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n-sided polygonal hybrid finite elements with unified fundamental solution kernels for topology optimization



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ABSTRACT

In topology optimization, the optimized design can be obtained based on spatial discretization of design domain using natural polygonal finite elements to reduce the influence of mesh geometry on topology optimization solutions. However, the natural polygonal finite elements require separate interpolants for each type of elements and involve troublesome domain integrals. In this study, an alternative n-sided polygonal hybrid finite element possessing multiple-node connection is formulated in a unified form to compress the checkerboard patterns caused by numerical instability in topology optimization. Different from the natural polygonal finite elements, the present polygonal hybrid finite elements involve two sets of independent displacement fields. The intra-element displacement field defined inside the element is approximated by the linear combination of the fundamental solution of the problem to achieve the purpose of the local satisfaction of the governing equations of the problem, but not the specific boundary conditions and the inter-element continuity conditions. To overcome such drawback, the inter-element displacement field defined over the entire element boundary is independently approximated by means of the conventional shape function interpolation. As a result, only line integrals along the element boundary are involved in the computation, whose dimension is reduced by one compared to the domain integrals in the natural polygonal finite elements, and more importantly, allowing us to flexibly construct any polygons from Voronoi tessellations in discretizing complex design domains using same fundamental solution kernels. Numerical results obtained indicate that the present *n*-sided polygonal hybrid finite elements can produce more accurate displacement solutions and smaller mean compliance, compared to the standard finite elements and the natural polygonal finite elements.

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1. Introduction

Topology optimization which optimizes the layout of continuum structure by adjusting material distribution within the design region is an effective tool to improve light-weight structure and design performance of many engineering applications such as aeronautics and aerospace engineering [1], additive manufacturing [2,3] and biomaterial design [4]. Currently, the commonly-used topology optimization theories include the homogenization method [5,6], the solid isotropic microstructure with penalization (SIMP) method [7–10], the bi-directional evolutionary structural optimization (BESO) method [11–14], and the level set method [15–18].

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In literature, most theories of topology optimization require inevitably the approximated solutions of displacements in each step of iteration. The prevailing numerical approach for such purpose is the conventional finite element method (FEM) in Galerkin form, in which the suitable interpolants are required to approximate the displacement field defined over the entire element domain. In practical applications, the shapes of basic finite element for two-dimensional analysis are usually limited to three-node linear triangle and four-node quadrilateral, which only possess one-node connection and are unfavourable to suppress checkerboard patterns in topology optimization [19]. Different from the basic finite elements, unstructured natural finite elements of polygons possessing higher degrees of geometric isotropy, i.e. plane polygonal elements with more than 4 sides, were developed based on Laplace/Wachspress basis functions over the past decade and can provide more flexible spatial discretization to accurately describe complex domains without introducing numerical instabilities [20–23]. Moreover, it was found that the employment of the natural polygonal finite elements has the unique benefit of suppressing the formation of checkerboard layout [19,24–27]. However, it should be mentioned that both the domain integral over arbitrarily-shaped polygonal region and the construction of interpolants for each type of polygons are troublesome for the implementation of the natural polygonal finite elements [23,28].

As an alternative to the unstructured natural polygonal finite element, the polygonal hybrid finite elements based on fundamental solution kernels have been successfully developed for the solutions of heat transfer in composite [29,30] and elasticity [31–33]. In this study, we consider the *n*-sided polygonal hybrid finite element mesh constructed from Voronoi tessellations for the discretization of two-dimensional design domain to improve the design of two-dimensional linear elastic continuum structures under the practical constraints. For each such element, there are two independent displacement fields: one is the intra-element displacement field defined inside the element and another is the inter-element displacement field defined over the element boundary. The intra-element displacement field is approximated in the form of linear combination of fundamental solutions of the problem so that it a priori satisfies the governing equations of the problem, but not, generally, the applied displacement and traction boundary conditions and the continuity conditions on the interface of neighbouring elements. Meanwhile, the approximated inter-element displacement field is introduced by the one-dimensional shape function interpolation to enforce the conformity requirement between adjacent elements. These two fields are connected by a double-variable hybrid functional in a weak sense to produce the solving system of equations in terms of nodal displacements, the so-called stiffness equation. It is necessary to note that the use of fundamental solutions of the problem for the approximation of intra-element displacement field makes the stiffness equation only involve the computation of line integrals along the element boundary, which are one dimension less than domain integrals in the standard finite elements and the natural polygonal finite elements. More importantly, it is allowed for the construction of hybrid finite elements in arbitrary polygonal forms more feasible so as to provide more generalized and simple applications for mesh discretization in design domain. Moreover, all types of polygonal hybrid finite elements are formulated in a unified form by same fundamental solution kernels, different to the natural polygonal finite elements. Also, the finite element discretization of design domain means that one can define different material properties for different elements, which is important for density-based topology optimization methods such as the BESO and SIMP methods. Besides, the induced stiffness matrix keeps symmetric and sparse in the computation, so the present polygonal elements are easily implemented into conventional finite elements. However, it should be noted that the dependence of fundamental solution for the present polygonal hybrid finite element may limit its applications to those problems without explicit expressions of fundamental solutions.

The paper is organized as follows: the methodology of the *n*-sided polygonal hybrid finite elements with fundamental solution kernels is formulated and verified by the Cook's problem in Section 2. In Section 3, the BESO theory used in the topology optimization is briefly described. The corresponding numerical test is given in Section 4 to assess the performance of the present polygonal hybrid finite elements in topology optimization by the benchmark problem. Finally, Section 5 presents some conclusions of this study.

2. The *n*-sided polygonal hybrid finite element scheme

In topology optimization, the information of the nodal displacement is commonly required for the computation of elementary sensitivity at each step of optimization iteration. To do so, the *n*-sided polygonal hybrid finite element (PHFE) formulation with fundamental solution kernels for plane elastic structures is presented in this section.

2.1. Formulation

- T

Considering a two-dimensional linear isotropic elastic problem defined in the design domain Ω bounded by the boundary Γ , its basic equations can be written as [34,35]

1	$\mathbf{L}^{1}\sigma = 0$	in Ω
	$\sigma = \mathbf{D}\varepsilon$	in Ω
ł	$\varepsilon = Lu$	in Ω
	$\mathbf{t} = \mathbf{\overline{t}}$	$on\Gamma_t$
	$\mathbf{u} = \mathbf{\bar{u}}$	$on\Gamma_u$

where **L** and **D** are respectively the differential operator matrix and the constitutive matrix for the stress-strain relation (see most of the textbooks of elasticity for details), **u** is the displacement vector, \boldsymbol{e} is the strain vector, $\boldsymbol{\sigma}$ is the stress vector,

(1)



Fig. 1. Illustration of (a) intra- and inter-element displacement fields for the present *n*-sided polygonal hybrid finite elements, (b) locations of source points and (c) Geometry interpolation from one side of the physical element to the mapped line element.

 $\mathbf{t} = \mathbf{A}\boldsymbol{\sigma}$ is the traction vector, and \mathbf{A} is the matrix consisting of outward unit normal components at a point on the boundary. $\mathbf{\tilde{u}}$ and $\mathbf{\tilde{t}}$ denote the prescribed values on the essential boundary Γ_t and the natural boundary Γ_t , respectively.

The design domain Ω can be discretized by a finite number of polygonal elements as shown in Fig. 1a in the context of topology optimization. The weak form of the constrained hybrid functional for a particular polygonal hybrid finite element,

say e, can be written as follows [36,37]

$$\Pi_{me} = \frac{1}{2} \int_{\Omega_{e}} (\mathbf{L}\mathbf{u})^{\mathrm{T}} \mathbf{D}(\mathbf{L}\mathbf{u}) \mathrm{d}\Omega - \int_{\Gamma_{e}^{t}} \mathbf{\tilde{t}}^{\mathrm{T}} \mathbf{\tilde{u}} \mathrm{d}\Gamma + \int_{\Gamma_{e}} \mathbf{t}^{\mathrm{T}} (\mathbf{\tilde{u}} - \mathbf{u}) \mathrm{d}\Gamma$$
(2)

where **u** is the intra-element displacement field defined within the element domain Ω_e , $\tilde{\mathbf{u}}$ is the compatible inter-element displacement field defined on the element boundary Γ_e , **t** is the traction field on Γ_e caused by the intra-element displacement field **u** and $\Gamma_e^t = \Gamma_t \cap \Gamma_e$ is the traction boundary of *e*th element.

In the formulation of PHFE, the intra-element displacement field at any point **x** in the interior of the element *e* is approximated by a linear combination of fundamental solutions centered at a series of source points \mathbf{x}_m^s ($m = 1, \dots, M$) for the local natural satisfaction of the given governing partial differential equations and not the boundary conditions and continuity conditions applied on its boundary Γ_e , that is,

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}(\mathbf{x})\mathbf{c}_{e}, \quad \mathbf{x} \in \Omega_{e}$$
(3)

where

 $\mathbf{c}_e = \left\{ c_{11} \quad c_{21} | \cdots | c_{1M} \quad c_{2M} \right\}^{\mathrm{T}}$ $\tag{4}$

is the unknown vector for the source intensity, and

$$\mathbf{N} = \begin{bmatrix} u_{11}^*(\mathbf{x}, \mathbf{x}_1^s) & u_{21}^*(\mathbf{x}, \mathbf{x}_1^s) | \cdots | u_{11}^*(\mathbf{x}, \mathbf{x}_M^s) & u_{21}^*(\mathbf{x}, \mathbf{x}_M^s) \\ u_{12}^*(\mathbf{x}, \mathbf{x}_1^s) & u_{22}^*(\mathbf{x}, \mathbf{x}_1^s) | \cdots | u_{12}^*(\mathbf{x}, \mathbf{x}_M^s) & u_{22}^*(\mathbf{x}, \mathbf{x}_M^s) \end{bmatrix}$$
(5)

is the matrix consisting of the displacement fundamental solution $u_{li}^*(\mathbf{x}, \mathbf{x}_m^s)(l, i = 1, 2)$ which indicates the induced displacement component along the *i*-direction at the field point \mathbf{x} due to the unit force along the *i*-direction at the source point \mathbf{x}_m^s , and can be found in most of textbooks on boundary element method (BEM), e.g. see [38].

As indicated in Fig. 1b, these source points are distributed on the pseudo boundary geometrically similar to the element physical boundary Γ_e , as was done in the classic method of fundamental solution [39-41], and their locations on the pseudo boundary can be generated by a simple geometrical expression with a dimensionless parameter $\gamma > 0$

$$\mathbf{x}^{s} = \mathbf{x}_{b} + \gamma \left(\mathbf{x}_{b} - \mathbf{x}_{c} \right) \tag{6}$$

where \mathbf{x}^{s} , \mathbf{x}_{b} , \mathbf{x}_{c} represent the coordinates of source point, node and centre of the element, respectively.

Correspondingly, the interior stress field σ and the traction field **t** on Γ_e can be naturally derived by the assumed intraelement displacement field using the classic elastic theory, i.e.,

$$\sigma = \mathbf{D}(\mathbf{L}\mathbf{u}) = \mathbf{T}\mathbf{c}_{e} \tag{7}$$

$$\mathbf{t} = \mathbf{A}\boldsymbol{\sigma} = \mathbf{A}\mathbf{T}\mathbf{c}_e = \mathbf{Q}\mathbf{c}_e \tag{8}$$

where $\mathbf{T} = \mathbf{DLN}$ and $\mathbf{Q} = \mathbf{AT}$.

In order to enforce the conformity of the displacement field on the common interface of adjacent elements and apply the specific boundary conditions, the inter-element displacement field $\tilde{\mathbf{u}}$ at any point $\mathbf{x} \in \Gamma_e$ can be interpolated with respect to the unknown elementary nodal displacement vector \mathbf{d}_e in the form of

$$\tilde{\mathbf{u}}(\mathbf{x}) = \tilde{\mathbf{N}}(\mathbf{x})\mathbf{d}_e, \quad \mathbf{x} \in \Gamma_e \tag{9}$$

where $\tilde{\mathbf{N}}$ denotes the matrix consisting of lower-dimensional shape functions, i.e. one-dimensional shape functions for twodimensional problems under consideration. For instance, for the polygonal element #2 consisting of six global nodes, as shown in Fig. 1a, when \mathbf{x} locates at its second side connecting the global nodes 1 and 6, the matrix $\tilde{\mathbf{N}}_e$ and the vector \mathbf{d}_e related to this physical line element can be written as

$$\tilde{\mathbf{N}}_{e} = \begin{bmatrix} 0 & \tilde{N}_{1} & \tilde{N}_{2} & 0 & 0 \end{bmatrix}, \mathbf{d}_{e} = \begin{bmatrix} \mathbf{d}_{10} & \mathbf{d}_{1} & \mathbf{d}_{6} & \mathbf{d}_{7} & \mathbf{d}_{8} & \mathbf{d}_{9} \end{bmatrix}^{\mathrm{T}}$$
(10)

with

$$\tilde{N}_1(\xi) = \frac{1-\xi}{2}, \quad \tilde{N}_2(\xi) = \frac{1+\xi}{2} \quad (-1 \le \xi \le 1)$$
(11)

In Eq. (11), the linear shape functions $\tilde{N}_1(\xi)$ and $\tilde{N}_2(\xi)$ are also used for the element geometry interpolation through the following relation

$$\mathbf{x}(\xi) = N_1(\xi)\mathbf{x}_1 + N_2(\xi)\mathbf{x}_6 \tag{12}$$

which maps the natural ξ coordinate of any point in the mapped line element to the actual **x** coordinate of the point in the physical line element, as indicated in Fig. 1c.

After applying the integration by parts and the divergence theorem to the hybrid functional Π_{me} and considering the fact that the intra-element displacement field a priori satisfies the governing equations of the problem, we finally have

$$\Pi_{me} = -\frac{1}{2} \int_{\Gamma_e} \mathbf{u}^{\mathrm{T}} \mathbf{t} \mathrm{d}\Gamma - \int_{\Gamma_e^{\mathrm{t}}} \mathbf{\tilde{t}}^{\mathrm{T}} \mathbf{\tilde{u}} \mathrm{d}\Gamma + \int_{\Gamma_e} \mathbf{t}^{\mathrm{T}} \mathbf{\tilde{u}} \mathrm{d}\Gamma$$
(13)

Algorithm 1

The overall structure of the present method implementation.

- Input: Discretization data
- 1: For each element do
- 1a: Compute element stiffness matrix
- 1b: Compute element load vector
- 2: Assemble the element stiffness matrix and load vector into the global ones
- 3: Introduce the specified boundary conditions
- 4: Solve the resulting matrix problem to obtain nodal displacements
- 5: Evaluate displacements and stresses in the element
- Output: Displacement and stress solutions

which includes three-line integrals over the element boundary only.

Subsequently, substituting Eqs. (3), (8) and (9) into the reduced hybrid functional (13) gives

$$\Pi_{me} = -\frac{1}{2} \mathbf{c}_e^{\mathrm{T}} \mathbf{H}_e \mathbf{c}_e - \mathbf{d}_e^{\mathrm{T}} \mathbf{g}_e + \mathbf{c}_e^{\mathrm{T}} \mathbf{G}_e \mathbf{d}_e \tag{14}$$

where

$$\mathbf{H}_{e} = \int_{\Gamma_{e}} \mathbf{Q}^{\mathrm{T}} \mathbf{N} \mathrm{d}\Gamma, \, \mathbf{G}_{e} = \int_{\Gamma_{e}} \mathbf{Q}^{\mathrm{T}} \tilde{\mathbf{N}} \mathrm{d}\Gamma, \, \mathbf{g}_{e} = \int_{\Gamma_{e}^{t}} \tilde{\mathbf{N}}^{\mathrm{T}} \tilde{\mathbf{t}} \mathrm{d}\Gamma$$
(15)

In the practical numerical implementation, the matrices \mathbf{H}_e and \mathbf{G}_e can be evaluated by means of side-by-side Gaussian quadrature over the element boundary and the vector \mathbf{g}_e can be evaluated by the same numerical quadrature scheme along the specified traction boundary, that is,

$$\mathbf{H}_{e} = \sum_{s=1}^{ns} \left[\int_{-1}^{1} \mathbf{Q}^{\mathsf{T}}(\mathbf{x}(\xi)) \mathbf{N}(\mathbf{x}(\xi)) J(\xi) d\xi \right] = \sum_{s=1}^{ns} \sum_{k=1}^{ng} w_{k} \mathbf{Q}^{\mathsf{T}}(\mathbf{x}(\xi_{k})) \mathbf{N}(\mathbf{x}(\xi_{k})) J(\xi_{k})
\mathbf{G}_{e} = \sum_{s=1}^{ns} \left[\int_{-1}^{1} \mathbf{Q}^{\mathsf{T}}(\mathbf{x}(\xi)) \tilde{\mathbf{N}}(\mathbf{x}(\xi)) J(\xi) d\xi \right] = \sum_{s=1}^{ns} \sum_{k=1}^{ng} w_{k} \mathbf{Q}^{\mathsf{T}}(\mathbf{x}(\xi_{k})) \tilde{\mathbf{N}}(\mathbf{x}(\xi_{k})) J(\xi_{k})
\mathbf{g}_{e} = \int_{-1}^{1} \tilde{\mathbf{N}}^{\mathsf{T}}(\mathbf{x}(\xi)) \tilde{\mathbf{t}} J(\xi) d\xi = \sum_{k=1}^{ng} w_{k} \tilde{\mathbf{N}}^{\mathsf{T}}(\mathbf{x}(\xi_{k})) \tilde{\mathbf{t}} J(\xi_{k})$$
(16)

where *ns* is the number of sides of polygonal element, *ng* is the number of Gaussian quadrature points on each element side, w_k is the weighting factors for the *k*th Gaussian points ξ_k and $J(\xi_k)$ is the corresponding Jacobian coefficient to the geometrical mapping equation given in (12).

The stationary of Π_{me} in Eq. (14) with respect to the unknown coefficient vector \mathbf{c}_e and the nodal displacement vector \mathbf{d}_e yields the following optional relationship between \mathbf{c}_e and \mathbf{d}_e

$$\mathbf{c}_e = \mathbf{H}_e^{-1} \mathbf{G}_e \mathbf{d}_e \tag{17}$$

and the element displacement-load equation, or the so-called element stiffness equation

$$\mathbf{K}_{e}\mathbf{d}_{e}=\mathbf{g}_{e} \tag{18}$$

where

$$\mathbf{K}_{e} = \mathbf{G}_{e}^{\mathrm{T}} \mathbf{H}_{e}^{-1} \mathbf{G}_{e} \tag{19}$$

is the symmetric element stiffness matrix relating the force vector to the displacement vector.

After assembling the element stiffness matrix element by element, the global stiffness equation can be derived as

$$\mathbf{K}\mathbf{u} = \mathbf{f} \tag{20}$$

where \mathbf{f} is the nodal force vector, \mathbf{u} is the nodal displacement vector, and \mathbf{K} is the sparse and symmetric global stiffness matrix.

From the procedure above, it can be seen that the employment of fundamental solutions of the problem in the intraelement displacement field can convert the domain integral in the hybrid functional into the boundary line integrals, which obviously decreases the complexity of the integration by one dimension and can be more easily evaluated in practice than the domain integrals, especially in the natural polygonal finite elements [28]. Furthermore, such feature allows for greater flexibility in constructing general (possibly non-convex) polygonal elements of arbitrary shapes with same kernel functions, the fundamental solutions of problem here, for discretizing complex geometry domains. This means that we can establish a family of *n*-sided polygonal hybrid finite element ($n \ge 3$) in a unified form for topology optimization, as shown in Fig. 2. Conversely, the natural polygonal finite element requires different interpolants for different element types. Besides, it is indicated from the solution procedure that the overall structure of the implementation of the present method given in Algorithm 1 is same as that of the standard FEM. The major difference between them is the way of evaluating element stiffness matrices.



Fig. 2. Family tree of *n*-sided polygonal hybrid finite element.



Fig. 3. Cook's problem.

2.2. Verification

To validate the present polygonal hybrid finite element in a simple way, the Cook's problem in plane stress state [19,42] is taken into consideration. It is assumed that the left edge of the beam is fixed without rotation constraint and the right edge is subjected to uniformly distributed shearing loads, as shown in Fig. 3. For this problem, the deflection at the midpoint of

Table 1	
Comparison of deflection results at the midpoint of the loaded edge for G	Cook's problem.

Number of elements	Polygonal hybrid FE	Quadrilaterial FE	Polygonal FE	Reference value
19	23.116	25.183	21.9240	
65	23.801	24.195	23.4488	23.96
148	23.880	24.087	1	

the loaded edge is evaluated by several meshes with the conventional quadrilateral finite elements implemented by ABAQUS and the present n-sided polygonal hybrid finite elements to assess the performance of the present elements. The results are compared with the available reference value 23.96 [42]. Besides, the results by the present elements are also compared to those in Reference [19] by the 16 and 64 natural polygonal finite elements with Laplace interpolants.

Results in Table 1 indicate that the three types of elements can produce better results with the increase of number of elements. However, the present polygonal hybrid finite elements are not as stiff as the quadrilateral elements and the natural polygonal finite elements and behave the best accuracy, especially with coarser meshes. Similar conclusions can be found in literature [30,31,43]. Since the checkerboard patterns in topology optimization are mainly caused by the artificial stiffness in the conventional finite element approximation, so the present polygonal hybrid finite elements are expected to be able to compress such pathologies.

Besides, it is observed that both the present polygonal hybrid finite elements and the natural polygonal finite elements have more nodes than the conventional quadrilateral finite elements with the same number of elements. For example, 19, 65 and 148 polygonal elements have 39, 132 and 296 nodes, respectively, but the same number of quadrilateral elements just have 28, 83 and 175 nodes, respectively. This can be attributed to the multi-node connection of polygonal mesh. Thus, the polygonal elements inevitably lead to the bigger size of the solving system than that for the quadrilateral elements. Here, the results with the same number of degrees of freedom (DOFs) generated by the standard quadrilateral elements and the present hybrid polygonal elements are also presented for comparison. The related numbers of DOFs are 78, 264 and 592, which correspond to 39, 132 and 296 nodes, respectively. It is observed that total of 19, 65, 148 hybrid polygonal elements shown in Fig. 4 and 26, 111, 261 quadrilateral elements generated in ABAQUS are involved to produce these DOFs. Correspondingly, the resulted midpoint deflections are 23.116, 23.801, 23.880 for the present hybrid polygonal elements and 24.639, 23.995, 24.022 for the quadrilateral elements, respectively. It is found that similar accuracy is obtained for the present hybrid polygonal element and the standard quadrilateral element under the same number of DOFs, which is obviously higher than that for the natural polygonal elements, as indicated in Table 1. However, one has to use more quadrilateral elements to fulfil such high accuracy.

Additionally, to demonstrate the reasonable choice of the parameter γ in Eq. (6), we investigate the variation of the deflection at the midpoint of the loaded edge when the value of the parameter γ changes from 0.1 to 20. The results in Fig. 5 indicate that there is a large range to produce stable results for the three polygonal meshes given in Fig. 4. Here, we take $\gamma = 10$ for the following computation, unless otherwise stated.

Finally, the capability of the present hybrid finite element formulation for handling the non-convex elements is assessed. Fig. 6 depicts the non-convex polygonal element discretization of the computational domain. Total 12 elements and 35 nodes are included. Correspondingly, the deflection at the midpoint of the loaded edge is 23.143, which is very close to that obtained from the convex polygonal element discretization.

2.3. Comparison to the virtual element method (VEM)

Of noteworthy mentioning, the present hybrid polygonal finite element employs fundamental solutions as basis functions for the intra-element displacement field, in such a way that the local domain integrals of the variational functional can be transformed into local boundary integrals. There are other numerical frameworks in literature that reduce the domain discretisation of continuum problems to only polytopal meshes. The increasingly popular Virtual Element Method (VEM) [44–47] is one such example.

Although the overall implementation structure of the VEM [48] is much the same as that of the standard FEM and the present hybrid FEM, the computation of element stiffness matrices in the VEM is different. In contrast to the present method, the VEM, originating from mimetic finite difference method [49], employs projection operators [50] onto the polynomial space, in such a way that the discrete bilinear forms of the local domain integrals satisfying the polynomial consistency and stability properties can be approximated by the DOFs specified on the polytopal meshes. These DOFs of the unknown fields implicitly define shape functions in the virtual element space including the polynomial subspace of prescribed degree. This favourable approach in effect bypasses the explicit computation of canonical basis functions for trial and test spaces on each local element and generalizes the standard conforming finite element formulation to produce general mesh partitions consisting of (possibly non-convex) polygons of general shapes by relaxing the conformity condition, and been applied for linear elasticity problems [51,52], plate bending problems [53] and potential problems [54,55].

In analogy to the VEM, in which the total degrees of freedom can be separated into boundary degrees of freedom and internal degrees of freedom [56], the present method can obtain the same total DOFs by varying the order of polygonal elements and the number of source points associated to the intra-element displacement field for the desired boundary and



Fig. 4. Quadrilateral and polygonal discretization with 19, 65, and 148 elements.

internal DOFs, respectively. Notwithstanding the fundamental differences of the numerical frameworks, both the present method and the VEM excel in their flexibilities of adopting to general polygonal and non-convex meshes.

3. Formulation of topology optimization

In the study, the compliance minimization problems described by the BESO theory [13] are considered to couple with the present polygonal hybrid finite element for providing an alternative to topology optimization. For the consideration of completeness of the paper, the BESO theory is briefly reviewed in this section.

3.1. Optimization model

To obtain an optimized design of a two-dimensional continuum structure with maximum stiffness and prescribed material volume, the topology optimization problem can be stated as follows [13]

Minimize
$$c = \frac{1}{2} \sum_{i=1}^{N} \mathbf{u}_i^{\mathrm{T}} \mathbf{K}_i \mathbf{u}_i$$

Subject to : $V^* - \sum_{i=1}^{N} V_i \rho_i = 0$
 $\rho_i = \rho_{\min} \text{ or } 1$ (21)

where the objective function *c* is the mean compliance, \mathbf{K}_i and \mathbf{u}_i are the stiffness matrix and the nodal displacement vector of the *i*th element, respectively. V^* is the prescribed objective material volume, V_i is the volume of the *i*th element, and *N*



Fig. 5. Illustration of the effect of the parameter γ on the deflection at the midpoint of the loaded edge.

is the total number of elements in the discretized structure domain. The binary design variable ρ_i is the relative density of the *i*th element, which is restricted to be either 1 or ρ_{\min} in the soft-kill BESO theory [13]. Its upper bound represents a solid element ($\rho_i = 1$) while the lower bound implies a void element ($\rho_i = \rho_{\min}$). For practical reason, ρ_{\min} is chosen to be greater than zero, i.e. $\rho_{\min} = 0.001$, to ensure that none of the elements are completely removed from the design domain or otherwise issue of singularity will arise during numerical simulation.

In the density-based BESO method, the Young's modulus of each element is supposed to be a function of the element density, i.e.

$$E_i(\rho_i) = \rho_i^p E_0 \tag{22}$$

where E_0 denotes the real elastic modulus of the solid material and p > 1 is the penalty exponent. The parameter p is introduced here so as to diminish the impact caused by elements with small density, i.e. $\rho_i = \rho_{\min}$ is applied to the total structural stiffness. Usually, p = 3 is set for the topology optimization.

Under the assumption that the Poisson's ratio will not change during the optimization, the elementary stiffness matrix \mathbf{K}_i in Eq. (21) can be expressed as

$$\mathbf{K}_i = \rho_i^p \mathbf{K}_i^0 \tag{23}$$

where \mathbf{K}_{i}^{0} stands for the elementary stiffness matrix corresponding to the elastic modulus E_{0} .

3.2. Elementary sensitivity and filter scheme

In the BESO theory, the elementary sensitivity number is defined as the relative sensitivity of the objective function in terms of the discrete design variable ρ in the *i*th element

$$\alpha_{i} = \left(-\frac{1}{p}\frac{\partial c}{\partial \rho_{i}}\right) / V_{i} = \begin{cases} \frac{1}{2V_{i}}\mathbf{u}_{i}^{T}\mathbf{K}_{i}^{0}\mathbf{u}_{i} & \text{for } \rho_{i} = 1\\ \frac{\rho_{\min}^{\rho-1}}{2V_{i}}\mathbf{u}_{i}^{T}\mathbf{K}_{i}^{0}\mathbf{u}_{i} & \text{for } \rho_{i} = \rho_{\min} \end{cases}$$
(24)

When interpreting the solutions from topology optimization, a mesh-independent filtering scheme of sensitivities is desired to suppress the formation of checkerboard pattern in the simulation [12,13]. This filtering technique as adopted in the BESO method firstly evaluates the specific nodal sensitivity number through the computation of the elemental design sensitivities with respect to an arbitrary node. The nodal sensitivities in the fixed neighbourhood are later distributed back to make modification for the final elementary sensitivities. The detailed procedure is given below.



Fig. 6. Non-convex polygonal discretization with 12 elements and 35 nodes.



Fig. 7. Schematic diagram of filter scheme.

Firstly, for an arbitrary node j as shown in Fig. 7, its sensitivity number can be evaluated by averaging the elementary sensitivities with respect to its surrounding elements

$$\beta_j = \sum_{m=1}^M w_m \alpha_m \tag{25}$$



Fig. 8. Design domain of the simply supported beam structure.

where *M* is the number of elements connected to the nodal point *j*, and w_m is the weight factor related to the *m*th element and is defined by

$$w_{m} = \frac{1}{M-1} \left(1 - \frac{r_{mj}}{\sum\limits_{m=1}^{M} r_{mj}} \right)$$
(26)

where r_{mi} is the distance between the center of the *m*th element and the *j*th node.

The elementary sensitivity number of the *i*th element is then updated by averaging the sensitivity values of the surrounding nodes whose distance to the geometric center of the *i*th element is less than or equal to the filter radius r_{min} as shown in Fig. 7,

$$\alpha_{i} = \frac{\sum\limits_{j=1}^{J} \omega(r_{ij})\beta_{j}}{\sum\limits_{i=1}^{J} \omega(r_{ij})}$$
(27)

where J is the total number of nodes included in the circular subdomain Ω_i inside the radius r_{\min} and $\omega(r_{ij})$ is the linear weight factor defined as

$$\omega(r_{ij}) = r_{\min} - r_{ij} \tag{28}$$

where r_{ij} is the distance between the center of the *i*th element and the *j*th node.

To improve the performance of the BESO, the filtered elementary sensitivities can be averaged again with information from the previous and the current steps,

$$\alpha_i = \frac{\alpha_i^{(k)} + \alpha_i^{(k-1)}}{2} \tag{29}$$

where (k) is the current iteration step and (k-1) is the previous iteration step. The $\alpha_i^{(k)}$ of the current iteration (k) is then replaced by the updated elementary sensitivity in Eq. (29) for the next solution step.

After the element sensitivity numbers are evaluated, the elements are sorted according to the values of their sensitivity numbers (from the highest to the lowest) for performing added or removed operation of elements at the current iteration (k). For the *i*th element, the removal criteria is

$$\alpha_i \le \alpha_{del}^{th} \tag{30}$$

where the relative density ρ_i will be switched to ρ_{\min} and the additive criteria is

$$\alpha_i > \alpha_{add}^{th} \tag{31}$$

where the relative density ρ_i will be switched to 1. It is also noted that α_{del}^{th} and α_{add}^{th} are the threshold numbers for removing and adding elements and can be determined by the steps given in literature [13].



Fig. 9. Mesh configurations for the simply supported beam with (a) 5000 4-sided (regular square) hybrid finite elements and (b) 5000 polygonal hybrid finite elements.

3.3. Stopping criteria

Before the elements are removed from or added to the current design, the target volume for the next iteration (k+1) shall be evaluated first. Since the volumetric constraint V^* could be greater or smaller than the volume of the initial guess design, the target volume may decrease or increase at each step until the optimized design reaches the volumetric constraint V^* . The evolution of the volume fraction of the design domain can be stated by the following simple expression

$$V^{(k+1)} = \max(V^{(k)}(1 \pm \delta), V^*)$$
(32)

where δ is the evolutionary volume ratio.

The topology optimization is solved iteratively until the mean compliance change is lower than a specified tolerance τ and the continuum structure reaches the specified volume fraction. The mean compliance change can be computed using

$$\frac{\left|\sum_{i=1}^{T} \left(c^{(k-i+1)} - c^{(k-T-i+1)} \right)\right|}{\sum_{i=1}^{T} c^{(k-i+1)}} \le \tau$$
(33)

where *k* is the current iteration number, τ is an allowable tolerance and *T* is an integer number. Normally, *T*=5 can be employed [13].



Fig. 10. Evolution histories of the volume fraction and the mean compliance for the simply supported beam using the regular square hybrid finite elements.



Fig. 11. Evolution of topology configuration for the simply supported beam using the regular square hybrid finite elements.



Fig. 12. Evolution histories of the volume fraction and the mean compliance for the simply supported beam by BESO2D with same mesh and control parameters.

4. Numerical examples

Based on the formulations of the polygonal hybrid finite element and the BESO theory described above, the algorithm written in MATLAB codes is developed to assess the performance of the present *n*-sided polygonal hybrid finite element for topology optimization. In this work, a typical two-dimensional benchmark example of a Michell type elastic beam structure taken from the BESO2D[®] manual is considered as the design domain, as shown in Fig. 8. The free software BESO2D[®] was developed by Xie's group at RMIT based on the BESO theory and can be downloaded from the website¹ The structure is simply supported at two ends of the bottom edge and a 100 N point load is applied at the bottom center. The elastic modulus and the Poisson's ratio of the material are 210 GPa and 0.30, respectively. The available material is specified to cover 60% of the original design domain.

To control the quality of the topology optimization iterative process, the allowable tolerance of stopping criteria is set as $\tau = 0.001(0.1\%)$, the penalty exponent is p = 3, and the evolutionary volume ratio δ which controls the volume fraction change between any two consecutive iterations is chosen as 0.02 (2%).

4.1. Verification of the present method

For the numerical simulation of optimised structure, the original design domain is firstly meshed by 100×50 4-sided square hybrid finite elements with size of 0.1 m along the individual element edge, as shown in Fig. 9a. The total number of nodes are 5151 while the filter radius is taken to be 3 times of the element size, that is $r_{min} = 0.3$ m. Fig. 10 shows the evolution histories of the volume fraction and the mean compliance. As expected, the mean compliance of the design domain is inversely proportional to the material removal and both of which converge to the stable values after a mere of 35 iterations. Correspondingly, the mean compliance increases linearly during the first 26 iterations and converges to 3.044×10^{-7} Nm. It is vice versa for the material removal that the volume fraction decreases linearly until it reaches the

¹ http://www.rmit.edu.au/research/research-institutes-centres-and-groups/research-centres/centre-for-innovative-structures-and-materials/software.



Fig. 13. Evolution histories of the volume fraction and the mean compliance for the simply supported beam using the *n*-sided polygonal hybrid finite elements.



Fig. 14. Evolution of topology configuration for the simply supported beam using 5000 n-sided polygonal hybrid elements.



Fig. 15. Influence of the penalty exponent with $r_{\min} = 0.3$ m and $\delta = 2\%$.

objective volume, i.e. 60% volume of the original design domain. Fig. 11 displays the evolution history of topologies at selected iteration steps and the final optimized design (see Fig. 11d) is obtained after the 35th iteration.

To verify the present method of coupling the BESO theory and the fundamental-solution-based hybrid finite element formulation, the above problem using exactly the same mesh and optimization parameters are solved by the BESO2D with the standard quadrilateral finite elements. The evolution histories and the final topology is given in Fig. 12, which is very similar to the present optimized solution in Fig. 11d. It is observed that the convergent mean compliance from the BESO2D is about 3.2×10^{-7} Nm which is evidently higher than that $(3.044 \times 10^{-7}$ Nm) from the present method. The difference may be attributed to the replacement of the standard finite element by the 4-sided hybrid finite element in the present method, which has been proven to have better accuracy than the traditional finite element, as shown in Section 2.2.

Next, to address the polygonal geometric feature of the spatial discretization, the full design domain is discretized by the *n*-sided convex polygonal hybrid finite elements and its performance on topology optimization is investigated. Fig. 9b displays the mesh configuration with 5000 polygonal hybrid finite elements produced by PolyMesher [57]. Total 7 quadrilateral elements, 583 pentagonal elements, 4098 hexagonal elements and 312 heptagonal elements are included in the meshed model. With the same optimal parameter setting, i.e. $\tau = 0.001$, $\delta = 2\%$, p = 3 and $r_{min} = 0.3$ m, the design domain under the specific displacement constraints and load condition is optimized using the developed polygonal hybrid finite elements. As seen in Fig. 13, the mean compliance converges after 35 iterations and has a value of 3.099×10^{-7} Nm, which is apparently lower than results obtained from the BESO2D (3.2×10^{-7} Nm). Fig. 14 shows the variations of topology configuration at selected iteration steps. It is observed that the topological optimum design from the present *n*-sided polygonal hybrid finite elements, square finite elements, which implies the mesh-independence of the present method. This completes the verification of the present hybrid finite element formulation for topology optimization. Besides, it is necessary to point out that the slight non-symmetry is observed from Fig. 14. This can be attributed to the non-symmetric mesh configuration of the polygonal elements in the design domain (see Fig. 9b), which usually includes the combination of multiple types of polygons. However, it is found that both the regular square elements produce almost same final optimized design.



Fig. 16. Converged mean compliance and topologies for different volume constraints with $r_{\min} = 0.3$ m, $\delta = 0.2$ and p = 3.

4.2. Parameter study of optimal parameters

It has long been revealed that the final optimized topology of continuum structures is critically controlled by factors such as the filter radius, the evolutionary volume ratio and the penalty exponent. It is therefore desirable to investigate the influence of these free parameters on the present simulation outcomes. In this section, the parameter study of the three control parameters is performed and the effects of them on the final topological design are assessed for potential applications. It is noted that only the *n*-sided polygonal hybrid finite elements as shown in Fig. 9b are considered for assessing the numerical performance of the present method. For the purpose of comparison, the results from the BESO2D using 5000 standard square plane elements are referred in the parameter study.

4.2.1. Influence of the filter radius

As it is presumed that the filter radius plays a significant impact on the number of nodes contributed to the checkerboard pattern phenomenon, we first investigate the variation of the filter radius with respect to the optimum topological design. In this parameter study, the evolutionary volume ratio δ is unchanged at 2% and the penalty exponent p is set as 3. The filter radius r_{min} are separately taken to be of 0.1 m, 0.3 m, 0.5 m, and 0.7 m. Table 2 shows the converged topological designs and the corresponding mean compliances at various filter radii. It is found that the predicted mean compliance increases with an enlarged filter radius for both the standard quadrilateral finite elements in the BESO2D and the present polygonal hybrid finite elements. To put the numerical performance into perspective, the present elements using the fundamental solution kernels can produce mean compliance of smaller magnitude than the standard quadrilateral finite elements. We observe from Table 2 that the setting of the filter radius would affect the outcome of an optimum topological design. For instance, the filter radius of extreme magnitudes will not be able to produce the desired optimized topology for both the present method and the referenced BESO2D. When the filter radius is chosen to be very small, e.g. $r_{min} = 0.1 \text{ m}$, the neighbouring elements will have little contribution to suppress the checkerboard pattern issue. Interestingly, it is found that the resulting topologies from the BESO2D and the present method are significantly different for $r_{\rm min} = 0.1$ m. It can be argued that the polygonal elements are less regular than the square elements and thus, a relatively less consistent number of nodes could be included inside the filter circle and consequently deteriorating the elementary sensitivity number. When the r_{min} is increased to 0.2 m and 0.3 m, the polygonal hybrid finite elements are shown to give much stable solutions which are almost identical to the optimum topological designs. However, the application of a larger filter radius does not seem to benefit for a better topological design.



Table 2 Influence of the filter radius with $\delta = 2\%$ and p = 3.

4.2.2. Influence of the evolutionary volume ratio

The evolutionary volume ratio is also considered as one of the key parameters which dictates the change of the volume fraction during the optimization iterations. To investigate the effect of the evolutionary volume ratio on the resulting topology, two of the optimization parameters are set to be constants: $r_{min} = 0.3$ m and p = 3. The maximum value of the parameter δ is now 8%, instead of the previous 2%. Table 3 indicates that the resulting topologies from the BESO2D and the present method are almost identical with respect to the different evolutionary volume ratios where the evolutionary volume ratio is found to be proportional to the mean compliance. It is observed that the predicted mean compliance from the BESO2D is always higher than that from the present method. Notwithstanding that a better topological design can be achieved by employing a smaller δ value, i.e. $\delta = 2\%$ as seen in Table 3. But such improvement often trades off with an increase of computational cost as indicated by the higher demand of the iteration number. For example, a 6% drop of the evolutionary volume ratio δ from 8% to 2% in the simulation would result in a steep increase of the iteration number from 17 to 35 units.

4.2.3. Influence of the penalty exponent

Finally, the effect of the penalty exponent p is investigated in this section. For practical reason, p is set to be greater than 1.0 in the numerical simulation. It is assumed that the parameters r_{\min} and δ are retained at 0.3 m and 2%, respectively while the penalty exponent p varies in the interval of 1.5 and 5.0. Results in Fig. 15 indicate that the polygonal hybrid finite elements can produce stable results when $p \ge 1.5$, and that both the predicted mean compliances and the resulting





topological designs are similar to the optimized configuration with p = 3, suggesting that the influence of the penalty exponent p is negligible for $p \ge 1.5$. Similar conclusion can found in the book of Huang and Xie [13].

4.3. Topologies for various volume constraints

It should be emphasised that the above optimization is carried out by setting the volumetric constraint at 60%. In this section, the topologies of various volumetric constraints are examined to better understand the performance of the present polygonal hybrid finite element for topology optimization. Fig. 16 illustrates the variation of the mean compliance for different specified volumetric constraints whose values are up to 30%. In this figure, the relative mean compliance is computed by referring to the mean compliance (2.523×10^{-7} Nm) corresponding to the original full design domain ($V^* = 100\%$). As expected, the converged mean compliance increases as more materials are removed from the design domain. It can be observed from the progression of topological designs that the internal supporting bars and the outer rim are getting thinner as the available material covering the design domain reduces up to 40%, and later the number of inner supporting bars decrease to three when the volumetric constraint is taken to be 30%. The present method using the polygonal hybrid finite elements is shown to capture this change of topology well. However, it's worth noting that the extremely smaller

volume fractions such as 10% and 20% may not produce acceptable topologies for the current example by using both the present polygonal hybrid finite elements and the uniform grid of square finite element in the BESO2D software, because there might be discontinuous material distributions in the final topologies. The corresponding mean compliances can reach to a very bigger value (theoretically infinite), compared to the normal magnitude (10^{-7} Nm) of the mean compliance. In fact, an acceptable optimized result of a given problem is closely related to the applied constraint conditions such as the specified volume constraint, loads etc.

5. Conclusions

This paper presents a generalized fundamental-solution-based *n*-sided polygonal hybrid finite element formulation including line integrals only for solutions of discrete topology optimization problems. The use of unstructured polygonal hybrid finite elements possessing higher degree of geometry isotropy is contributed to compressing the checkerboard patterns in topology optimization. Simulated results from the Cook's problem and the benchmark example have validated the accuracy and the feasibility of the present *n*-sided hybrid finite elements for topology optimization. It can be concluded that (1)the present *n*-sided polygonal hybrid finite element family tree (n = 3, 4, 5, ...) provides a unifying paradigm for topology optimization, which uses same kernel functions for approximating intra-element displacement field. This extremely simplifies the construction of interpolants in the element domain; (2) the feature of line integration over the element boundary not only brings the reduction in integration dimensionality, but also allows for greater flexibility using polygonal mesh for the discretization of complex design domains; (3) the present *n*-sided polygonal hybrid finite element can produce better accuracy of displacement solution than the conventional quadrilateral finite elements and the natural polygonal finite elements; (4) the predicted mean compliance in the present method is always lower than that from the referenced method of BESO2D, in which the uniform grid of standard linear square finite elements is employed; (5) the present hybrid finite element has demonstrated mesh-independence by altering the mesh configuration from the square hybrid finite elements to the n-side polygonal hybrid finite elements; and (6) the filter radius and the evolutionary volume ratio are shown to be more effective than the penalty exponent for providing an optimum topological design.

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