Connecting face hitting sets in planar graphs

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Abstract

We show that any face hitting set of size n of a connected planar graph with a minimum degree of at least 3 is contained in a connected subgraph of size $5n-6$. Furthermore we show that this bound is tight by providing a lower bound in the form of a family of graphs. This improves the previously known upper and lower bound of $11n-18$ and $3n$ respectively by Grigoriev and Sitters. Our proof is valid for simple graphs with loops and generalizes to graphs embedded in surfaces of arbitrary genus.

1 Introduction

This note is concerned with the relationship among hitting sets of planar graphs. Two prominent hitting sets of planar graphs are the *face hitting set*, i.e., a set of vertices of a planar graph that meets every face (of some planar embedding), and the cycle hitting set better known as *feedback vertex set*, i.e., a set of vertices that has a nontrivial intersection with every cycle of the graph. Since every face of a planar graph is a cycle, every feedback vertex set is also a face hitting set. Conversely, every face hitting set is contained in a feedback vertex set whose size is at most twice as large as the size of the face hitting set [6].

The computational problem asking for the feedback vertex set number (i.e., computing the minimal size of a feedback vertex set in a graph) is well studied. In particular for planar graphs, it is known that computing the feedback vertex set number is $\mathcal{NP}$-complete [5], is sub-exponentially fixed-parameter tractable [4, 7], has a linear kernel [2] and has a PTAS [4].

The problem of computing small connected feedback vertex sets in connected planar graphs was introduced in [1]. A set of vertices is connected, if the subgraph induced by the vertex set is connected. This definition led to two new types of hitting sets of connected planar graphs: The connected feedback vertex sets and the connected face hitting sets. In fact, these two definitions coincide [6].

Misra, Philip, Raman, Saurabh and Sikdar [8] have recently investigated complexity theoretic properties of computing connected feedback vertex sets in general graphs. For the related Steiner tree problem in planar graphs, which asks to connect any given set of vertices in planar graphs, Borradaile, Klein and Mathieu [3] give an approximation scheme.

In connected planar graphs in general, there is no relationship between the size of the smallest connected and the smallest unconnected face hitting set (or feedback vertex set): Subdivisions of edges may force connected subgraphs to be arbitrarily large. However, Grigoriev and Sitters [6] show that any face hitting set S of a connected planar graph with a minimum degree of at least 3 is contained in a connected subgraph of size at most $11|S| - 18$. They thereby show that the size of the smallest face hitting set of such a graph is at most 11 times smaller than the size of the smallest connected face hitting set. They also show that there exist approximation schemes for both the connected and unconnected face hitting set.

We show that every face hitting set S of a connected planar graph G with a minimum degree of at least 3 is contained in a connected subgraph of G of size at most $5|S|-6$. Our concise proof improves over the multiplicative factor of 11 obtained in the proof of Sitters and Grigoriev [6].

They conjecture that the correct multiplicative factor is 3, by giving a lower bound for this. We show that in fact 5 is the correct multiplicative factor by giving a lower bound in the form of a family of graphs that matches our upper bound. The bound of $5|S| - 6$ is therefore tight.

Our proof is furthermore valid for graphs with loops and generalizes to arbitrary genus, showing that in any graph embedded in a surface of genus g every face hitting set S is contained in a connected subgraph of size of at most $5|S| - 6 + 4g$.

2 Connected versus unconnected face hitting sets

Theorem 1. *Let G be a connected planar graph with a minimum degree of at least 3 and let $S \subseteq V(G)$ be a face hitting set (of some embedding). Then there is a set $\overline{S} \supseteq S$ of vertices of size $|\overline{S}| \leq 5|S| - 6$ which induces a connected subgraph of G.*

We will divide the vertices in $\overline{S}$ into three types, which we need to count separately:

Definition 1. For a graph G and a nonempty subset of its vertices $S \subseteq V(G)$ we define $N(S)$ to be the vertices of $V(G)\backslash S$ which are neighbors of S. We further define $R(S) = V(G)\backslash(S \cup N(S))$ to be the *residual vertices with respect to S.*

Hence any nonempty set S partitions the vertices into the disjoint sets S, its neighbors $N(S)$, and the residual vertices $R(S)$.

Proof of Theorem 1. To prove the theorem, we first show that we can restrict ourselves to the case where vertices not contained in S have degree 3, and every face of G contains exactly one vertex from S. Figures 1 and 2 show the respective reductions.

Degree reduction of vertices outside of S: Suppose G is a connected planar graph and S a face hitting set in some embedding, that we fix from now on. Inductively, it suffices to reduce the degree of one vertex not contained in S. Consider a vertex $v \in N(S) \cup R(S)$ of degree $\deg(v) > 3$. We replace v by two adjacent vertices v' and v'' obtaining a new graph G'. We partition the edges that were incident with v into edges incident with v' and edges incident with v'' while maintaining planarity: Suppose u_1 and u_2 are neighbors of v that are consecutive in the circular ordering of the neighbors of v. We let u_1 and u_2 be the neighbors of v'; all other former neighbors of v become neighbors of v'' (see Figure 1).

In the new graph G' the vertices in S still form a face hitting set and the connection cost of S (i.e., the size of the smallest connected subgraph containing S) is not smaller than in the original graph G, since by contracting the edge between v' and v'' any connected subgraph of G' maps to a connected subgraph of G of at most equal size. For the degrees of the new vertices we have $\deg(v') = 3$ and $\deg(v'') = \deg(v) - 1$.

W.l.o.g. we thus assume from now on that every vertex not contained in S has degree 3.

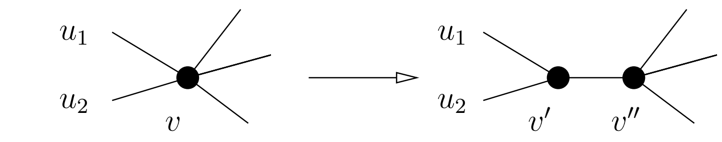

Figure 1: Splitting a vertex to reduce the degree: The vertex $v \notin S$ is replaced by two vertices v' and v''. The two consecutive neighbors u_1 and u_2 of v become neighbors of v', the other neighbors of v become neighbors of v''. This reduces the degree of the node v, in the sense that $\deg(v') = 3$ and $\deg(v'') = \deg(v) - 1$.

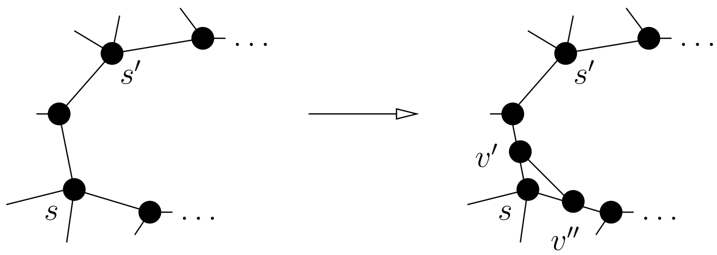

Figure 2: Removing multiple vertices s, s' from S on one face: The edges adjacent to the vertex s are subdivided and the newly introduced vertices are connected by an edge. The walk around the face that meets the vertices s, v' and v'' contains only one occurrence of a vertex from S. The walk around the other created face contains one occurrence of a vertex from S less than the original face.

Removal of multiple vertices from S that are incident with the same face: We manipulate the graph so that it only contains one vertex of S on every walk around the boundary of a face. Suppose there is a walk around a face f that meets two (possibly equal) vertices s and s' of S. (Here, the starting vertex and the ending vertex of the walk, which are equal, are not counted as two occurrences.) Let e_1 and e_2 be two consecutive edges of the walk around the face f which are both incident with s. Since s has a degree of at least 3, the edges e_1 and e_2 are distinct. We subdivide the edges e_1 and e_2 by introducing the vertices v' and v'' and connect the newly added vertices by a new edge. Preserving planarity, this new edge divides the face into two faces. One face contains at least s and the other face contains at least s' (see Figure 2). As in the previous reduction, the connection cost of S in the new graph is not smaller than in the original graph: any connected subgraph in the new graph maps to a connected subgraph in the original graph whose size is not larger, when we identify the vertices s, v' and v''. With this process, we inductively decrease the number of incidences with vertices from S when walking around the boundary of a face. Note that by introducing new faces in this way, the set S remains a face hitting set, and that the newly added vertices have degree 3.

Due to the two reductions we now assume w.l.o.g. that all vertices in the original graph G not contained in S have degree 3 and every walk around a face meets exactly one vertex from S.

Observe that this implies that any vertex in $N(S)$ is adjacent to exactly one vertex in S: Indeed, since a vertex in $N(S)$ has degree 3, any two of its neighbors are incident with a common face. In other words the observation says that the vertices from S have a pairwise distance of at least 3.

Note that G is loopless after applying the reductions: The existence of a loop at a vertex v of degree 3 implies the existence of a face incident only with this vertex, which implies that v is in S. The existence of a loop at a vertex $s \in S$ of degree at least 3 implies the existence of a face-walk which meets s multiple times.

We now analyze the structure of the graph $G - S$ after the reductions.

The graph $G - S$ has exactly one connected component: The graph $G - S$ is not empty: Any graph of minimum degree 3 has a face with more than one vertex. Those vertices cannot all be contained in S after the reduction. Recall that vertices from S are not adjacent. Thus, to show connectivity of $G-S$, it suffices to show that any two neighbors of a vertex $s \in S$ are connected by a path in $G - S$. For this it suffices to show that two neighbors that occur consecutively in the circular ordering of the neighbors of $s \in S$ are connected in $G-S$. But this is true: the vertex s and its two consecutive neighbors lie on a common face. The boundary of this face defines two walks between the two neighbors of s, one via s and one that avoids s. Since vertices in S do not lie on a common face, the latter walk does not contain any vertex from S, and thus only contains vertices from $G - S$.

Bound on the number of residual vertices: We now show that $|R(S)| \leq \max\{2|S| - 4, 0\}$. If $R(S) = \{\}$ there is nothing to show. We thus assume otherwise and consider the graph G' obtained by repeatedly contracting vertices of degree 2 in $G-S$, i.e., removing vertices of degree 2 and connecting their neighbors.

The graph G' is connected and 3-regular (and, as opposed to our original graph G, may contain multi-edges). It contains $|V(G')| = |R(S)|$ vertices, since the vertices in $N(S)$ are exactly the vertices of degree 2 in $G-S$. (Here we use the observation that vertices in $N(S)$ are only adjacent to one vertex of S and rely on the absence of multi-edges in G). Note that since $R(S) \neq \{\}$ and since $G - S$ is connected, every cycle of $G - S$ must contain a vertex of degree larger than 2, which means it contains a vertex from $R(S)$. Therefore faces do not vanish when vertices of degree two are replaced.

The number of faces of G' is $|S|$, i.e., $|F(G')| = |F(G - S)| = |S|$: Consider the embedding of $G-S$ obtained by deleting the vertices in S in the embedding of G. On the one hand a face f of $G-S$ consists of merged faces of G and since S is a face hitting set at least one vertex from S was contained in f prior to deletion. Hence $|F(G - S)| \leq |S|$. On the other hand, consider a vertex $s \in S$. Removal of s and its incident edges merges all faces that are incident with s. Since vertices from S do not lie on a common face in G, all vertices different from s that lie on a common face with s are vertices of $G - S$. In $G - S$ these vertices thus bound a face f which contained s prior to its deletion. The face f is composed exclusively of faces incident with s, and thus did not contain a vertex from $S \setminus \{s\}$ prior to deletion. We conclude that at most one vertex from S has been deleted within every face of $G-S$. Hence $|F(G-S)| \geq |S|$. Combining the two results shows that the deleted vertices from S and the faces of $G - S$ (or equivalently the faces of G') are in one-to-one correspondence.

Since G' is a 3-regular multi-graph, it holds that $2|E(G')| = 3|V(G')|$. Substituting this into Euler's formula $|V(G')| - |E(G')| + |F(G')| = 2$ we obtain $|V(G')| = 2|F(G')| - 4$, which leads to $|R(S)| = 2|S| - 4$ according to the equalities above.

Bound on the distance between vertices in S, ignoring vertices in $R(S)$: We now show that for any nonempty set $S' \subsetneq S$ there is a vertex $s \in S \setminus S'$ and a path in G from s to a vertex in S', that contains at most two vertices from $N(S)$: Let $s \in S \setminus S'$ and p be a path from s to some vertex in S' for which the number of vertices from $N(S)$ is minimal. Suppose p contains at least 3 vertices from $N(S)$. Consider a vertex $v \in N(S)$ on the path p which is not the first nor the last vertex among the vertices from $N(S)$. By definition of $N(S)$, the vertex v is adjacent to a vertex $s^* \in S$. This vertex s^* is either in S' or in $S \setminus S'$. In both cases we can shorten the path p, and thereby reduce the number of vertices from $N(S)$ on the path p, by either starting in s^* or ending in s^*.

Finally we conclude the theorem by the following counting argument: We successively connect all vertices from S by using at most 2 additional vertices from $N(S)$ for each vertex, except for the first one, and an arbitrary number of vertices from $R(S)$. Together we get at most $|S| + 2(|S| - 1) + |R(S)| \leq |S| + 2(|S| - 1) + \max\{2|S| - 4, 0\} = \max\{5|S| - 6, 1\}$ vertices. Since $|S| \geq 2$ for any face hitting set S of a simple planar graph with minimum degree at least 3, this simplifies to $5|S| - 6$. $\square$

Two steps in the proof require the planarity of the graph: the application of Euler's formula and the fact that boundaries of faces are cycles. Both can be reformulated, so that the proof extends to graphs embedded in an arbitrary surface: Using Euler's formula for general graphs on arbitrary surfaces and observing that we can perform the reduction also, when the boundary of a face decomposes into more than one connected component, we obtain that any face hitting set in a connected graph with minimum degree 3 embedded on a surface of genus g is contained in a connected subgraph of size at most $5|S| - 6 + 4g$.

Since any feedback vertex set is also a face hitting set the theorem shows that under the same assumptions the conclusion of the theorem also holds for feedback vertex sets. In general however, for minor closed graphs, there is no relation between feedback vertex sets and their connected variant: Consider a ternary rooted tree together with 6 additional vertices, two for each of the three subtrees of the root, whose neighbors are exactly the leaves of their corresponding subtree. The graph has treewidth 3, a feedback vertex set of size 6, and every connected feedback vertex set must be connected to the root and to some vertex within the last two levels of the tree. This shows an arbitrarily large multiplicative factor between the feedback vertex set number and the connected feedback vertex set number in graphs of bounded treewidth.

As René Sitters pointed out to us, the ratio is also unbounded when considering planar multigraphs. A tree with an additional vertex which is connected to every leaf by two edges demonstrates this fact.

We conclude with a family of graphs that provides a lower bound that matches the upper bound from Theorem 1, showing that both bounds are tight.

Theorem 2. *For $n \in \{2, 3, \ldots\}$ there exists an embedded connected planar graph G_n of minimum degree 3 whose smallest face hitting set has size n and whose smallest connected face hitting set has size $5n - 6$.*

Proof. To construct G_n with $n \in \{2, 3, \ldots\}$ we start with a tree T on $n-1$ vertices of maximum degree 3. We replace every vertex by the wheel W_8, with 8 spokes and 9 vertices. (See Figure 3). More precisely, suppose the center nodes of the wheels, the hubs, are $s_1, \ldots, s_{n-1}$. For every edge $(u, v) \in E(T)$, we subdivide an outer edge of the wheels that correspond to u and v. These newly created vertices are then connected. The choice of the wheel's outer edge to be subdivided is done in such a way that no two adjacent outer edges are subdivided. We call the set of vertices introduced by these subdivisions R (indicating that they will be the residual vertices with respect to the unique minimum face hitting set).

Finally, we pick another non-adjacent outer edge e in a wheel corresponding to a leaf in T. We replace e by a path of length 5 with an additional vertex s_0 adjacent to all inner vertices of the path, as also shown in Figure 3.

The unique smallest face hitting set in the graph is the set $s_0, \ldots, s_{n-1}$: It contains exactly the vertices of degree larger than 3, and every face is incident with only one vertex from the set. Therefore any smaller set covers less faces.

Every connected face hitting set contains all vertices from R. There are $2(n - 2)$ such vertices, since every edge of T contributes two vertices of R. Furthermore every connected face hitting set contains for every vertex in R a connection vertex on the outer edges of the wheel in which it is contained. Since vertices of R within one wheel are at a distance of at

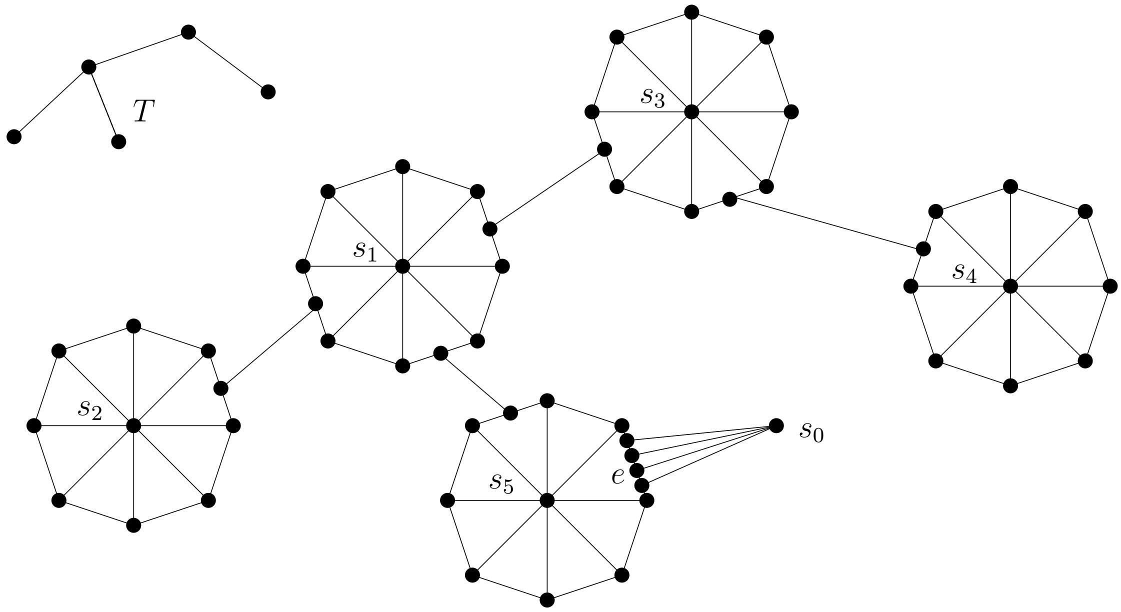

Figure 3: The lower bound construction. Upper left: Generating tree T. Remaining figure: T replaced by insertion of $n-1$ wheels W_8 with hubs $s_1, \ldots, s_{n-1}$. An additional vertex s_0 is connected to a wheel corresponding to a leaf in the original tree T. In this wheel s_0 is adjacent to the inner vertices of the path that replaces edge e.

least 3, these connection vertices are distinct. The connected face hitting set thus contains at least $|R| = 2(n-2)$ connection vertices.

Finally, every minimal connected face hitting set in the graph contains the hubs of every wheel: Suppose there is a minimal connected face hitting set that does not contain hub s_i. Consider the wheel corresponding to s_i. In this wheel, the minimal connected face hitting set contains more than one outer vertex in addition to the vertices from R and their connection vertices. Replacing these additional vertices with s_i reduces the number of vertices in the connected face hitting and yields a contradiction. (Note that the outer face remains covered by a vertex in R.) A similar argument can be used to see that s_0 together with two additional connection vertices is contained in every minimal connected face hitting set.

Together this shows that at least $2(n-2) + 2(n-2) + n - 1 + 3 = 5n - 6$ vertices are in every connected face hitting set of G_n. (That there is indeed a connected face hitting set of size $5n-6$ follows of course from Theorem 1). $\square$

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