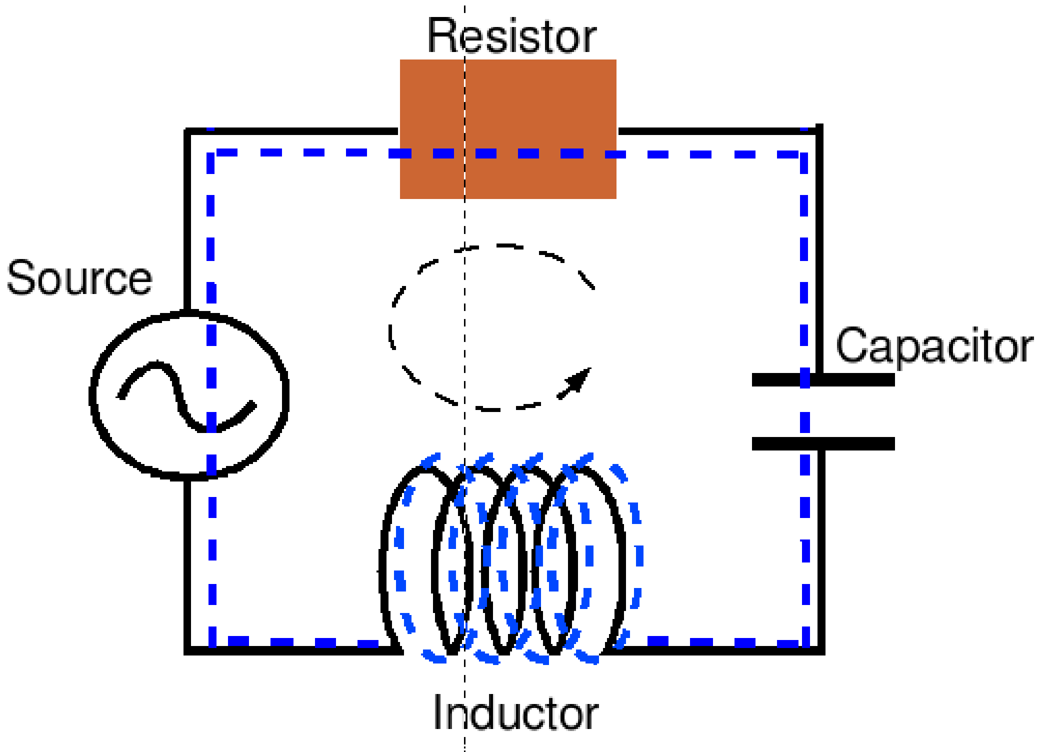
IMPEDANCE and NETWORKS

- Kirchoff's laws
- Charge inside metals
- Skin effect
- Impedance, Resistance, Capacitance, Inductance
- Mutual Inductance, Transformers
- Stray impedance

Kirchhoff's Voltage Law

- Kirchhoff's voltage law is based on **Faraday's law.**

$$\oint_{\gamma} \mathbf{E}.\mathbf{dl} = -\int_{A} \frac{\partial \mathbf{B}}{\partial t}.\mathbf{dA}$$



Charge inside Metals

- **Under no conditions can charge appear inside a metal**
- We showed that ...

$$\frac{\partial \mathbf{E}}{\partial t} = -\frac{\sigma}{\epsilon_0}\mathbf{E}$$

- The solution to this equation is $\mathbf{E} = \mathbf{E_0} \exp\left(-\sigma/\epsilon_0 t\right)$
- For copper $\sigma/\epsilon_0 = 6.55 \times 10^{18} s^{-1}$. So that surplus charge must decay in about 10^{-19} seconds.

Skin Depth

- **Electromagnetic waves, j, E, B, ... only penetrate a distance δ into a metal. Check the magnitude of δ in lab and web exercises.**
- The wave equation for metals simplifies to...

$$\frac{\partial^2 E_y(z)}{\partial z^2} = j\omega\sigma\mu_0\, E_y(z)$$

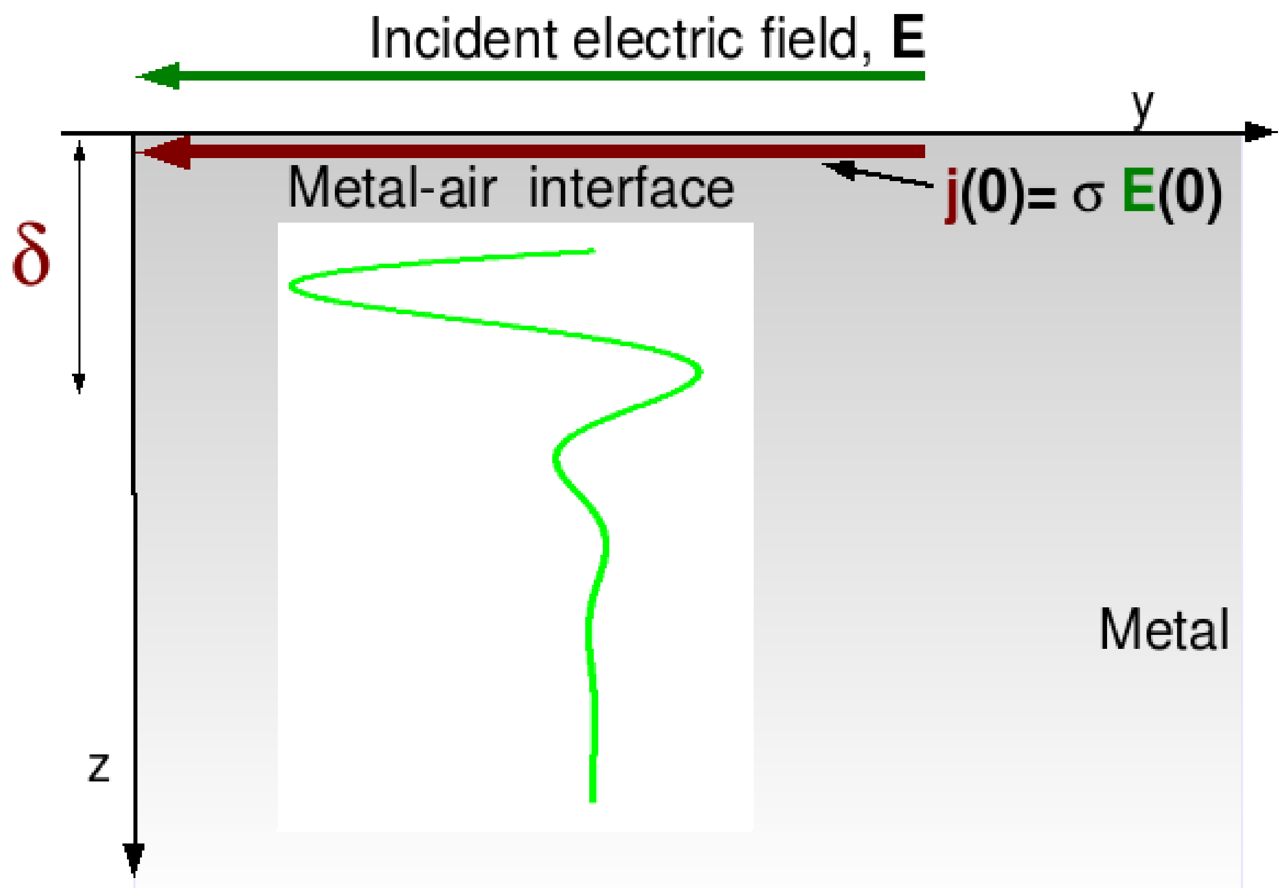
- The solution...

$$E_y(z) = \exp\left(-\frac{1+j}{\delta}\, z\right)$$

- where δ the **skin depth** is given by...

$$\delta = \sqrt{\frac{2}{\omega\sigma\mu_0}}$$

Skin Depth



Incident electric field, **E**
y
Metal-air interface
j(0)= σ E(0)
δ
Metal
z

Impedance per Square

- By integrating the formula for the electric field inside a metal,

$$E_y(z) = \exp\left(-\frac{1+j}{\delta}z\right)$$

to find the current per unit width I_s we defined the impedance per square as

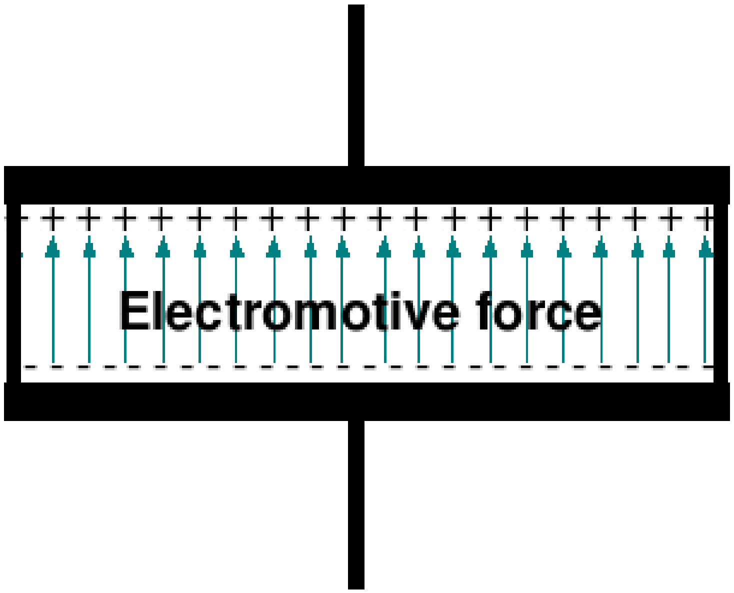
$$Z_s = E_y(0)/I_s = \frac{1+j}{\sigma\delta} = \sqrt{\frac{\pi\mu_0 f}{\sigma}}(1+j)$$

- For a wire of radius, a, length L and circumference $2\pi a$, we obtain

$$Z = \frac{L}{2\pi a}Z_s$$

The Voltage Source

- We can think of voltage sources as capacitors of near infinite capacitance.
- Later we also include some series impedance.
- The lines indicate the direction of the **electromotive force** driving positive current through the positive terminals.

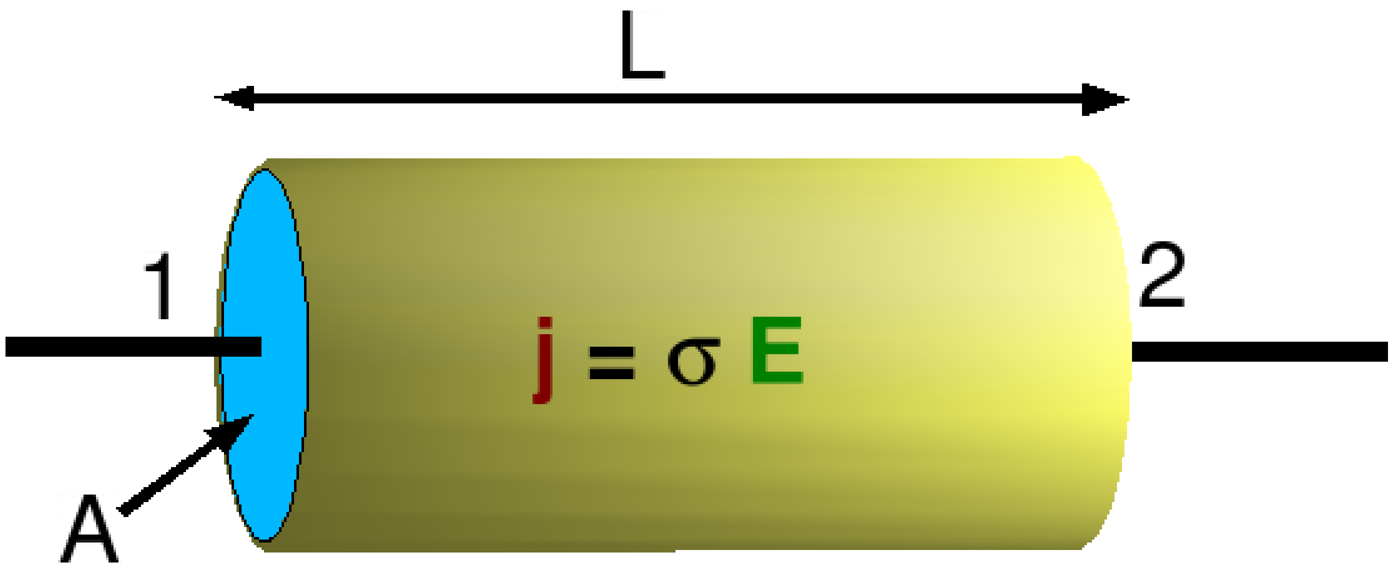


Resistance at Very Low Frequency

- At very low frequency $\delta = \infty$ and the current flows uniformly over the cross-section.

$$V = \int_1^2 E.dl = \frac{JL}{\sigma} = \frac{L}{\sigma A} I = RI$$

- Also define resistivity $\eta = 1/\sigma$.

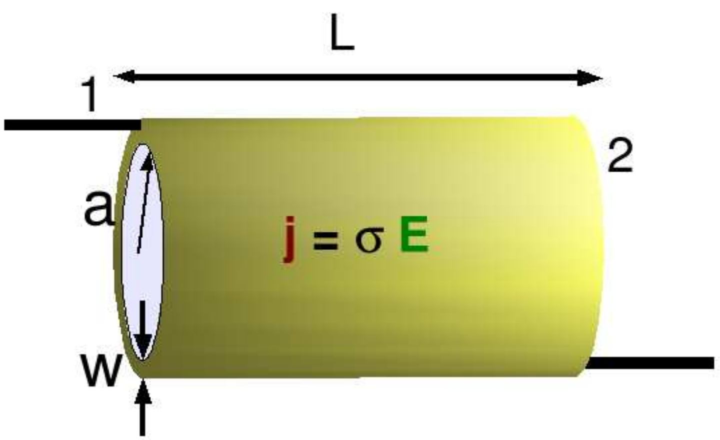


Resistance at Low Frequency: How can we make resistors for radio?

- At low frequency δ is finite and we know that if δ is on the order of the wire radius, then the impedance becomes inductive.
- However if the wire thickness $w \ll \delta$ then the current density will be uniform.

$$V = \int_1^2 E.dl = \frac{JL}{\sigma} = \frac{L}{2\pi a w \sigma} I$$

- Here the impedance is both **resistive and frequency independent**.



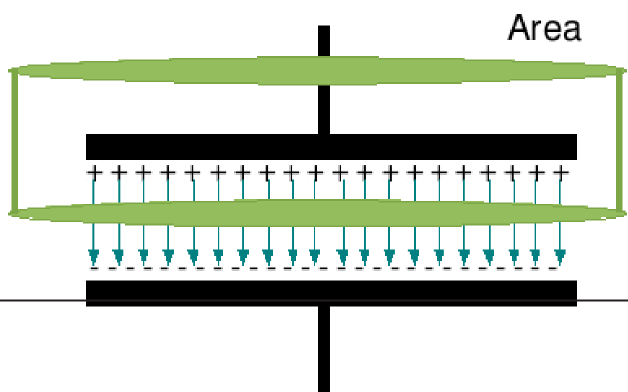
THE AUSTRALIAN NATIONAL UNIVERSITY

The Capacitor at Low Frequencies 1

- We have already seen that a capacitor specifies the voltage developed by a charges on a pair of plates: $q = CV$.
- The impedance of a capacitor can be derived from the current into the positive plate

$$I = \oint \mathbf{j}.\mathbf{dA} = -\frac{\partial q}{\partial t} = -C\frac{\partial V}{\partial t} = j\omega CV$$

- For a parallel plate capacitor: $E = \epsilon_0 \epsilon_r q_s / d$ where q_s is the charge per unit area, ϵ_r is the relative dielectic constant and d is the plate separation.
- The capacitance is therefore given by $C = \epsilon_0 \epsilon_r A/d$

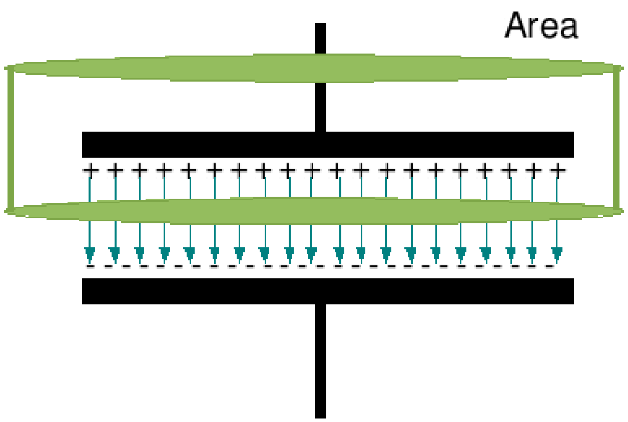


The Capacitor at Low Frequencies 2

- The impedance of a capacitor

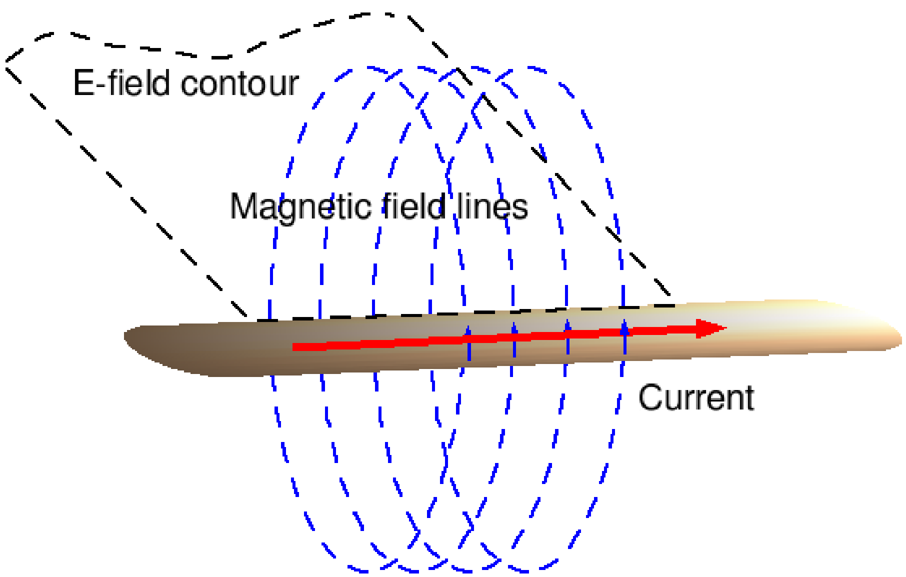
$$Z = \frac{1}{j\omega C}$$

- Valid for the capacitor itself provided that the component is physically small compared to a wavelength. Otherwise we have a **resonant cavity**
- However resistors and capacitors suffer from other physical limitations due to **parasitic** or **stray impedances** that arise due to their construction.



The Inductance and External Impedance

- So far we have looked at that part of the impedance of a wire which results from the external E-field driving a current inside the wire. Both capacitors and resistors develop their impedance **internally**.
- However the current in a circuit induces a magnetic field outside the wire which acts back on the circuit through Faraday's law.

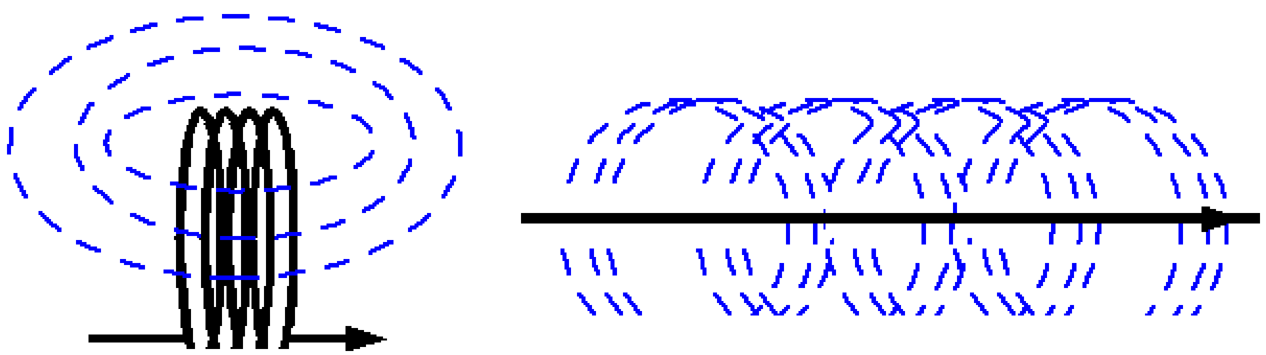


Inductance 1

- **Faraday's law.**

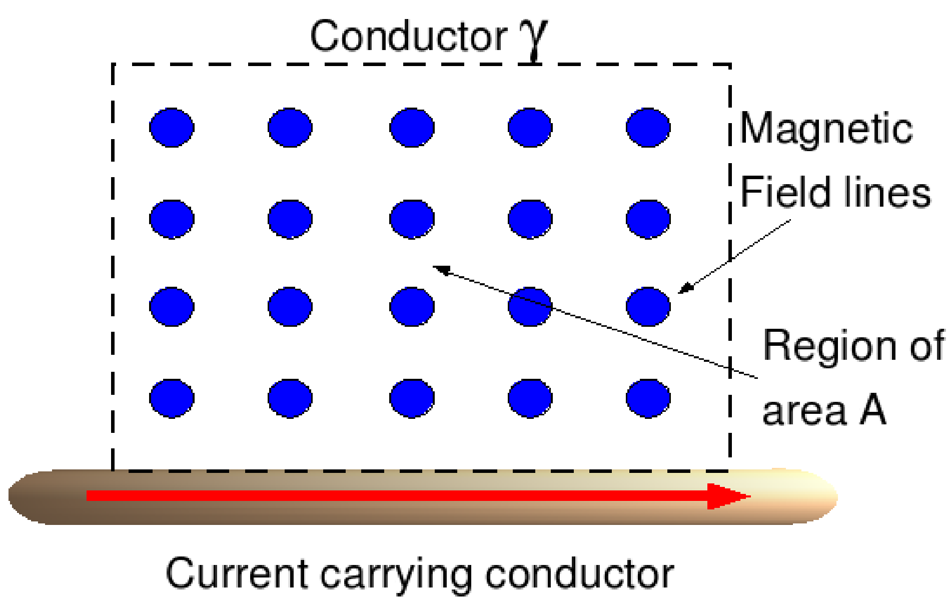
$$\oint_\gamma \mathbf{E}.\mathbf{dl} = -\int_A \frac{\partial \mathbf{B}}{\partial t}.\mathbf{dA}$$

- An inductor is a coil of wire in which the magnetic flux is bunched up.
- To see how the flux exists in the circuit, just unravel the inductor.



Inductance 2

- To define inductance, integrate Faraday's law keeping the contour **outside the wire** consistent with the separation into internal and external impedance.



Inductance 3

- To obtain an expression for the magnetic field use Ampere's law (neglecting displacement current in a small circuit).
- Since the current, I, is the same everywhere in the circuit, the magnetic field in the space surrounding the wire is given by,

$$\mathbf{B} = I\mathbf{G}(\mathbf{x})$$

where $\mathbf{G}(\mathbf{x})$ is a factor depending on the circuit or inductor geometry.

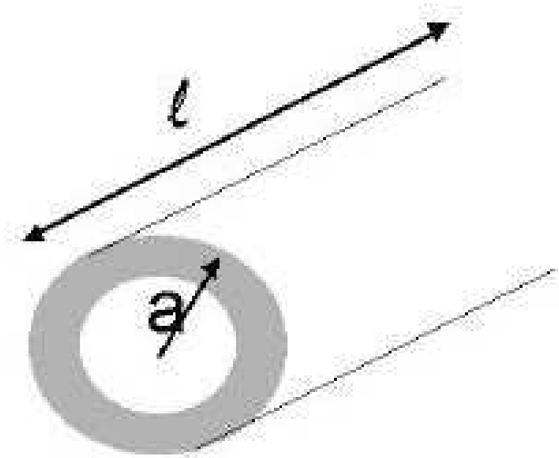
$$\oint_{\gamma} \mathbf{E}.\mathbf{dl} = -\left[\int_{A} \mathbf{G}(\mathbf{x}).\mathbf{dA}\right]\frac{\partial I}{\partial t} = -L\frac{dI}{dt} = j\omega L\, I$$

- An oscillating current in a circuit induces an electromotive force given by $j\omega LI$ for monochromatic oscillations.

Inductance 4

- Inductance is sometimes referred to as **self inductance** due to the fact that the circuit is acting on itself
- The most important case is that of a cylindrical straight conductor where the circuit of integration is precisely that above

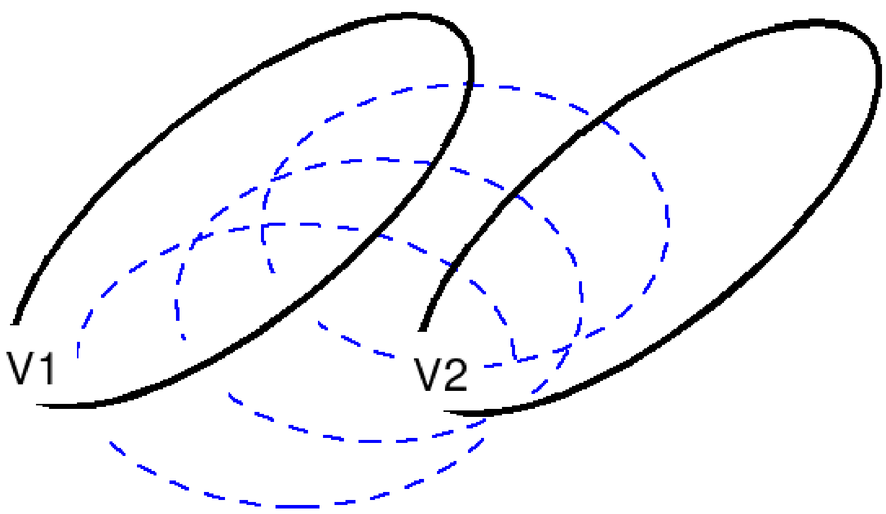
$$L = 0.002l\left[log_e\left(\frac{2l}{a}\right) - \frac{3}{4}\right]: \quad l\,(cm)\; for\; L\,(\mu H)$$



Mutual Coupling

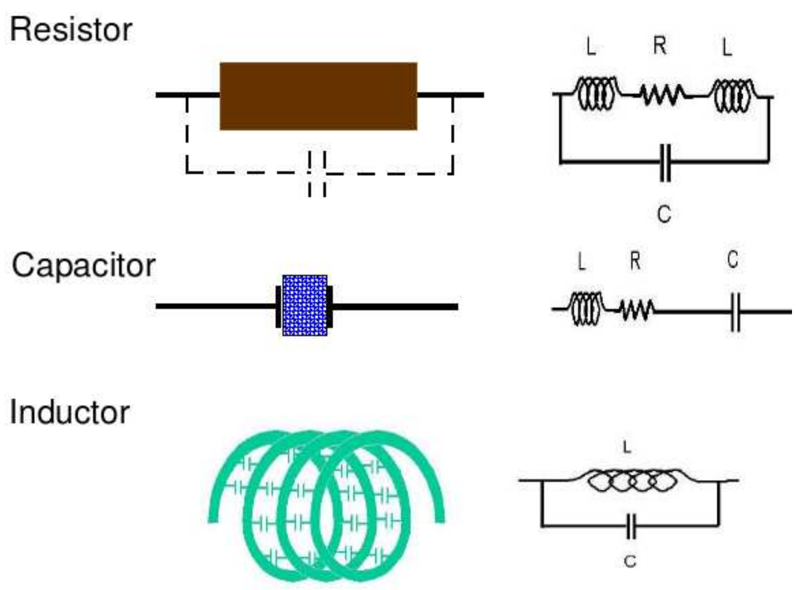
- The magnetic fields of **different** circuits induce electromotive forces in each other.
- **Mutual Inductance, M**

$$V_1 = M\frac{dI_2}{dt} \quad V_2 = M\frac{dI_1}{dt}$$



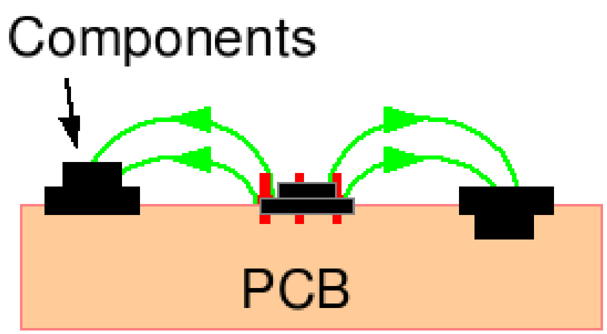
Stray Impedance 1

- Both component structure and external circuit wiring complicate the simple pictures of resistance, capacitance and inductance
- Careful scrutiny of the component geometry reveals hidden or stray impedances that are **usually non-negligible** at radiofrequency.

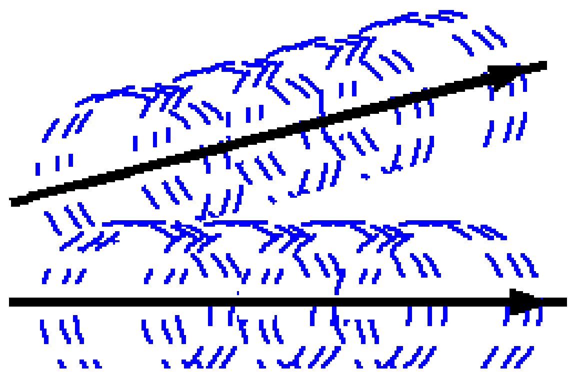


Stray Impedance 2

- On a more general level, the mere presence of an electric field in the space surrounding a circuit constitutes a **capacitive coupling** between adjacent **components**.
- Additionally, the mere presence of a changing magnetic field in a circuit constitutes an **inductive or mutual coupling** between adjacent **circuits**.



Capacitive coupling


Mutual coupling