$ come from the partition of $\Phi_t(z_t)$. They are

$$\begin{aligned} f_1(z_k) = {} & \frac{a}{2}\left[q_0 F_1(\zeta_k, \zeta_t^*) + q_0 F_2(\zeta_k, \zeta_t^*) + \bar{q}_0 F_1(\zeta_k^{-1}, \bar{\zeta}_t^*) + \bar{q}_0 F_2(\zeta_k^{-1}, \bar{\zeta}_t^*)\right] \\ & + \frac{e_{j1} a \gamma}{2}\left[q_0 F_3(\zeta_k, \zeta_t^*) + q_0 F_4(\zeta_k, \zeta_t^*) + \bar{q}_0 F_3(\zeta_k^{-1}, \bar{\zeta}_t^*) + \bar{q}_0 F_4(\zeta_k^{-1}, \bar{\zeta}_t^*)\right], \\ f_2(z_k) = {} & \frac{\mathrm{i}\mu_k b}{2}\left[-q_0 F_1(\zeta_k, \zeta_t^*) + q_0 F_2(\zeta_k, \zeta_t^*) + \bar{q}_0 F_1(\zeta_k^{-1}, \bar{\zeta}_t^*) - \bar{q}_0 F_2(\zeta_k^{-1}, \bar{\zeta}_t^*)\right] \\ & + \frac{\mathrm{i} e_{j1} \mu_k b \gamma}{2}\left[q_0 F_3(\zeta_k, \zeta_t^*) - q_0 F_4(\zeta_k, \zeta_t^*) - \bar{q}_0 F_3(\zeta_k^{-1}, \bar{\zeta}_t^*) + \bar{q}_0 F_4(\zeta_k^{-1}, \bar{\zeta}_t^*)\right] \quad , \end{aligned} \tag{50}$$

where the subscripts 1 and 2 are the indices for the different possible functions.

The electroelastic potentials $\Phi_k(z_k)$ can thus be chosen as

$$\Phi_k(z_k) = f_1(z_k) q_{k1} + f_2(z_k) q_{k2} \quad , \tag{51}$$

where q_{ki} are complex numbers to be determined from Eq. (35).

The substitution of Eqs. (45) and (51) into Eq. (35), yields

$$\begin{Bmatrix} q_{11} \\ q_{21} \\ q_{31} \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ t_1^* & t_2^* & t_3^* \end{bmatrix}^{-1} \begin{Bmatrix} -d_2 \\ -\tau d_2 \\ d_3 + \mathrm{i}\varepsilon_0 c_3 \end{Bmatrix} \quad , \tag{52}$$

$$\begin{Bmatrix} q_{12} \\ q_{22} \\ q_{32} \end{Bmatrix} = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \\ \mu_1^2 & \mu_2^2 & \mu_3^2 \\ \mu_1 t_1^* & \mu_2 t_2^* & \mu_3 t_3^* \end{bmatrix}^{-1} \begin{Bmatrix} -d_2 \\ -\tau d_2 \\ d_3 + \mathrm{i}\varepsilon_0 c_3 \end{Bmatrix} \tau \quad , \tag{53}$$

where $t_k^* = \chi_k - \mathrm{i}\varepsilon_0 t_k$.

4
Interaction between a crack and a hole

To illustrate the application of the proposed Green's functions, consider an infinite piezoelectric plate with a crack of length of $2c$ and holes of various shapes subjected to traction-charge $\mathbf{t}_0$ and heat flux h_0 on the crack faces. The central point of the crack is denoted by $z_k^0 = x_{10} + \mu_k x_{20}$, and its orientation angle is denoted by α. The geometry of the configuration of the crack-hole system is shown in Fig. 2. The orientation of the crack may be arbitrary. The mathematical statement of this problem can be stated more precisely as follows:

$$h_n = h_0, \quad \mathbf{t}_n = \mathbf{t}_0 \quad \text{on crack faces} \ , \tag{54}$$

$$h_n = \mathbf{t}_n = 0 \quad \text{on the hole boundary} \ , \tag{55}$$

$$h_i = \Pi_i = 0 \quad i = 1, 2; \ \text{at infinity} \ . \tag{56}$$

The boundary conditions (54) can be satisfied by redefining the discrete Green's functions q_0 in Eq. (45) in terms of distributing Green's functions $q_0(\xi)$ defined along the crack line, $z_t = z_t^0 + \eta z_t^*$, $\hat{z}_t = z_t^0 + \xi z_t^*$, where $z_t^0 = x_{10} + \tau x_{20}$, $z_t^* = \cos\alpha + \tau \sin\alpha$. In this case, the load parameter q_0, which has appeared in Sec. 3, should be taken as $\theta_0(\xi)/4\pi\mathrm{i}$. Enforcing the satisfaction of the applied heat flux conditions on the crack faces, a system of singular integral equations for the Green's function is obtained as

$$\frac{1}{\pi}\mathrm{Re}\left[\int_{-c}^{c}\left[\frac{1}{\eta - \xi} + K_0(\eta, \xi)\right]\theta_0(\xi)\mathrm{d}\xi\right] = -\frac{2h_0}{k} \ , \tag{57}$$

where K_0 is Holder-continuous along $-c \leq \xi \leq c$ and given by

$$K_0(\eta, \xi) = -z_t^*\left[\frac{1}{z_p}\frac{\partial z_p}{\partial \zeta_t} - \frac{1}{\zeta_t(1 - \xi_t\bar{\zeta}_t^*)}\right]\frac{\partial \zeta_t}{\partial z_t} \ , \tag{58}$$

where

$$z_p = a_{1t} - \frac{a_{2t}}{\zeta_t\zeta_t^*} + \left(a_{3t} - \frac{a_{4t}}{\zeta_t^j\zeta_t^{*j}}\right)\sum_{k=0}^{j-1}\zeta_t^k\zeta_t^{*j-k-1} \ , \tag{59}$$

During the derivation of Eq. (57), the following relation has been employed:

$$\ln(\zeta_t - \zeta_t^*) = \ln(z_t - \hat{z}_t) - \ln z_p \ , \tag{60}$$

For single-valued temperature around a closed contour surrounding the whole crack, the following auxiliary condition has to be satisfied:

$$\int_{-c}^{c}\theta_0(\xi)\mathrm{d}\xi = 0 \ . \tag{61}$$

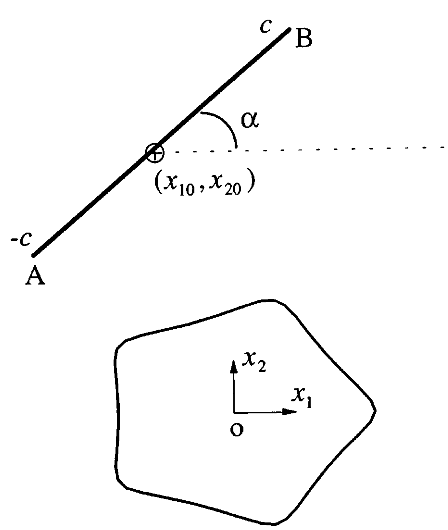

Fig. 2. Geometry of a crack-hole system

The singular integral Eq. (57) for the temperature discontinuity density, combined with Eq. (61), can be solved numerically [20]. Since the solution for the function $\theta_0(\xi)$ has a square-root singularity at both crack tips, it is more efficient for the numerical calculations by letting

$$\theta_0(\xi) = \frac{\Theta(\xi)}{\sqrt{c^2 - \xi^2}} \quad , \tag{62}$$

where $\Theta(\xi)$ is a regular function defined in a closed interval $|\xi| \leq c$. Once the function $\Theta(\xi)$ has been found, the corresponding SED can be given, from Eqs. (16) and (21), in the form

$$\Pi_1(\eta) = -\frac{1}{2\pi}\int_{-c}^{c} \mathrm{Im}\left\{\mathbf{BP}\left[\langle f'_{*1}(z_k)\rangle \mathbf{q}_1 + \langle f'_{*2}(z_k)\rangle \mathbf{q}_2\right] + \tau \mathbf{d}\Phi'_*(z_t)\right\}\theta_0(\xi)\mathrm{d}\xi \quad , \tag{63}$$

$$\Pi_2(\eta) = -\frac{1}{2\pi}\int_{-c}^{c} \mathrm{Im}\left\{\mathbf{B}\left[\langle f'_{*1}(z_k)\rangle \mathbf{q}_1 + \langle f'_{*2}(z_k)\rangle \mathbf{q}_2\right] + \mathbf{d}\Phi'_*(z_t)\right\}\theta_0(\xi)\mathrm{d}\xi \quad , \tag{64}$$

where

$$\mathbf{q}_i = \{q_{1i}\ q_{2i}\ q_{3i}\}^{\mathrm{T}}, \quad \langle f(z_k)\rangle = \mathrm{diag}[f(z_1) f(z_2) f(z_3)] \quad (i = 1, 2) \quad , \tag{65}$$

$$\begin{aligned} f_{*1}(z_k) = {} & \frac{a}{2}\left[F_1(\zeta_k, \zeta_t^*) + F_2(\zeta_k, \zeta_t^*) - F_1(\zeta_k^{-1}, \bar{\zeta}_t^*) - F_2(\zeta_k^{-1}, \bar{\zeta}_t^*)\right] \\ & + \frac{e_{j1} a\gamma}{2}\left[F_3(\zeta_k, \zeta_t^*) + F_4(\zeta_k, \zeta_t^*) - F_3(\zeta_k^{-1}, \bar{\zeta}_t^*) - F_4(\zeta_k^{-1}, \bar{\zeta}_t^*)\right], \\ f_{*2}(z_k) = {} & \frac{\mathrm{i}\mu_k b}{2}\left[-F_1(\zeta_k, \zeta_t^*) + F_2(\zeta_k, \zeta_t^*) - F_1(\zeta_k^{-1}, \bar{\zeta}_t^*) + F_2(\zeta_k^{-1}, \bar{\zeta}_t^*)\right] \\ & + \frac{\mathrm{i}e_{j1}\mu_k b\gamma}{2}\left[F_3(\zeta_k, \zeta_t^*) - F_4(\zeta_k, \zeta_t^*) + F_3(\zeta_k^{-1}, \bar{\zeta}_t^*) - F_4(\zeta_k^{-1}, \bar{\zeta}_t^*)\right] \quad , \end{aligned} \tag{66}$$

$$\begin{aligned} \Phi_*(z_t) = {} & a_{1t}\left[F_1(\zeta_t, \zeta_t^*) - F_2(\zeta_t^{-1}, \bar{\zeta}_t^*)\right] + a_{2t}\left[F_2(\zeta_t, \zeta_t^*) - F_1(\zeta_t^{-1}, \bar{\zeta}_t^*)\right] \\ & + e_{j1} a_{3t}\left[F_3(\zeta_t, \zeta_t^*) - F_4(\zeta_t^{-1}, \bar{\zeta}_t^*)\right] + a_{4t}\left[F_4(\zeta_t, \zeta_t^*) - F_3(\zeta_t^{-1}, \bar{\zeta}_t^*)\right] \quad , \end{aligned} \tag{67}$$

Thus, the traction-charge vector on the crack faces induced by h_0 is of the form

$$\mathbf{t}_n^0(\eta) = -\Pi_1(\eta)\sin\alpha + \Pi_2(\eta)\cos\alpha \quad , \tag{68}$$

Generally, $\mathbf{t}_n^0(\eta) \neq \mathbf{t}_0$ on the crack faces $|\eta| \leq c$. To satisfy the condition $(54)_2$ on the crack faces, we must superpose a solution of the corresponding isothermal problem with a traction-charge vector equal and opposite to $(\mathbf{t}_n^0 - \mathbf{t}_0)$ in the range $|\eta| \leq c$. The electroelastic solution for a single dislocation of strength $\mathbf{b}_0$ in an infinite plate with a hole is thus required. The solutions have been given in the literature [16]. They are

$$\Pi_1(\eta) = -\frac{1}{\pi}\mathrm{Im}\left\{\mathbf{BP}\left[\left\langle\frac{1}{\zeta_m - \zeta_m^*}\right\rangle \mathbf{G}_* - \sum_{m=1}^{\infty} m\langle\zeta_k^{-n-1}\rangle(\mathbf{E}_m\mathbf{G}_* - \mathbf{F}_m\bar{\mathbf{G}}_*)\right]\right\}\mathbf{b}_0 \quad , \tag{69}$$

$$\Pi_2(\eta) = \frac{1}{\pi}\mathrm{Im}\left\{\mathbf{B}\left[\left\langle\frac{1}{\zeta_k - \zeta_k^*}\right\rangle \mathbf{G}_* - \sum_{m=1}^{\infty} m\langle\zeta_k^{-m-1}\rangle(\mathbf{E}_m\mathbf{G}_* - \mathbf{F}_m\bar{\mathbf{G}}_*)\right]\right\}\mathbf{b}_0 \quad , \tag{70}$$

where

$$\mathbf{G}_* = \left(\bar{\mathbf{A}}^{-1}\mathbf{A} - \bar{\mathbf{B}}^{-1}\mathbf{B}\right)^{-1}\bar{\mathbf{A}}^{-1} \quad , \tag{71}$$

$$\mathbf{E}_i = \left(\mathbf{P}_i - \mathbf{Q}_i\bar{\mathbf{P}}_i^{-1}\bar{\mathbf{Q}}_i\right)^{-1}\left(\mathbf{E}_i^* - \mathbf{Q}_i\bar{\mathbf{P}}_i^{-1}\bar{\mathbf{F}}_i^*\right) \quad , \tag{72}$$

$$\mathbf{F}_i = \left(\mathbf{P}_i - \mathbf{Q}_i\bar{\mathbf{P}}_i^{-1}\bar{\mathbf{Q}}_i\right)^{-1}\left(\mathbf{F}_i^* - \mathbf{Q}_i\bar{\mathbf{P}}_i^{-1}\bar{\mathbf{E}}_i^*\right) \quad , \tag{73}$$

$$\mathbf{P}_i = \begin{bmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \chi_{1i} & \chi_{2i} & \chi_{3i} \end{bmatrix}, \quad \mathbf{Q}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t_1^* & t_2^* & t_3^* \end{bmatrix} , \tag{74}$$

$$\mathbf{E}_i^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{1i}^* & b_{2i}^* & b_{3i}^* \end{bmatrix}, \quad \mathbf{F}_i^* = \begin{bmatrix} -\bar{e}_{1i} & -\bar{e}_{2i} & -\bar{e}_{3i} \\ -\bar{e}_{1i}\bar{\mu}_1 & -\bar{e}_{2i}\bar{\mu}_2 & -\bar{e}_{3i}\bar{\mu}_3 \\ b_{1i}^{**} & b_{2i}^{**} & b_{3i}^{**} \end{bmatrix} , \tag{75}$$

in which, s. [16],

$$e_{km} = -\frac{\zeta_k^{*-m}}{m}, \quad b_{ki}^* = \left[\chi_k - \frac{\mathrm{i}\varepsilon_0\left(1 + r_{in}^{4i}\right)}{1 - r_{in}^{4i}} t_k\right] e_{ki}, \quad b_{ki}^{**} = \frac{2\mathrm{i}\varepsilon_0 r_{in}^{2i}}{1 - r_{in}^{4i}} \bar{t}_k \bar{e}_{ki} , \tag{76}$$

$$r_{in} = \begin{cases} \sqrt{\frac{1-e}{1+e}} & \text{for} \quad j = 1 \\ \sqrt[j+1]{j\gamma} & \text{for} \quad j \neq 1 , \end{cases} \tag{77}$$

Using Eqs. (68)–(70), the boundary condition $(54)_2$ can be expressed by

$$\frac{1}{\pi}\mathrm{Im}\left\{\int_{-c}^{c}\left[\frac{\mathbf{BG}_*}{\eta - \xi} + \mathbf{K}_0(\eta, \xi)\right]\mathbf{b}_0(\xi)\mathrm{d}\xi\right\} = \mathbf{t}_0 - \mathbf{t}_n^0(\eta) , \tag{78}$$

where

$$\mathbf{K}_0(\eta, \xi) = -\mathbf{B}\left[\left\langle z_k^* \frac{\partial z_{kp}/\partial \zeta_k}{z_{kp}} \frac{\partial \zeta_k}{\partial z_k}\right\rangle \mathbf{G}_* + \sum_{m=1}^{\infty} m \left\langle z_k^* \zeta_k^{-m-1} \frac{\partial \zeta_k}{\partial z_k}\right\rangle (\mathbf{E}_m \mathbf{G}_* - \mathbf{F}_m \mathbf{G}_*)\right] , \tag{79}$$

and z_j^*, η and ξ are defined by

$$z_j^* = \cos\alpha + \mu_j \sin\alpha , \tag{80}$$

$$z_{kp} = a_{1k} - \frac{a_{2k}}{\zeta_k \zeta_k^*} + \left(a_{3k} - \frac{a_{4k}}{\zeta_k^j \zeta_k^{*j}}\right) \sum_{m=0}^{j=1} \zeta_k^m \zeta_k^{*j-m-1} , \tag{81}$$

Here $\mathbf{K}_0(\eta, \xi)$ is a kernel function of the singular integral equations, and it is Holder-continuous along $-c \leq \xi \leq c$.

For single-valued displacements and the electric potential around a closed contour surrounding the whole crack, the following conditions have also to be satisfied:

$$\int_{-c}^{c} \mathbf{b}_0(\xi)\mathrm{d}\xi = 0 . \tag{82}$$

As it was done previously, let

$$\mathbf{b}_0(\xi) = \frac{\Theta(\xi)}{\sqrt{c^2 - \xi^2}} . \tag{83}$$

Once the function $\Theta(\xi)$ has been found from Eqs. (78) and (82), the stresses and electric displacement, $\Pi_n(\eta) = \{\sigma_{nm}\sigma_{nn}D_n\}^{\mathrm{T}}$ in a local coordinate corresponding to the crack line, can be expressed in the form

$$\Pi_n(\eta) = \frac{1}{\pi}\Omega(\alpha)\mathrm{Im}\left\{\int_{-c}^{c}\left[\frac{\mathbf{BG}_*}{\eta - \xi} + \mathbf{K}_0(\eta, \xi)\right]\mathbf{b}_0(\xi)\mathrm{d}\xi\right\} , \tag{84}$$

where the (3×3)-matrix $\Omega(\alpha)$, whose components are the cosine of the angle between the local coordinates and global coordinates, is in the form

$$\Omega(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \ . \tag{85}$$

Using Eq. (84), we can evaluate the stress intensity factors $\mathbf{K}^* = (K_{II},\ K_I,\ K_D)^{\mathrm{T}}$ at the tips, e.g. at the right tip $\xi = c$ of the crack, by the following definition:

$$\mathbf{K}^* = \lim_{\xi \to c^*} \sqrt{2\pi(\xi - c)}\Pi_n(\xi) \ , \tag{86}$$

Combined with the results of Eq. (84), one then leads to

$$\mathbf{K}^* \approx \sqrt{\frac{\pi}{c}}\Omega(\alpha)\mathrm{Im}(\mathbf{BG}_*)\Theta(c) \ . \tag{87}$$

5
Numerical examples

As an illustration, consider a piezoelectric ceramic ($BaTiO_3$) plate with a crack of length $2c$ and a square hole, comp. Fig. 2, in which $x_{10} = 0,\ x_{20} = 3c,\ j = 3,\ e = 1,\ a = 1.8c$ and $\gamma = 0.2c$. The material properties of the plate can be found in [15] as

$$\begin{aligned} &c_{11} = 150\,\mathrm{GPa},\ c_{12} = 66\,\mathrm{GPa},\ c_{22} = 146\,\mathrm{GPa},\ c_{33} = 44\,\mathrm{GPa}, \\ &\alpha_{11} = 8.53 \times 10^{-6}\,\mathrm{K}^{-1},\ \alpha_{22} = 1.99 \times 10^{-6}\,\mathrm{K}^{-1},\ \lambda_2 = 0.133 \times 10^5\,\mathrm{NC}^{-1}\,\mathrm{K}^{-1}, \\ &e_{21} = -4.35\,\mathrm{Cm}^{-2},\ e_{22} = 17.5\,\mathrm{Cm}^{-2},\ e_{13} = 11.4\,\mathrm{Cm}^{-2},\ \kappa_{11} = 1115\,\kappa_0, \\ &\kappa_{22} = 1260\,\kappa_0,\ \kappa_0 = 8.85 \times 10^{-12}\,\mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}\ (=\text{permitivity of the free space}) \ . \end{aligned} \tag{88}$$

Since the values of the coefficient of heat conduction for $BaTiO_3$ could not be found in the literature, the values $k_{22}/k_{11} = 1.5,\ k_{12} = 0$ and $k_{11} = 1$ W/mK are assumed.

Figure 3 shows the numerical results for the coefficients of stress intensity factors β_i versus the crack orientation α, where β_i are defined by

$$\begin{aligned} K_I(B) &= \frac{1}{k}h_{20}c\sqrt{\pi c}\gamma_{22}\beta_1(\alpha), \\ K_{II}(B) &= \frac{1}{k}h_{20}c\sqrt{\pi c}\gamma_{11}\beta_2(\alpha), \\ K_D(B) &= \frac{1}{k}h_{20}c\sqrt{\pi c}g_2\beta_D(\alpha) \ , \end{aligned} \tag{89}$$

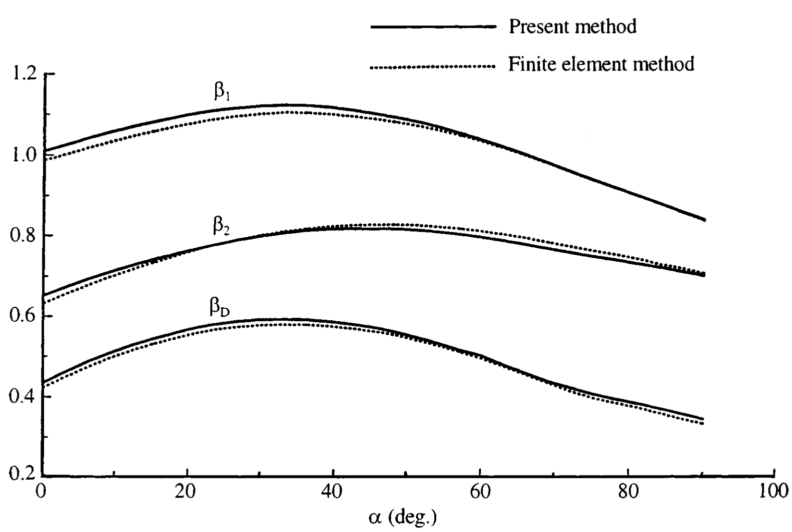

Fig. 3. Coefficients β_i $(i = 1, 2, D)$, comp. Eqs. (89), vs crack orientation α

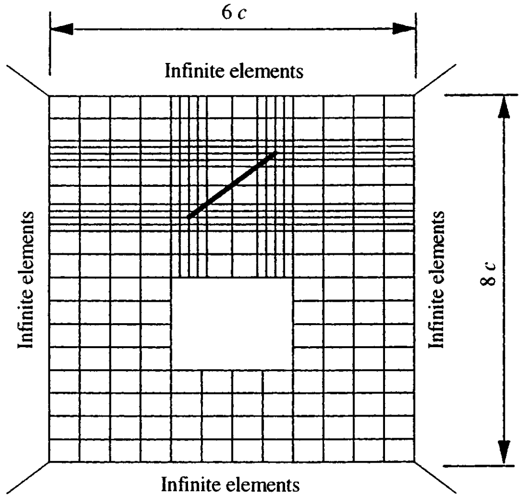

Fig. 4. A typical mesh for finite element analysis

Numerical results for such a problem are not yet available in the literature. For comparison, the finite element method (FEM) is used to obtain the corresponding results. In the FEM analysis, the configuration of a particular element mesh is shown in Fig. 4. It can be seen from Fig. 3 that all the coefficients β_i $(i = 1, 2, D)$ are not very sensitive to the crack orientation α, although they slightly vary with it. It is also found from Fig. 3 that the numerical results obtained from the two models (FEM and the proposed method) are in good agreement.

6
Conclusion

The problem of a hole embedded in a transversely isotropic piezoelectric solid subjected to thermal and electroelastic loads has been addressed within the framework of in-plane electroelastic interactions. A unified analytical thermoelectroelastic Green's function for the hole problem has been derived through the use of the Lekhnitskii's formalism, conformal mapping and exact electric boundary conditions. Using the solution developed, a system of singular integral equations for the unknown temperature discontinuity and EDEP dislocation defined on crack faces has been developed to study the interaction between a crack and a hole. Numerical results of the SED intensity factors for an infinite plate with one crack and a square hole are presented to illustrate the application of the proposed formulation. The results show that all the coefficients β_i $(i = 1, 2, D)$ of the SED intensity factors are not very sensitive to the crack orientation α, but slightly vary with it. It is also found that the numerical results obtained from the two model (FEM and the proposed method) are in good agreement.

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