Linear Quantum Systems

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Outline

- Background and Motivation
- Linear Quantum Models
- Dissipation Properties
- H-Infinity Controller Synthesis
- Robust Stability
- Examples

Background and Motivation

- The need for robustness in quantum control systems
- The “H-infinity” approach to robust control system design

Linear Quantum Models

We consider noncommutative stochastic systems of the form

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t) \\ dz(t) &= Cx(t)dt + Ddw(t) \end{aligned}$$

where A, B, C and D are real matrices, and

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

is a vector of possibly noncommutative plant variables.

The initial system variables $x(0)$ are Gaussian with state ρ, and satisfy the commutation relations

$$[x_j(0), x_k(0)] = C^{xx}_{jk} = 2i\Theta_{jk}, \quad j, k = 1, \ldots, n,$$

where Σ is a real antisymmetric matrix.

For example, a system with one classical variable and two conjugate quantum variables is characterized by

$$\Theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

The vector quantity w describes input channels and is assumed to admit the decomposition

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t)$$

where $\beta_w(t)$ is the self-adjoint finite variation part, and $\tilde{w}(t)$ is the (Gaussian) noise part of $w(t)$ with Ito table

$$d\tilde{w}(t)d\tilde{w}^T(t) = F_{\tilde{w}}dt,$$

where $F_{\tilde{w}}$ is a non-negative Hermitian matrix.

For instance,

$$F_{\tilde{w}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{bmatrix}$$

describes a noise vector with one classical component and a pair of conjugate quantum Gaussian noises.

Physical systems

Preservation of commutation relations imposes constraints on plant matrices A, B, C, D:

Assume $[x_i(t), \beta_{w,j}(t)] = C_{ij}^{x\beta_w}$ for all $t \geq 0$. Then

$$[x_i(0), x_j(0)] = C_{ij}^{xx} \text{ implies } [x_i(t), x_j(t)] = C_{ij}^{xx}$$

for all $t \geq 0$ if and only if

$$AC^{xx} + C^{xx}A^T - B(C^{x\beta_w})^T + C^{x\beta_w}B^T + 2BT_{\tilde{w}}B^T = 0.$$

Physical realizability

When seeking to build controllers from linear models we need to know when such a model is physically realizable.

We regard all classical systems as physically realizable, at least approximately via classical electronics.

We shall say that a system is *physically realizable* if it corresponds to a completely positive evolution.

The commutation relations can be used to characterize physical realizability, for example:

Assume that Σ is invertible and n is even, and $C^{x\beta_w} = 0$. Then the system is physically realizable if and only if the condition

$$AC^{xx} + C^{xx}A^T + 2BT_{\tilde{w}}B^T = 0$$

holds.

Dissipation Properties

- We wish to characterize the effect of inputs on system behavior

  e.g. effect of disturbances on performance

- We generalize the notions of "dissipation" widely used in classical control and formalized by J.C. Willems, 1972.

- This keeps track of power flows and stored energy

Definition

Given an operator valued quadratic form

$$r(x, \beta_w) = [x^T \beta_w^T] R \begin{bmatrix} x \\ \beta_w \end{bmatrix}$$

where

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}$$

is a given real symmetric matrix.

We say the system is *dissipative with supply rate* $r(x, \beta_w)$ if there exists a positive operator valued quadratic form $V(x) = x^T X x$ (where X is a real positive definite symmetric matrix) and a constant $\lambda > 0$ such that

$$\langle V(x(t)) \rangle + \int_0^t \langle r(x(s), \beta_w(s)) \rangle ds \leq \langle V(x(0)) \rangle + \lambda t \quad \forall t > 0,$$

for all Gaussian ρ.

We say that the system is *strictly dissipative* if there exists a constant $\epsilon > 0$ such that the above inequality holds with the matrix R replaced by the matrix $R + \epsilon I$.

Theorem

The system is dissipative with supply rate $r(x, \beta_w)$ *if and only if there exists a real positive definite symmetric matrix* X *such that the following matrix inequality is satisfied:*

$$\begin{pmatrix} A^T X + XA + R_{11} & R_{12} + XB \\ B^T X + R_{12}^T & R_{22} \end{pmatrix} \leq 0.$$

Furthermore, the system is strictly dissipative if and only if there exists a real positive definite symmetric matrix X *such that the following matrix inequality is satisfied:*

$$\begin{pmatrix} A^T X + XA + R_{11} & R_{12} + XB \\ B^T X + R_{12}^T & R_{22} \end{pmatrix} < 0$$

Moreover, if either of the above holds then the required constant $\lambda \geq 0$ can be chosen as

$$\lambda = \mathrm{tr}\left[\left[\begin{array}{c} B^T \\ G^T \end{array}\right] X \left[\begin{array}{cc} B & G \end{array}\right] F\right]$$

where the matrix F is defined by the following relation:

$$F dt = \left[\begin{array}{c} dw \\ dv \end{array}\right] \left[\begin{array}{cc} dw^T & dv^T \end{array}\right].$$

The proof depends on the following identity

$$\langle V(x(t)\rangle \triangleq (\rho \otimes \phi)(V(x(t))) = \langle \rho, E_0[V(x(t))]\rangle,$$

where E_0 denotes expectation with respect to ϕ, and on the following

Lemma

Consider a real symmetric matrix X and corresponding operator valued quadratic form $\eta^T X \eta$. Then the following statements are equivalent:

1. There exists a constant $\lambda \geq 0$ such that

$$\langle \rho, \eta^T X \eta \rangle \leq \lambda$$

   for all Gaussian states ρ.

2. The matrix X is negative semidefinite.

For H^∞ control, we need the following special case:

Definition

The system is said to be *Bounded Real* if it is dissipative with supply rate

$$
\begin{aligned}
r(x, \beta_w) &= \beta_z^T \beta_z - \beta_w^T \beta_w \\
&= [x^T \beta_w^T] \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - I \end{bmatrix} \begin{bmatrix} x \\ \beta_w \end{bmatrix}.
\end{aligned}
$$

It is *Strictly Bounded Real* if it is strictly dissipative with this supply rate.

Corollary

The system is bounded real if and only if there exists a positive definite symmetric matrix $X \in \mathbf{R}^{n\times n}$ such that the following matrix inequality is satisfied:

$$\left(\begin{array}{cc} A^T X + XA + C^T C & C^T D + XB \\ B^T X + D^T C & D^T D - I \end{array} \right) \leq 0.$$

Furthermore, the quantum stochastic system is strictly bounded real if and only if there exists a positive definite symmetric matrix $X \in \mathbf{R}^{n\times n}$ such that the following matrix inequality is satisfied:

$$\left(\begin{array}{cc} A^T X + XA + C^T C & C^T D + XB \\ B^T X + D^T C & D^T D - I \end{array} \right) < 0.$$

Moreover, in both cases the required constant $\lambda \geq 0$ can be chosen as

$$\lambda = \mathrm{tr}\left[\left[\begin{array}{c} B^T \\ G^T \end{array} \right] X \left[\begin{array}{cc} B & G \end{array} \right] F \right].$$

Corollary

The following statements are equivalent

1. *The system is strictly bounded real.*
2. A *is a stable matrix and* $\|C(sI - A)^{-1}B + D\|_\infty < 1$.
3. $I - D^T D > 0$ *and there exists a positive definite matrix* $X > 0$ *such that*

$$\begin{aligned} &A^T X + XA + C^T C \\ &\quad + (XB + C^T D)(I - D^T D)^{-1}(B^T X + D^T C) < 0. \end{aligned}$$

4. *The algebraic Riccati equation*

$$\begin{aligned} &A^T X + XA + C^T C \\ &\quad + (XB + C^T D)(I - D^T D)^{-1}(B^T X + D^T C) = 0 \end{aligned}$$

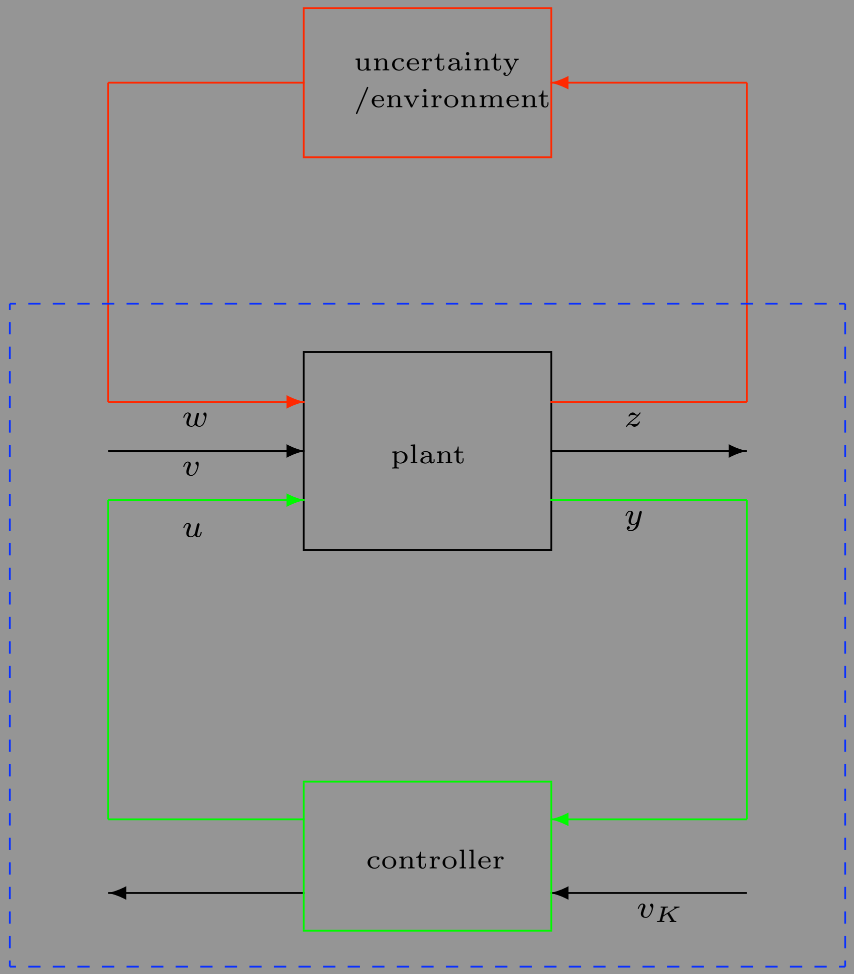
   has a stabilizing solution $X \geq 0$.

Furthermore, if these statements hold then $0 < X$.

H-Infinity Controller Synthesis

Problem:

- Given a plant and gain g, find a controller such that the closed loop system from w to z has gain less than g.



(to reduce effect of environment/uncertainty)

Idea of solution:

- Form a closed loop system consisting of the plant and controller (of prescribed form - a standard approach)
- Apply the strict bounded real lemma
- This (partly) determines linear equations for controller
- Complete, if possible, physical realization of controller

Plant model

$$
\begin{aligned}
dx(t) &= Ax(t)dt + B_0 dv(t) + B_1 dw(t) + B_2 du(t); \\
& \quad x(0) = x; \\
dz(t) &= C_1 x(t)dt + D_{12} du(t); \\
dy(t) &= C_2 x(t)dt + D_{20} dv(t) + D_{21} dw(t)
\end{aligned}
$$

Controller model

$$
\begin{aligned}
d\xi(t) &= A_K\xi(t)dt + B_{K1}dv_K(t) + B_K dy(t) \\
du(t) &= C_K\xi(t)dt + B_{K0}dv_K(t)
\end{aligned}
$$

Closed loop system

Write

$$\eta(t) = \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}.$$

Then

$$\begin{aligned} d\eta(t) &= \tilde{A}\eta(t)dt + \tilde{B}dw(t) + \tilde{G}d\tilde{v}(t) \\ dz(t) &= \tilde{C}\eta(t)dt + \tilde{D}d\tilde{v}(t) \end{aligned}$$

where

$$\begin{aligned} \tilde{v}(t) &= \begin{bmatrix} v(t) \\ v_K(t) \end{bmatrix}; \quad \tilde{A} = \begin{bmatrix} A & B_2C_K \\ B_KC_2 & A_K \end{bmatrix}; \\ \tilde{B} &= \begin{bmatrix} B_1 \\ B_KD_{21} \end{bmatrix}; \quad \tilde{G} = \begin{bmatrix} B_0 & B_2B_{K0} \\ B_KD_{20} & B_{K1} \end{bmatrix}; \\ \tilde{C} &= \begin{bmatrix} C_1 & D_{12}C_K \end{bmatrix}; \quad \tilde{D} = \begin{bmatrix} 0 & D_{12}B_{K0} \end{bmatrix}. \end{aligned}$$

Assumptions

1. $D_{12}^T D_{12} = E_1 > 0$.
2. $D_{21} D_{21}^T = E_2 > 0$.
3. The matrix $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ is full rank for all $\omega \geq 0$.
4. The matrix $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ is full rank for all $\omega \geq 0$.

Riccati Equations

The following Riccati equations will be used to construct the controller:

$$
\begin{aligned}
(A - B_2 E_1^{-1} D_{12}^T C_1)^T X + X(A - B_2 E_1^{-1} D_{12}^T C_1) \\
+ X(B_1 B_1^T - g^2 B_2 E_1^{-1} B_2') X \\
+ g^{-2} C_1^T (I - D_{12} E_1^{-1} D_{12}^T) C_1 = 0;
\end{aligned}
$$

$$
\begin{aligned}
(A - B_1 D_{21}^T E_2^{-1} C_2) Y + Y(A - B_1 D_{21}^T E_2^{-1} C_2) \\
+ Y(g^{-2} C_1^T C_1 - C_2^T E_2^{-1} C_2) Y \\
+ B_1 (I - D_{21}^T E_2^{-1} D_{21}) B_1^T = 0.
\end{aligned}
$$

Controller

The controller matrices A_K, B_K and C_K are determined as follows:

$$\begin{aligned} A_K &= A + B_2C_K - B_KC_2 + (B_1 - B_KD_{21})B_1^TX; \\ B_K &= (I - YX)^{-1}(YC_2^T + B_1D_{21}^T)E_2^{-1}; \\ C_K &= -E_1^{-1}(g^2B_2^TX + D_{12}^TC_1). \end{aligned}$$

The remaining controller matrices B_{K0}, B_{K1} and any required noise sources v_K will be discussed below.

Theorem

Necessity. *If there exists a controller of the above form such that the resulting closed loop system is strictly bounded real with disturbance attenuation* g, *then the above Riccati equations will have solutions* $X \geq 0$ *and* $Y \geq 0$ *satisfying*

1. $A - B_2 E_1^{-1} D_{12}^T C_1 + (B_1 B_1^T - g^2 B_2 E_1^{-1} B_2')X$ *is a stability matrix.*
2. $A - B_1 D_{21}^T E_2^{-1} C_2 + Y(g^{-2} C_1^T C_1 - C_2^T E_2^{-1} C_2)$ *is a stability matrix.*
3. *The matrix* XY *has a spectral radius strictly less than one.*

Sufficiency. *Suppose the Riccati equations have solutions $X \geq 0$ and $Y \geq 0$ satisfying*

1. $A - B_2 E_1^{-1} D_{12}^T C_1 + (B_1 B_1^T - g^2 B_2 E_1^{-1} B_2')X$ *is a stability matrix.*
2. $A - B_1 D_{21}^T E_2^{-1} C_2 + Y(g^{-2} C_1^T C_1 - C_2^T E_2^{-1} C_2)$ *is a stability matrix.*
3. *The matrix XY has a spectral radius strictly less than one.*

If the controller is such that the matrices A_K, B_K, C_K are given by

$$
\begin{aligned}
A_K &= A + B_2 C_K - B_K C_2 + (B_1 - B_K D_{21}) B_1^T X; \\
B_K &= (I - YX)^{-1} (Y C_2^T + B_1 D_{21}^T) E_2^{-1}; \\
C_K &= -E_1^{-1} (g^2 B_2^T X + D_{12}^T C_1).
\end{aligned}
$$

then the resulting closed loop system will be strictly bounded real with disturbance attenuation g.

Physical Realization

Consider the case of a fully quantum controller, of dimension $n = \dim x = \dim \xi$ (even), and no classical degrees of freedom. We take

$$C^{\xi\xi} = 2i\,\mathrm{diag}(J, \ldots, J),$$

an $n \times n$ matrix, where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The input and output channels are also fully quantum, with

$$\begin{aligned} F_y &= I + i\,\mathrm{diag}(J, \ldots, J) \\ F_u &= I + i\,\mathrm{diag}(J, \ldots, J) \\ F_{v_K} &= I + i\,\mathrm{diag}(J, \ldots, J) \end{aligned}$$

We seek a Hamiltonian

$$H_{controller} = \frac{1}{2}\xi^T R \xi$$

and coupling operator vector

$$L_{controller} = \Lambda \xi$$

Here, the matrices $R \geq 0$ and Λ are to be determined and compatible with the controller matrices A_K, B_K and C_K given above.

General form

Edwards-Belavkin, 2005

$$d\xi = -iC^{\xi\xi}(R + \Im(\Lambda^\dagger \Lambda))\xi dt + C^{\xi\xi} \begin{bmatrix} -\Lambda^\dagger & \Lambda^T \end{bmatrix} \Gamma \begin{bmatrix} dv_K \\ dy \end{bmatrix}$$

$$du = P_{N_q^u}^T \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix} \left(\begin{bmatrix} \Lambda + \Lambda^* \\ -i\Lambda + i\Lambda^* \end{bmatrix} \xi dt + P_{n_c} \begin{bmatrix} dv_K \\ dy \end{bmatrix} \right)$$

$$\Sigma = \begin{bmatrix} I_{N_q^u \times N_q^u} & 0_{N_q^u \times (n_c - N_q^u)} \end{bmatrix}.$$

The basic idea is to add additional noise channels to ensure the commutation relations are preserved.

Theorem

The controller is fully quantum realizable if and only if

$$iC^{\xi\xi}A_K + iA_K^T(C^{\xi\xi})^T \geq 0$$

Furthermore, if this condition is satisfied then explicit formulas exist (omitted).

Robust Stability

- The strict bounded real property of the closed loop system obtained above guarantees stability robustness against real parameter uncertainties.
- We regard the true physical system as a perturbation of the nominal system used for design.

We suppose that the true closed loop quantum system is described by the equations

$$d\eta(t) = \bar{A}\eta(t)dt + \tilde{G}d\tilde{v}(t)$$

where $\bar{A} = \tilde{A} + \tilde{B}\Delta\tilde{C}$ and Δ is a constant matrix satisfying

$$\Delta^T\Delta \leq I.$$

This closed loop quantum system is said to be *mean square stable* if there exists a real positive definite matrix $X > 0$ and a constant $\lambda > 0$ such that

$$\langle \eta(t)^T X \eta(t) \rangle + \int_0^t \langle \eta(s)^T \eta(s) \rangle ds \leq \langle \eta(0)^T X \eta(0) \rangle + \lambda t \quad \forall t > 0$$

for all Gaussian ρ.

Lemma

The closed loop system is mean square stable if and only if the matrix $\bar{A}$ is asymptotically stable.

Proof: Uses above dissipation results and standard Lyapunov results.

Theorem

If the nominal closed loop system is strictly bounded real then the true closed loop system is mean square stable for all Δ satisfying

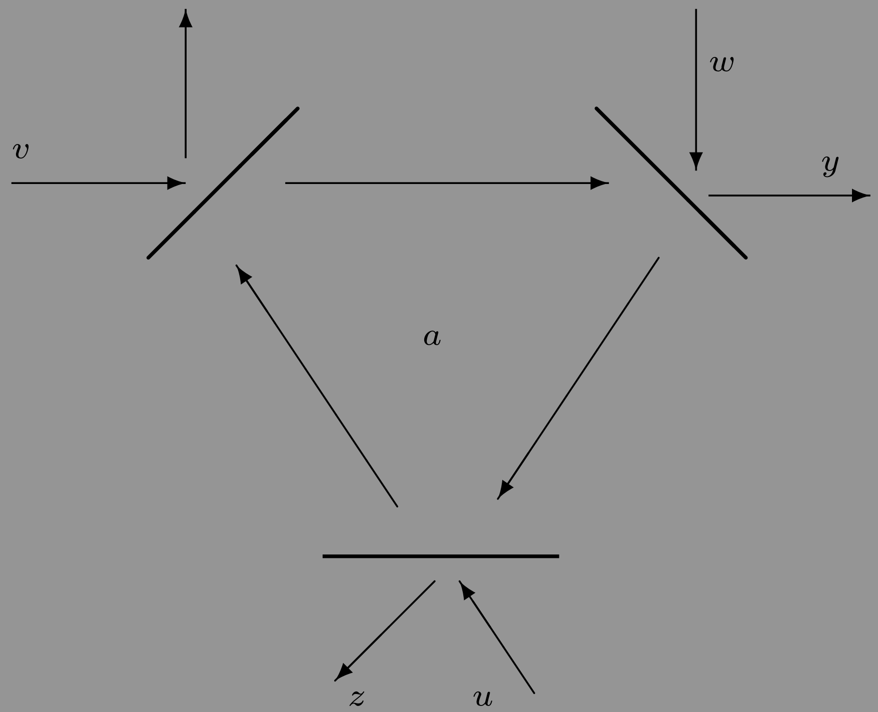
$$\Delta^T \Delta \leq I.$$

Proof: Uses above lemma and standard small gain theorem.

Examples from Quantum Optics

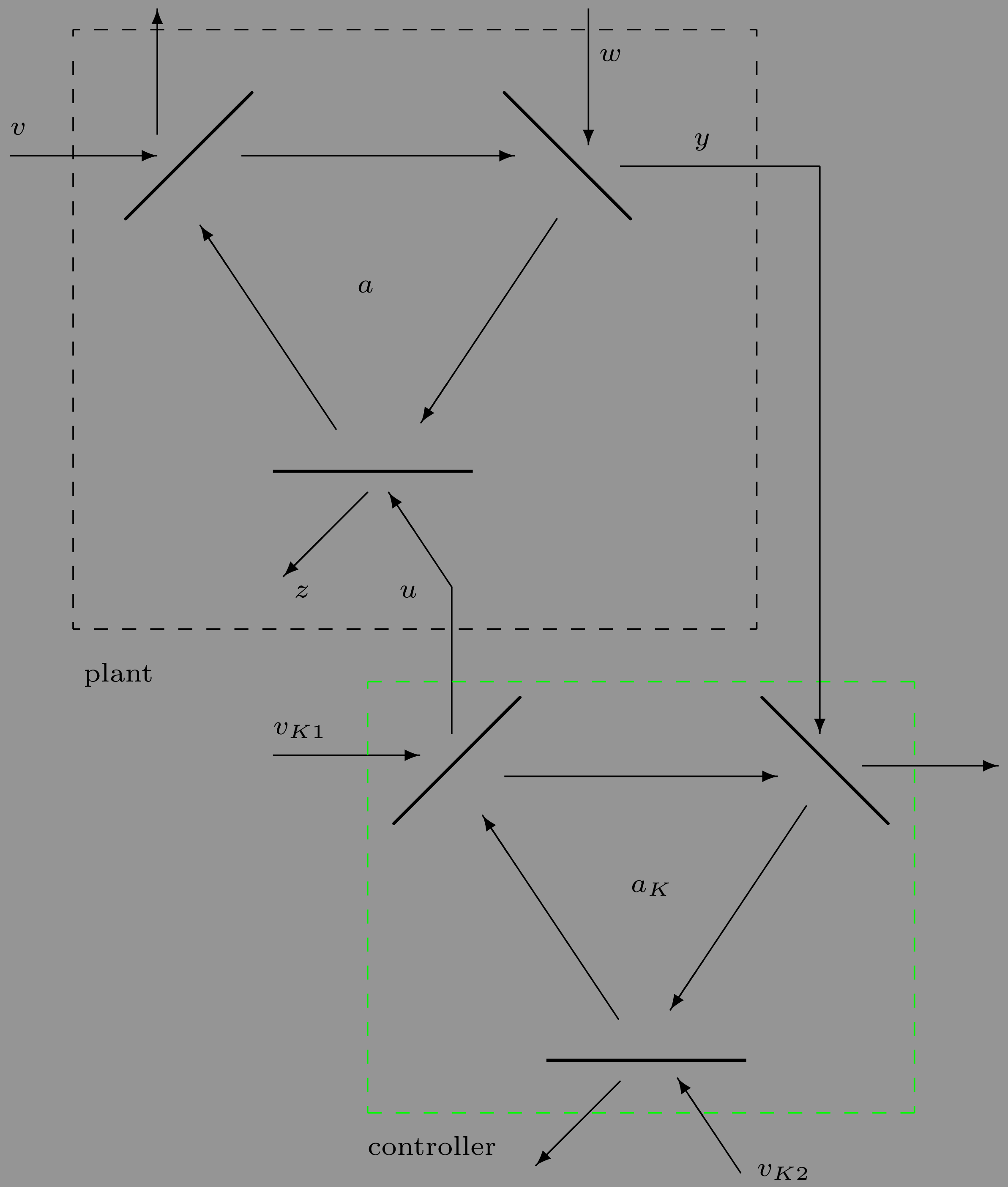
- H-infinity: Quantum controller
- H-infinity: Classical controller
- Robust stability: parameter uncertainty

Plant

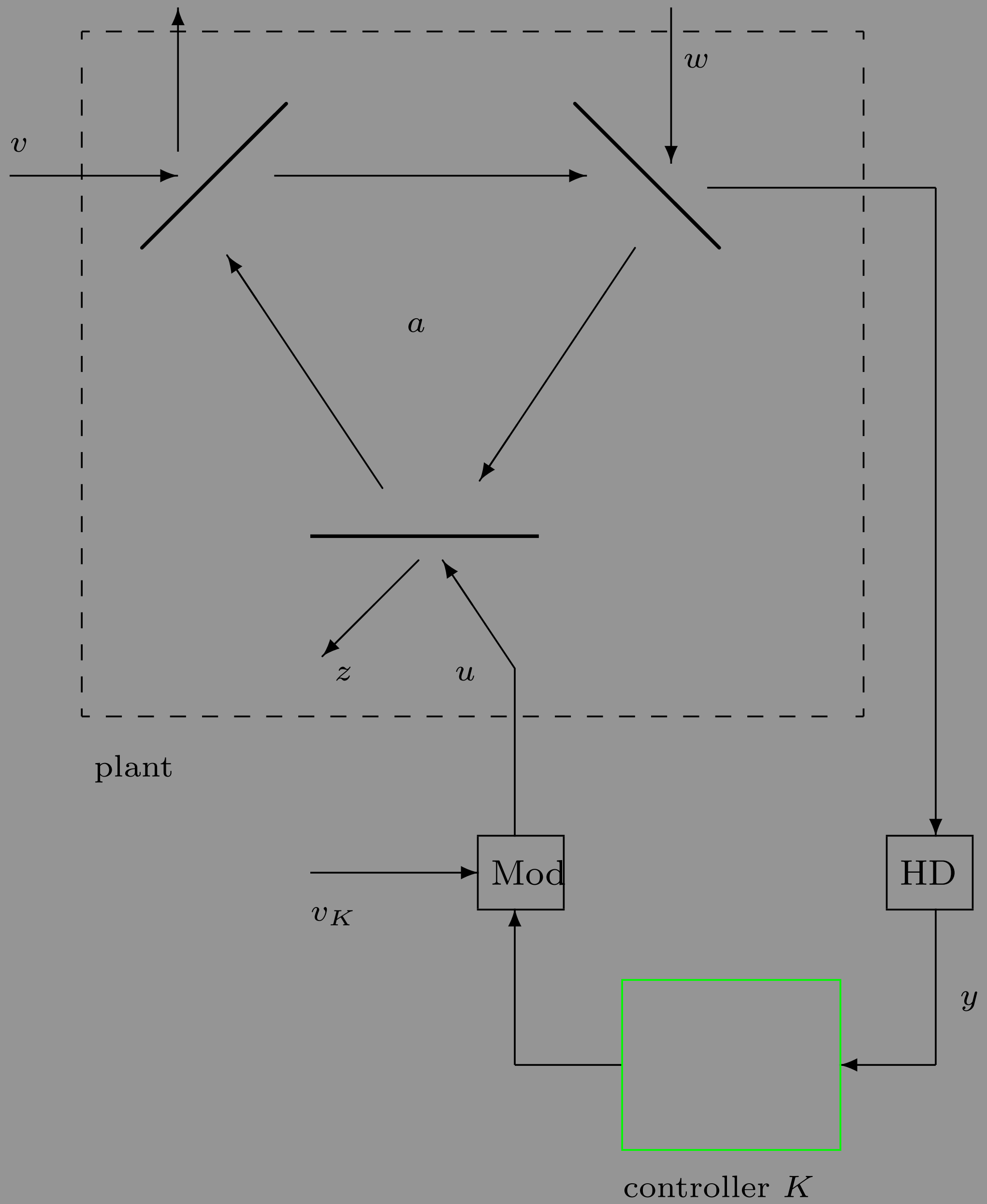


$$
\begin{aligned}
da &= -\frac{\gamma}{2} a dt - \sqrt{\kappa}\, dV - \sqrt{\kappa}\, dW - \sqrt{\kappa}\, dU \\
dZ &= \sqrt{\kappa}\, a dt + dU \\
dY &= \sqrt{\kappa}\, a dt + dW && \text{(quantum output)} \\
dY &= \sqrt{\kappa}\, (a + a^*) dt + d\tilde{w}_1 && \text{(classical output)}
\end{aligned}
$$

Closed loop with quantum controller



Closed loop with classical controller



Plant with parameter uncertainty

$$
\begin{aligned}
da &= -\frac{\gamma+\delta}{2} a dt - \sqrt{\kappa+\delta}\, dV - \sqrt{\kappa}\, dW - \sqrt{\kappa}\, dU \\
dZ &= \sqrt{\kappa}\, a dt + dU \\
dY &= \sqrt{\kappa}\, a dt + dW.
\end{aligned}
$$

where δ is a constant but unknown uncertain parameter satisfying a known bound

$$|\delta| \le \mu$$

To apply the robust stability results, we write

$$A = -\frac{\gamma}{2} I + \tilde{B}_1 \Delta \tilde{C}_1 \text{ where } \tilde{B}_1 = \frac{\mu}{2}, \quad \tilde{C}_1 = T^{-1}$$

and T is any non-singular matrix.

Then

$$\Delta = \frac{\delta}{\mu} I \implies \Delta^T \Delta \le I$$

and robust stability will follow.

Conclusions

- We have extended standard methods of H-infinity robust control to the domain of linear quantum systems
- The controllers may be quantum or classical
- There are interesting new realization questions
- The results provide the beginnings of useful robust control design methods for quantum technology