Algebraic Characterizations of Dissipativity for Linear Quantum Stochastic Systems

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- Problem Statement
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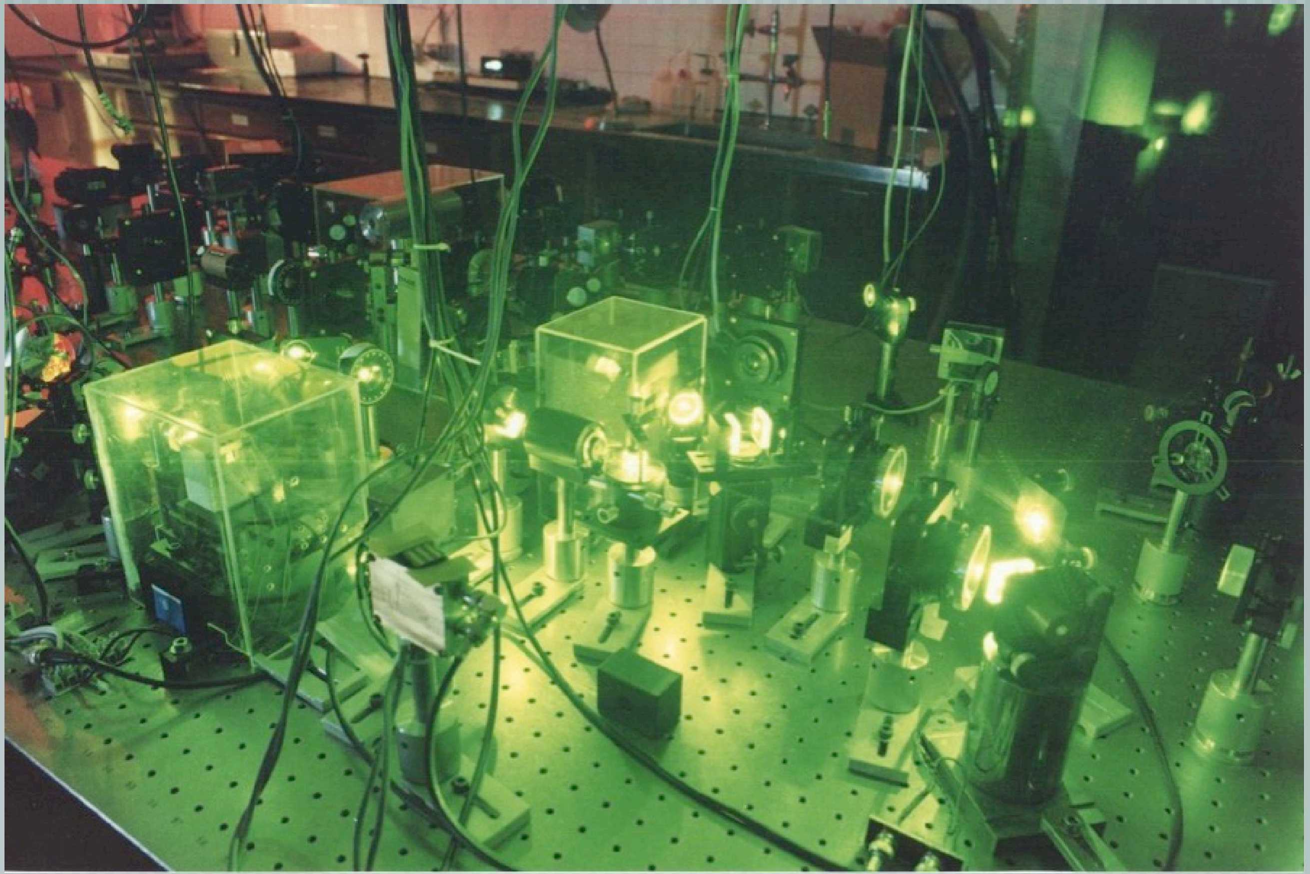
Motivation

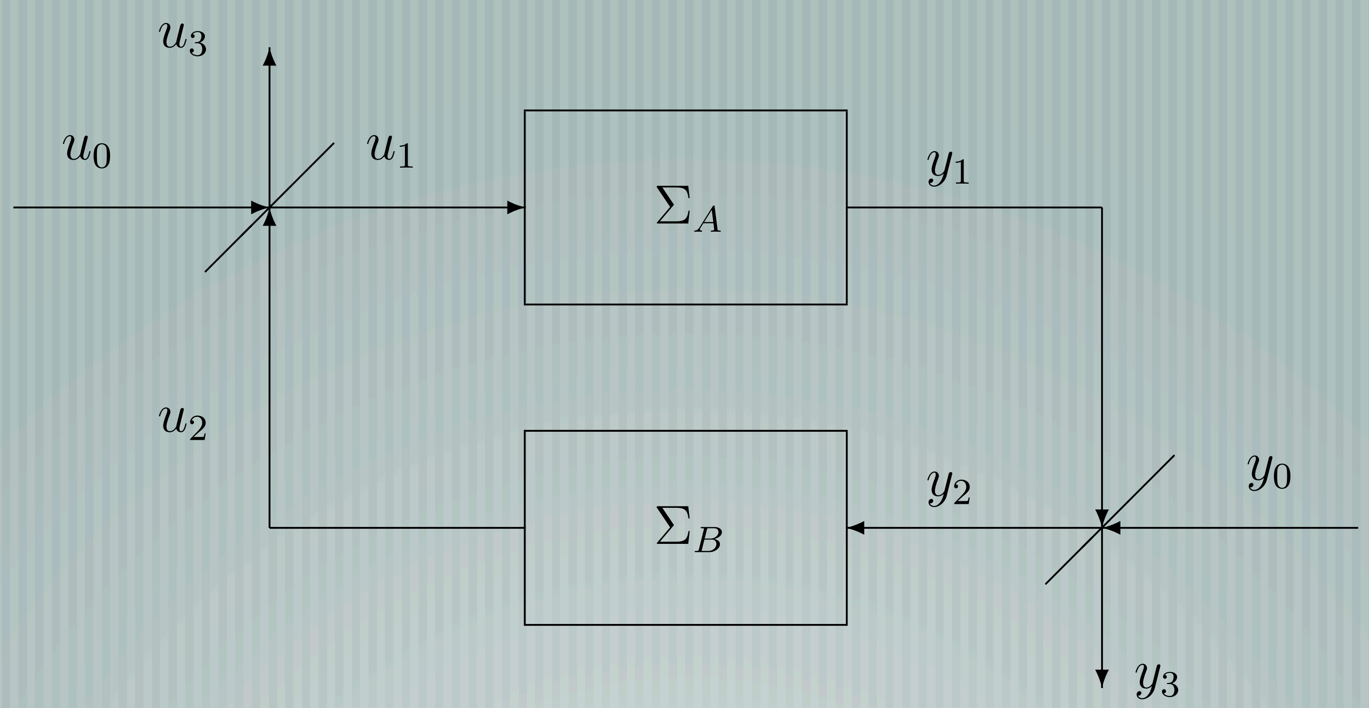
Linear quantum systems; e.g quantum optics



- Quantum systems, like all real world systems, are subject to noise and uncertainty
- This can lead to performance degradation, and even instability

Feedback Stability

Even when individual components are stable, feedback interconnections need not be.



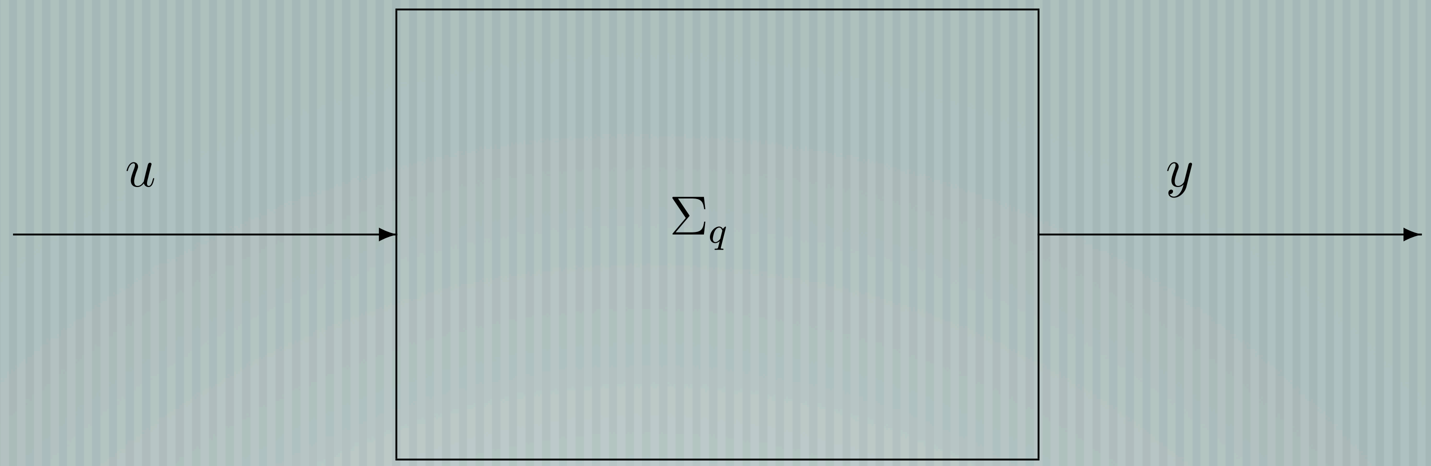
The small gain theorem asserts stability of the feedback loop if the loop gain is less than one:

$$g_A g_B < 1$$

Classical: Zames, Sandberg, 1960's

Quantum: D'Helon-James, 2006

Stability is quantified in a mean-square sense as follows.



$$
\begin{aligned}
dU(t) &= \beta_u(t)dt + dB_u(t) \\
dY(t) &= \beta_y(t)dt + dB_y(t)
\end{aligned}
$$

$$\| \beta_y \|_t^2 \leq \mu + \lambda t + g^2 \| \beta_u \|_t^2$$

- We seek system theoretic ways of characterizing dissipativity for quantum feedback networks
- This will provide a foundation for quantum robust stability analysis and design
- We focus (initially) on linear quantum systems, such as quantum optical systems

Problem Statement

Linear/Gaussian Quantum Model:

We consider systems described by noncommutative stochastic models of the form

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t) + Gdv(t) \quad x(0) = x, \\ dz(t) &= Cx(t)dt + Ddw(t) \end{aligned}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

is a vector of possibly noncommutative plant variables.

The “disturbance” input

$$w(t) = \begin{bmatrix} w_1(t) \\ \vdots \\ w_{n_w}(t) \end{bmatrix}$$

is assumed to admit the decomposition

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t)$$

where $\tilde{w}(t)$ is the noise part of $w(t)$ and $\beta_w(t)$ is the finite variation part of $w(t)$. Similarly, the “error” ouput

$$z(t) = \begin{bmatrix} z_1(t) \\ \vdots \\ z_{n_z}(t) \end{bmatrix}$$

is assumed to admit the decomposition

$$dz(t) = \beta_z(t)dt + d\tilde{z}(t)$$

and hence $\beta_z(t) = Cx(t) + D\beta_w(t)$.

The noise vectors

$$v(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_{n_v}(t) \end{bmatrix}; \ \tilde{w}(t) = \begin{bmatrix} \tilde{w}_1(t) \\ \vdots \\ \tilde{w}_{n_w}(t) \end{bmatrix};$$

are vectors of noncommutative Wiener processes, with Ito table

$$dv(t)dv^T(t) = F_v dt, \ d\tilde{w}(t)d\tilde{w}^T(t) = F_{\tilde{w}} dt,$$

where F_v and $F_{\tilde{w}}$ are non-negative Hermitian matrices;

This model is defined in a quantum probability space $(\mathscr{A}, \mathbb{P})$, where $\mathscr{A}$ is a von Neumann algebra and $\mathbb{P}$ is a state on this algebra.

E.g. $(\mathscr{A}, \mathbb{P}) = (\mathscr{B} \otimes \mathscr{W}, \rho \otimes \phi)$, where $\mathscr{B} = \mathscr{B}(\mathsf{h})$ is the algebra of system operators with state ρ, and $\mathscr{W} = \mathscr{B}(\mathsf{F})$ is the algebra of operators on Fock space with vacuum state ϕ.

In the above, $x(0) \in \mathscr{B}$, and the noises belong to $\mathscr{W}$.

E.g. single degree of freedom quantum particle, where

$$x = \begin{bmatrix} q \\ p \end{bmatrix},$$

with non-commuting position q and momentum p operators:

$$[q, p] = qp - pq = \hbar i$$

E.g. disturbance w represents a coherent optical field (a laser model):

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t)$$

Here, $\beta_w(t)$ is a complex valued function of time (modulation of the field), and

$$\tilde{w}(t) = \begin{bmatrix} Q(t) \\ P(t) \end{bmatrix}$$

is a standard quantum Wiener process (representing quantum noise), with Ito table:

$$d\tilde{w}(t)d\tilde{w}^T(t) = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} dt$$

Definition

Given an operator valued quadratic form

$$r(x,\beta_w) = [x^T \beta_w^T] R \begin{bmatrix} x \\ \beta_w \end{bmatrix}$$

where R is a given real symmetric matrix, we say the system (1) is *dissipative with supply rate* $r(x,\beta_w)$ if there exists a positive operator valued quadratic form $V(x) = x^T X x$ (where X is a real positive definite symmetric matrix) and a constant $\lambda > 0$ such that

$$\begin{aligned} &\langle V(x(t))\rangle + \int_0^t \langle r(x(s),\beta_w(s))\rangle ds \\ \leq \quad &\langle V(x(0))\rangle + \lambda t \quad \forall t > 0, \end{aligned}$$

for all Gaussian ρ. Here we use the shorthand notation

$$\langle \cdot \rangle \equiv \mathbb{P}(\cdot)$$

for expectation.

We say that the system (1) is *strictly dissipative* if there exists a constant $\epsilon > 0$ such that the above inequality holds with the matrix R replaced by the matrix $R + \epsilon I$.

Main Results

Theorem

Given a quadratic form $r(x, \beta_w)$ defined as above, then the quantum stochastic system is dissipative with supply rate $r(x, \beta_w)$ if and only if there exists a real positive definite symmetric matrix X such that the following matrix inequality is satisfied:

$$\begin{pmatrix} A^T X + XA + R_{11} & R_{12} + XB \\ B^T X + R_{12}^T & R_{22} \end{pmatrix} \leq 0.$$

Furtheremore, the system is strictly dissipative if and only if there exists a real positive definite symmetric matrix X such that the following matrix inequality is satisfied:

$$\begin{pmatrix} A^T X + XA + R_{11} & R_{12} + XB \\ B^T X + R_{12}^T & R_{22} \end{pmatrix} < 0$$

is satisfied.

Moreover, if either of the matrix inequalities holds then the required constant $\lambda \geq 0$ can be chosen as

$$\lambda = \mathrm{tr}\left[\left[\begin{array}{c} B^T \\ G^T \end{array}\right] X \left[\begin{array}{cc} B & G \end{array}\right] F\right]$$

where the matrix F is defined by the following Ito table

$$F dt = \left[\begin{array}{c} d\tilde{w}^T \\ dv^T \end{array}\right] \left[\begin{array}{cc} d\tilde{w} & dv \end{array}\right].$$

The proof depends on the following identity

$$\langle V(x(t)) \rangle = \langle \rho, E_0[V(x(t))] \rangle,$$

where E_0 denotes expectation with respect to ϕ, and on the following

Lemma

Consider a real symmetric matrix X and corresponding operator valued quadratic form $\eta^T X \eta$ for the system. Then the following statements are equivalent:

1. There exists a constant $\lambda \geq 0$ such that

$$\langle \rho, \eta^T X \eta \rangle \leq \lambda$$

for all Gaussian states ρ on $\mathscr{B}(\mathsf{h})$.

2. The matrix X is negative semidefinite.

Corollaries

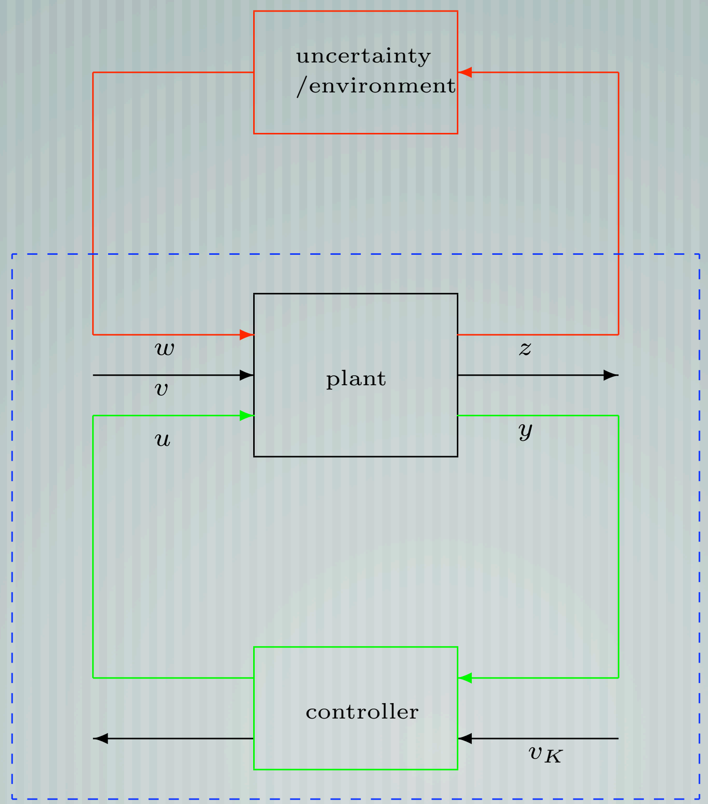
The following results follow from the above with suitable choices of supply rates:

- Bounded Real Lemma
- Positive Real
- Strict versions of BRL, PRL

Applications

The H-Infinity Robust Control Problem:

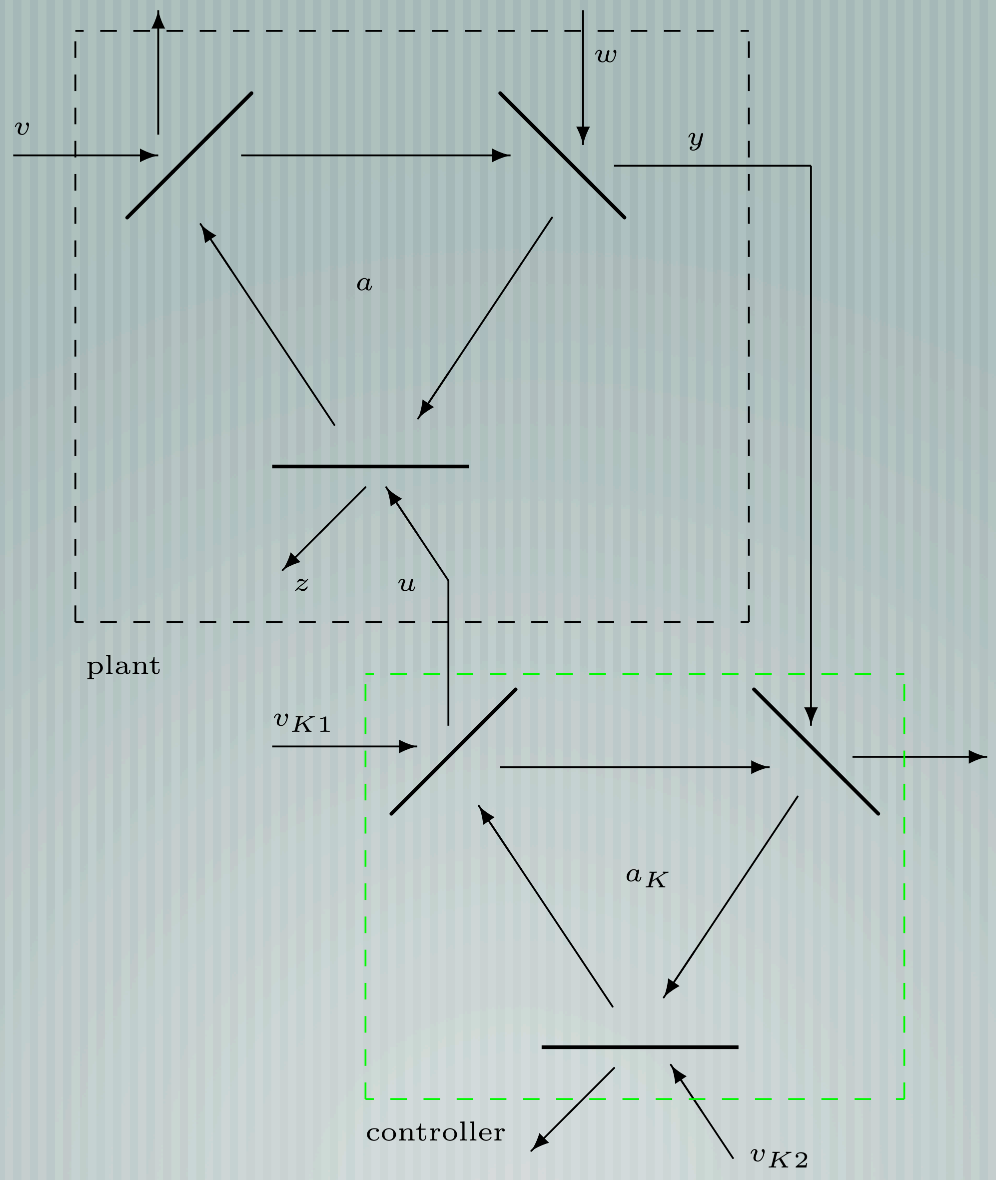
Given a system (plant), find another system (controller) so that the gain from w to z is small. This is one way of reducing the effect of uncertainty or environmental influences.



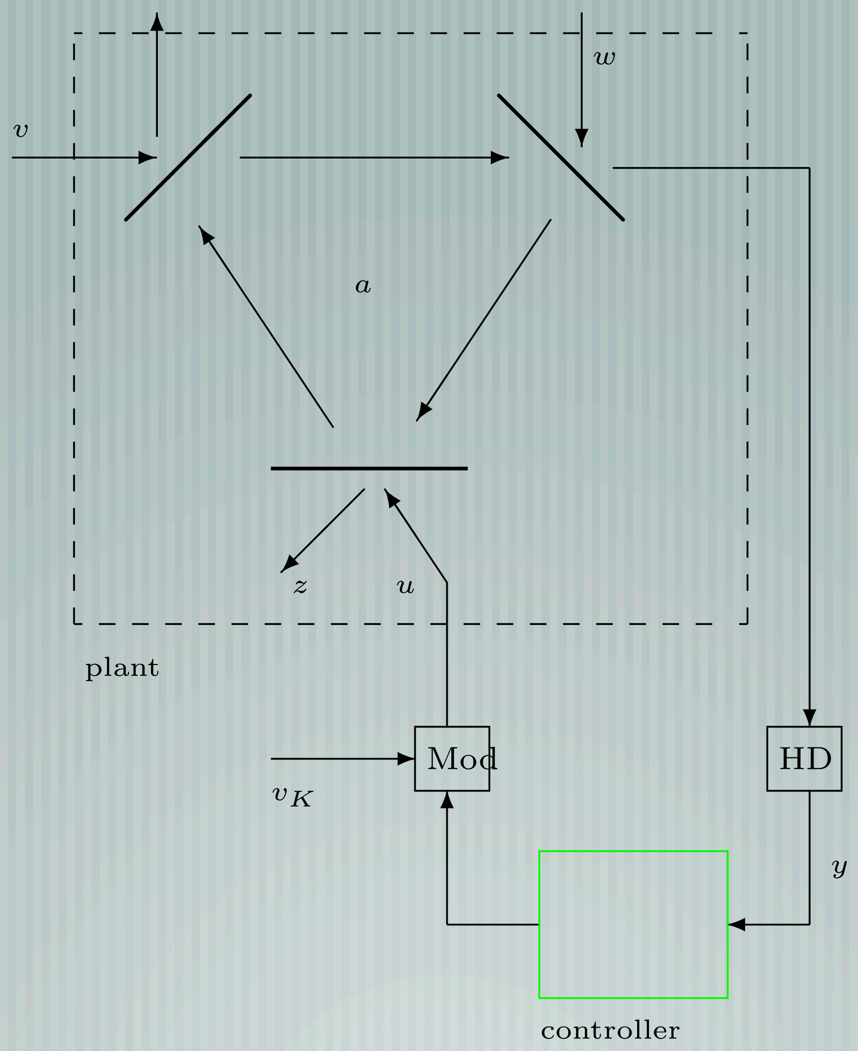
Classical: Zames, late 1970's

Quantum: D'Helon-James, 2005; James-Nurdin-Petersen, 2006

Example with Quantum Controller:



Example with Classical Controller:



Conclusions

- We have provided an algebraic characterization of dissipativity applicable to linear quantum stochastic systems
- Generalizations of the Bounded Real Lemma and Positive Real Lemma follow
- This work provides a foundation for quantum robust stability analysis and design