Robustness and Risk-Sensitive Control of Quantum Systems

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Introduction and Motivation

The issue: how to cope with model uncertainty and disturbances?

Robustness: *"good performance under nominal conditions, and acceptable performance in other than nominal conditions"*

A systematic approach to robust control design might include

- quantification of performance
- specification of a nominal model
- specification of a class of "other than nominal" models
- quantification of allowed deviations from nominal
- bound on performance in terms of quantified deviation

Some background:

- Late 70's - G. Zames initiates so-called H^∞ approach to robust control
- Intensive development of robust control for linear and nonlinear systems in 80's and 90's
- Links established among H^∞ control, dissipative systems, Riccati equations, dynamic games, stochastic risk-sensitive optimal control
- Typical H^∞ robust performance bound:

$$\int_0^T |z(t)|^2 dt \leq \beta(x_0) + \int_0^T |w(t)|^2 dt$$

performance quantity	initial bias	size of uncertainty

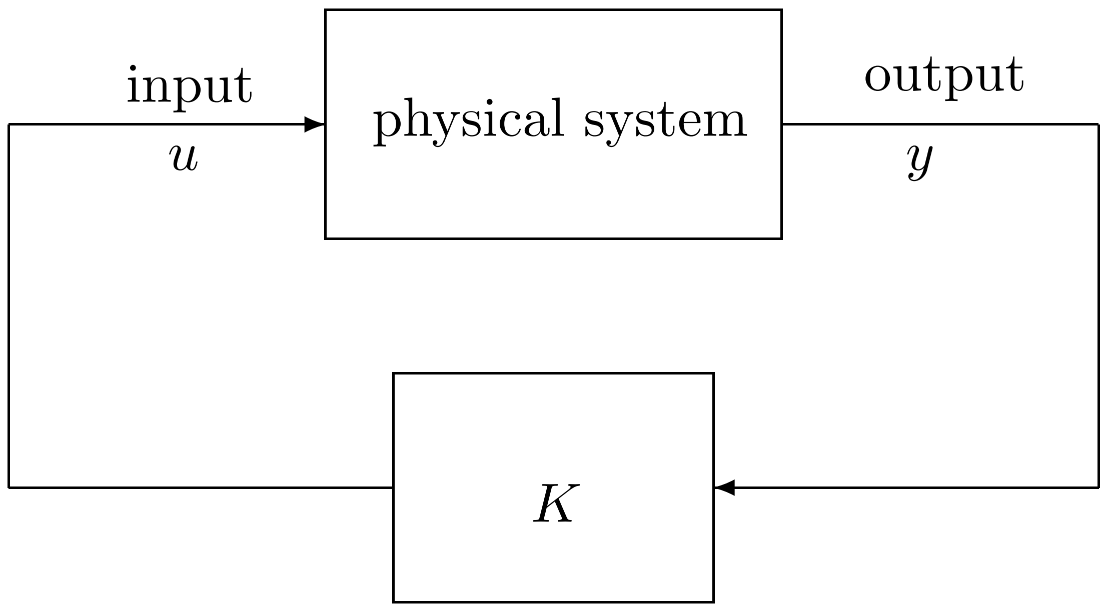
Some robust/quantum papers:

- A.C. Doherty, J. Doyle, H. Mabuchi, K. Jacobs, and S. Habib, Robust control in the quantum domain, Proc. IEEE CDC 2000.
- J.K. Stockton, J.M Geremia, A.C. Doherty, and H. Mabuchi, Robust quantum parameter estimation: coherent magnetometry with feedback to appear in Phys. Rev. A (2004).
- V. Protopopescu, R. Perez, C. D'Helon and J. Schmulen, Robust control of decoherence in realistic one-qubit quantum gates, 2003 J. Phys. A: Math. Gen. 36 2175-2189
- J. Beumee and H. Rabitz, Robust optimal control theory for selective vibrational excitation in molecules: A worst case analysis, J. Chem. Physics, 97(2) 1992.

Collaborators on this topic area:

- Ian Petersen (ADFA/UNSW)
- Stuart Wilson (ANU)
- Andrew Doherty (UQ)

Feedback Control of Quantum Systems



Discrete-time model of quantum system: (controlled stochastic master equation)

$$\omega_{k+1} = \Lambda_\Gamma(u_k, y_{k+1})\omega_k,$$

where (conditional state)

$$\omega = \text{ density operator}$$

(instrument)

$$\Gamma(u, y) = \text{ quantum operation}$$

$$p_\Gamma(y|u, \omega) = \langle \Gamma(u, y)\omega, I\rangle$$

$$\Lambda_\Gamma(u, y)\omega = \frac{\Gamma(u, y)\omega}{p(y|u, \omega)}$$

Feedback controller:

$$u = K(y)$$

where

$$K = \{K_0, K_1, \ldots, K_{M-1}\}$$

and

$$\begin{aligned} u_0 &= K_0 \\ u_1 &= K_1(y_1) \\ u_2 &= K_2(y_1, y_2) \quad \text{etc.} \end{aligned}$$

Probability distributions: (determined by instrument process)

$$\begin{aligned} \mathbf{P}^K_{\omega_0,0}(y_1, \ldots, y_M) &= \prod_{k=0}^{M-1} p_\Gamma(y_{k+1}|u_k, \omega_k) \\ &= \langle \prod_{k=0}^{M-1,\leftarrow} \Gamma(u_k, y_{k+1})\omega_0, I\rangle \end{aligned}$$

Optimal Control - Risk-Neutral

Minimize:

$$J_{\omega,0}(K) = \mathbf{E}^K_{\omega,0}[\sum_{i=0}^{M-1} \langle \omega_i, L(u_i) \rangle + \langle \omega_M, N \rangle]$$

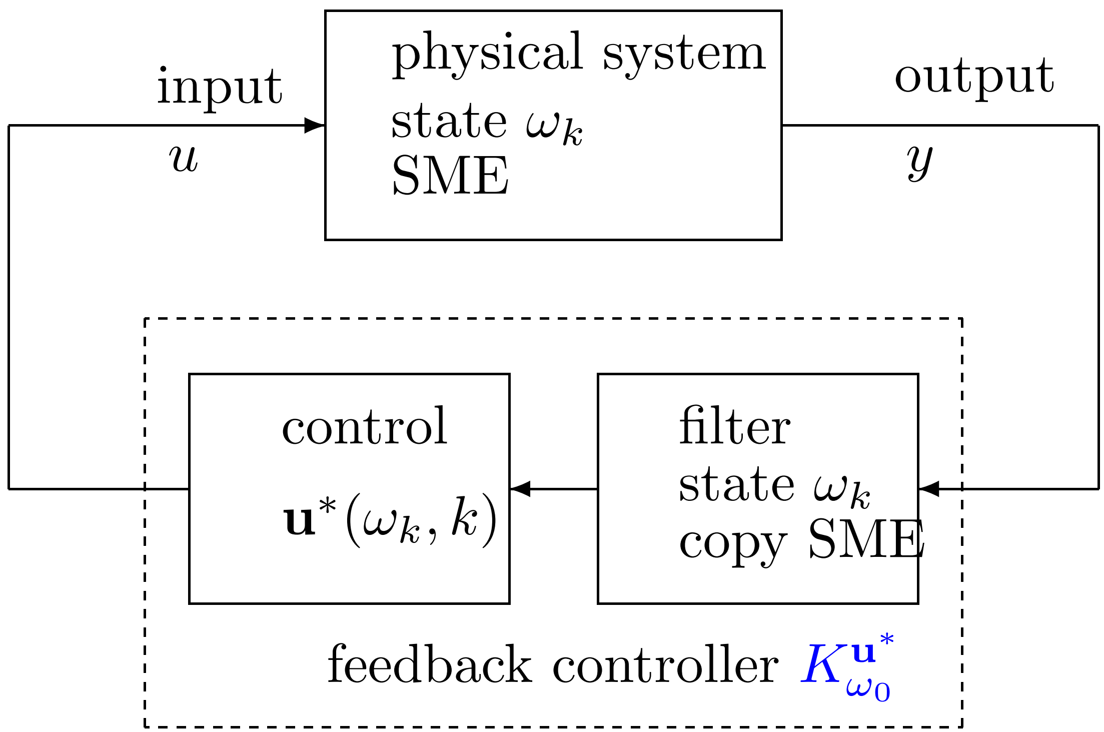
Dynamic programming:

$$\begin{aligned} V(\omega,k) &= \inf_{u\in\mathbf{U}} \{\langle \omega, L(u)\rangle + \sum_{y\in\mathbf{Y}} V(\Lambda_\Gamma(u,y)\omega, k+1) p(y|u,\omega)\} \\ V(\omega,M) &= \langle \omega, N \rangle \end{aligned}$$

Optimal controller: $K^{\mathbf{u}^*}_{\omega_0}$

$$\mathbf{u}^*(\omega,k) \in \operatorname*{argmin}_{u\in\mathbf{U}} \{\langle \omega, L(u)\rangle + \sum_{y\in\mathbf{Y}} V(\Lambda_\Gamma(u,y)\omega, k+1) p(y|u,\omega))\}$$

Optimal risk-neutral controller has a *separation structure*, with SME as filter:



Optimal Control - Risk-Sensitive

Two generalizations of classical risk-sensitive criteria are:

$$J^{1,\mu}_{\omega,0}(K) = \mathbf{E}^K_{\omega,0}[\exp\left(\sum_{k=0}^{M-1} \langle \omega_k, \mu L(u_k)\rangle + \langle \omega_M, \mu N\rangle\right)]$$

$$J^{2,\mu}_{\omega,0}(K) = \mathbf{E}^K_{\omega,0}[\prod_{k=0}^{M-1} \langle \omega_k, e^{\mu L(u_k)}\rangle \langle \omega_M, e^{\mu N}\rangle]$$

General framework: (for suitable operator valued cost $R(u)$)

$$J^{\mu}_{\hat{\omega},0}(K) = \sum_{y_{1,M}\in \mathbf{Y}^M} \langle \hat{\omega}, G_0\rangle$$

$$\begin{aligned} G_k &= R^\dagger(u_k)\Gamma^\dagger(u_k, y_{k+1})G_{k+1} \\ G_M &= F \end{aligned}$$

Operator valued costs for above criteria are:

$$R^1(u)\hat{\omega} = e^{\langle \hat{\omega}, \mu L(u) \rangle / \langle \hat{\omega}, 1 \rangle} \hat{\omega}$$

and

$$R^2(u)\hat{\omega} = \frac{\langle \hat{\omega}, e^{\mu L(u)} \rangle}{\langle \hat{\omega}, 1 \rangle} \hat{\omega}$$

[Another choice:

$$R^3(u)\hat{\omega} = \sum_c Z_c(u) \hat{\omega} Z_c(u)$$

(arises from a dilation of the quantum operation model)]

Solution is in terms of a *modified SME dynamics*:

$$\hat{\omega}_{k+1} = \Lambda_{\Gamma,R}(u_k, y_{k+1})\hat{\omega}_k$$

where (modified conditional state)

$$\hat{\omega} = \text{ unnormalized density operator}$$

$$\Gamma_R(u,y) = \Gamma(u,y)R(u) \qquad \text{(unnormalized)}$$

$$p_{\Gamma,R}(y|u,\hat{\omega}) = \frac{\langle \Gamma_R(u,y)\hat{\omega}, I\rangle}{\langle R(u)\hat{\omega}, I\rangle}$$

$$\Lambda_{\Gamma,R}(u,y)\hat{\omega} = \frac{\Gamma(u,y)\hat{\omega}}{p_{\Gamma,R}(y|u,\hat{\omega})}$$

Alternate representation of cost criterion in terms of modified conditional state:

$$J^{\mu}_{\hat{\omega},0}(K) = \mathbf{E}^{K}_{\hat{\omega},0}[\langle \hat{\omega}_M, F\rangle]$$

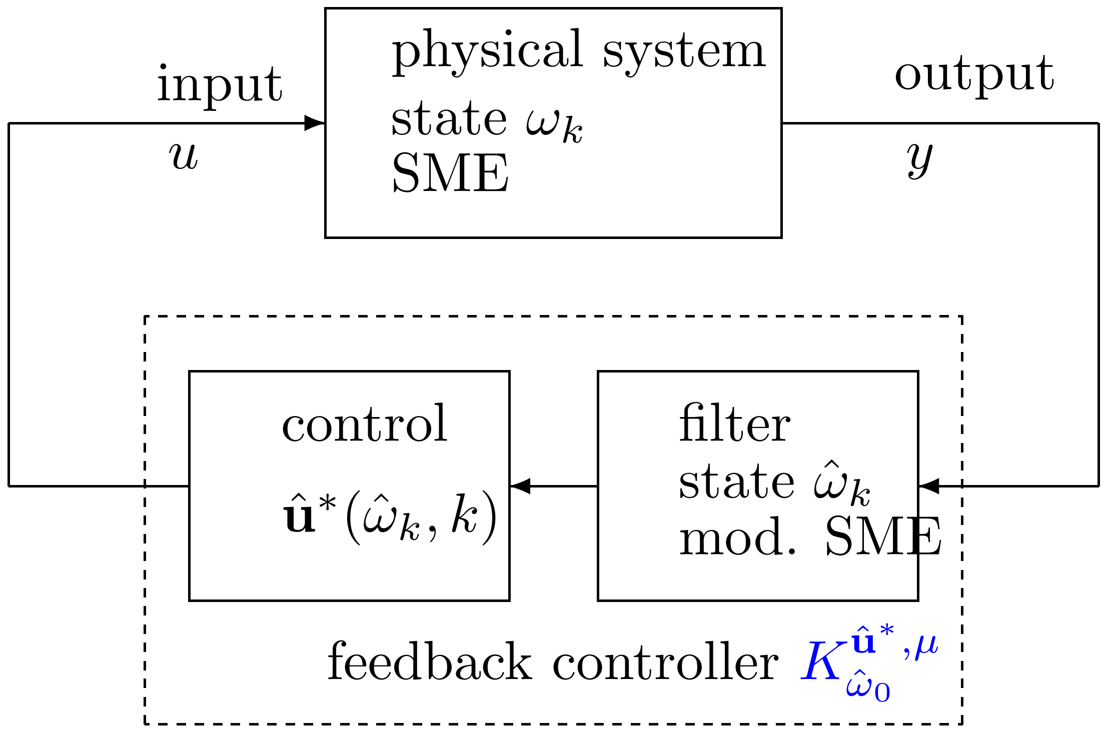
Dynamic programming:

$$\begin{aligned} W(\hat{\omega}, k) &= \inf_{u\in\mathbf{U}} \{ \sum_{y\in\mathbf{Y}} W(\Lambda_{\Gamma,R}(u,y)\hat{\omega}, k+1) p_{\Gamma,R}(y|u,\hat{\omega}) \} \\ W(\hat{\omega}, M) &= \langle \hat{\omega}, F\rangle \end{aligned}$$

Optimal control: $K^{\mathbf{u}^*,\mu}_{\omega_0}$

$$\hat{\mathbf{u}}^*(\hat{\omega}, k) \in \underset{u\in\mathbf{U}}{\operatorname{argmin}} \{ \sum_{y\in\mathbf{Y}} W(\Lambda_{\Gamma,R}(u,y)\hat{\omega}, k+1) p_{\Gamma,R}(y|u,\hat{\omega}) \}$$

Optimal risk-sensitive controller has a *separation structure*, with *modified SME* as filter:



Robustness Properties of Risk-Sensitive Control

Consider risk-sensitive criteria no. 2, and use to obtain controller.

$$J^{2,\mu}_{\omega,0}(K) = \mathbf{E}^{K}_{\omega,0}[\prod_{k=0}^{M-1} \langle \omega_k, e^{\mu L(u_k)} \rangle \langle \omega_M, e^{\mu N} \rangle]$$

Let $K^{\hat{\mathbf{u}}^*,\mu}_{\hat{\omega}_0}$ be the optimal risk-sensitive controller obtained using a *nominal model* (Kraus form)

$$\Gamma_{nom}(u,y)\omega = \sum_a \gamma_{nom,a}(u,y)\omega\gamma^{\dagger}_{nom,a}(u,y)$$

Let the *true model* be among the "other than nominal" models given by $(0 \le \lambda_a(u,y) \le c(u,y))$ (abs. cts.)

$$\Gamma_{true}(u,y)\omega = \sum_a \lambda_a(u,y)\gamma_{nom,a}(u,y)\omega\gamma^{\dagger}_{nom,a}(u,y)$$

Corresponding probability distributions: $(\mathbf{P}_{true} << \mathbf{P}_{nom})$

$$\mathbf{P}_{true}(y_1, \ldots, y_M) = \prod_{k=0}^{M-1} p_{\Gamma_{true}}(y_{k+1}|u_k, \omega_k^{true})$$

$$\mathbf{P}_{nom}(y_1, \ldots, y_M) = \prod_{k=0}^{M-1} p_{\Gamma_{nom}}(y_{k+1}|u_k, \omega_k^{nom})$$

Lemma

$$\frac{d\mathbf{P}_{true}}{d\mathbf{P}_{nom}}(y_1, \ldots, y_M) = \frac{\mathbf{P}_{true}(y_1, \ldots, y_M)}{\mathbf{P}_{nom}(y_1, \ldots, y_M)} = \prod_{k=0}^{M-1} f_{k+1}(y_{k+1}|y_1, \ldots, y_k)$$

where

$$f_{k+1}(y_{k+1}|y_1, \ldots, y_k) = \frac{p_{\Gamma_{true}}(y_{k+1}|u_k, \omega_k^{true})}{p_{\Gamma_{nom}}(y_{k+1}|u_k, \omega_k^{nom})} \geq 0$$

Duality formula: (free energy-relative entropy)

$$\log \mathbf{E}_{\mathbf{P}}[e^{Z}] = \sup_{\mathbf{Q}<<\mathbf{P}} \{\mathbf{E}_{\mathbf{Q}}[Z] - \mathfrak{R}(\mathbf{Q} \parallel \mathbf{P})\}$$

Lemma

$$\log \langle \omega, e^{\mu X} \rangle = \sup_{g(\cdot|X,\omega)} \{\mu \langle \omega, X g(X|u,\omega) \rangle - \mathfrak{C}_{\omega}(\tilde{X} \parallel X)\}$$

where $g(\cdot|X,\omega) \geq 0$,

$$\int g(x|X,\omega)\langle \omega, P_X(dx) \rangle = 1,$$

and

$$\mathfrak{C}_{\omega}(\tilde{X} \parallel X) = \langle \omega, g(X|X,\omega) \log g(X|X,\omega) \rangle.$$

Theorem (robustness bound)

$$\mathbf{E}^{true}_{\omega,0}[\sum_{k=0}^{M-1} \langle \omega_k, \tilde{L}_k(u_k)\rangle + \langle \omega_M, \tilde{N}_M\rangle]$$

$$\leq \frac{1}{\mu} \log J^{2,\mu}_{\omega,0}(K^{\hat{\mathbf{u}}^*,\mu}_{\hat{\omega}_0}) + \frac{1}{\mu}\mathfrak{R}(\mathbf{P}_{true} \parallel \mathbf{P}_{nom}) + \frac{1}{\mu}\mathfrak{C}(\tilde{L} \parallel L)$$

where

$$\tilde{L}_k(u_k) = L(u_k) g_k(L(u_k)|L(u_k), \omega_k), \quad \tilde{N}_M = N g_M(N|N, \omega_M)$$

$$\mathfrak{R}(\mathbf{P}_{true} \parallel \mathbf{P}_{nom}) = \mathbf{E}^{true}_{\omega,0}[\sum_{k=0}^{M-1} \log f_{k+1}(y_{k+1}|y_1, \ldots, y_k)]$$

$$\mathfrak{C}(\tilde{L} \parallel L) = \mathbf{E}^{true}_{\omega,0}[\sum_{k=0}^{M-1} \mathfrak{C}_{\omega_k}(\tilde{L}(u_k) \parallel L(u_k)) + \mathfrak{C}_{\omega_M}(\tilde{N} \parallel N)].$$

(Perturbed cost and cost entropy arise naturally in this quantum setup.)

Example

2-level system, noisy measurement of spin observable

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

want state to be spin up

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ or } |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

through a series of measurements and feedback control actions ($u = 0$ do nothing, or $u = 1$ flip)

Nominal model:

$$\Gamma_{nom}(u,y)\omega = q(y|-1)P_{-1}T^u\omega {T^u}^\dagger P_{-1} + q(y|1)P_1 T^u \omega {T^u}^\dagger P_1,$$

$$T^u = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } u = 0 \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } u = 1, \end{cases}$$

$$P_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and $(0 < \alpha < 1)$ (measurement error prob.)

$$\begin{aligned} q(-1|-1) &= q(1|1) &&= 1-\alpha \\ q(-1|1) &= q(1|-1) &&= \alpha. \end{aligned}$$

Cost observable:

$$L(u) = X^2 + c(u)I$$

and

$$N = X^2$$

where $X = \frac{1}{2}(A - I)$ and

$$c(0) = 0,\ c(1) = p,\ \text{with } p \geq 0$$

The expected value of X^2 is

$$\begin{aligned} \langle 1|X^2|1\rangle &= \mathrm{tr}[X^2|1\rangle\langle 1|] &= 0 \\ \langle -1|X^2|-1\rangle &= \mathrm{tr}[X^2|-1\rangle\langle -1|] &= 1 \end{aligned}$$

Robustness interpretation

Perturbed instrument and probability distribution:

$$\gamma_{nom,a}(u,y) = \sqrt{q(y|a)} P_a T^u$$

"True" value of meas. error $0 < \tilde{\alpha} < 1$. (RN deriv.)

$$\lambda_a(u,y) = \frac{\tilde{q}(y|a)}{q(y|a)}$$

Class of true models:

$$\Gamma_{true}(u,y)\omega = \tilde{q}(y|-1) P_{-1} T^u \omega T^{u\,\dagger} P_{-1} + \tilde{q}(y|1) P_1 T^u \omega T^{u\,\dagger} P_1$$

Perturbed cost:

$$\langle \omega, L(1) \rangle = (1+p)\omega_{11} + p\omega_{22}.$$

$$\langle \omega, \tilde{L}(1) \rangle = (1+\tilde{p})\omega_{11} + p(1 - \frac{1+\tilde{p}}{1+p}\omega_{11}).$$

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