Homomorphism-Homogeneous Graphs

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Abstract

We answer two open questions posed by Cameron and Nesetril concerning homomorphism-homogeneous graphs. In particular we show, by giving a characterization of these graphs, that extendability to monomorphism or to homomorphism leads to the same class of graphs when defining homomorphism-homogeneity.

Further we show that there are homomorphism-homogeneous graphs that do not contain the Rado graph as a spanning subgraph answering the second open question. We also treat the case of homomorphism-homogeneous graphs with loops allowed, showing that the corresponding decision problem is co-NP complete. Finally we extend the list of considered morphism-types and show that the graphs for which monomorphisms can be extended to epimorphisms are complements of homomorphism homogeneous graphs.

1 Introduction

The notion of regularity of a graph can range from degree regularity over transitivity all the way up to ultra-homogeneity. A graph G is *ultra-homogeneous* if every isomorphism between finite induced subgraphs of G extends to an automorphism of G, it is *homogeneous* if for any two isomorphic induced subgraphs of G there is some isomorphism that extends to an isomorphism of G. Gardiner [3] completed the classification of finite ultra-homogeneous graphs commenced by Sheehan [9]. Ronse [8] then showed that the finite ultra-homogeneous and homogeneous graphs coincide. Lachlan and Woodrow [6] perform the classification in the countable case.

In recent years the study of homomorphisms of graphs has received growing attention, Hell and Nešetřil exclusively and extensively treat this topic in their book on graphs and homomorphisms [4]. Cameron and Nešetřil [2] therefore introduced three closely connected variants of homogeneity for relational structures, which in the category of graphs specify to the following:

Definition 1. A graph belongs to the class

- **HH** if every homomorphism of a finite induced subgraph of G into G extends to a homomorphism from G to G.
- **MH** if every monomorphism of a finite induced subgraph of G into G extends to a homomorphism from G to G.
- **MM** if every monomorphism of a finite induced subgraph of G into G extends to a monomorphism from G to G.

Concerning these graphs, which are called homomorphism-homogeneous, they posed two questions in the loopless, simple, and undirected case: Is there a graph that is **MH** but not **HH**? Is there a countable **HH** graph that is not the disjoint union of complete graphs and does not contain the Rado graph (also called random graph) as a spanning subgraph? Recently there have been several results on homomorphism-homogeneous partially ordered sets. Cameron

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and Lockett [1] as well as Mašulović [7] both treat this topic. They classify homomorphism-homogeneous posets for strict and non-strict homomorphism, when the morphisms considered are homomorphisms, monomorphisms and isomorphisms. Ilić, Mašulović and Rajković [5] classify the finite homomorphism-homogeneous tournaments with loops allowed. However, the questions on graphs remained open. We answer the first question in the negative by showing that the classes **MH** and **HH** coincide. Additionally we show that, apart from disjoint unions of cliques, the class **MM** coincides with the other two as well. We then treat finite homomorphism-homogeneous graphs with loops allowed, showing that the recognition problem is co-NP complete. Afterwards we give a family of examples of homomorphism-homogeneous graphs that do not contain the Rado graph as a spanning subgraph, thereby answering the second open question in the affirmative. We conclude by analyzing under what circumstances morphisms can always be extended to epimorphisms, the graphs with this property are complements of homomorphism-homogeneous graphs.

The paper deals with simple undirected graphs that are apart from the graphs in Section 3 assumed to be loopless. For a graph G we denote by $V(G)$ and $E(G)$ the set of vertices and edges of G respectively.

2 Equivalence of the definitions

It turns out that, unlike for general Structures, for graphs the classes **HH** and **MH** coincide. It is easy to see that there are **MH** graphs which are not contained in the third class **MM**, but they must be disconnected graphs of very simple structure. More specifically, apart from disjoint unions of cliques of equal finite size the class **MM** coincides with the other two as well.

We are now going to prove this, by giving a characterization of **HH** graphs. In doing so we repeatedly use the following fact: Let G be a countable graph. If any homomorphism from any finite induced subgraph of G to G can be extended to any vertex, then any finite homomorphism can be extended to the whole graph. If monomorphisms can be extended to any vertex, such that the extension is still monomorphic, then the extension to the whole graph can be chosen monomorphic as well.

Definition 2. We say that a subset S of the vertices of a graph G (or a subgraph H) *has a cone* if there is a vertex $c \in V(G)$ that is adjacent to all vertices in S (resp. H). It has an *anti-cone* if there is a vertex $c \in V(G)$ that is adjacent to no vertex in S (resp. H). Any such vertex c is then called a cone (resp. anti-cone).

Theorem 3. *For a countable graph G the following statements are equivalent:*

1. *G is* **MH***.*
2. *If $\varphi: H \to H'$ is a monomorphism, where H and H' are finite induced subgraphs of G, which is surjective onto the vertices of H', and H has a cone, then H' also has a cone.*
3. *If $\varphi: H \to H'$ is a homomorphism, where H and H' are finite induced subgraphs of G, which is surjective onto the vertices of H', and H has a cone, then H' also has a cone.*
4. *G is* **HH***.*

Proof. (1. $\Rightarrow$ 2.) Let $\varphi: H \to H'$ be a monomorphism, where H and H' are finite induced subgraphs of G, which is surjective onto the vertices of H'. Further assume that H has a cone c in G. Since G is **MH** the monomorphism φ extends to the whole graphs, in particular the extension maps the cone-vertex c to a vertex which is adjacent to every vertex of $\varphi(V(H)) = V(H')$.

(2. $\Rightarrow$ 3.) Let $\varphi\colon H \to H'$ now be a homomorphism, where H and H' are again finite induced subgraphs of G, which is surjective onto the vertices of H'. For every vertex $v \in H'$ choose a pre-image $w_v \in \varphi^{-1}(v)$ of v, which exists by surjectivity of φ. Taking the induced subgraph on these pre-images we obtain a graph $\widetilde{H} \subseteq H \subseteq G$. The restriction of φ to $\widetilde{H}$ is a monomorphism onto the vertices of H. If H has a cone its subgraph $\widetilde{H}$ has a cone as well, by using (2.) we get that H' has a cone.

(3. $\Rightarrow$ 4.) Let $\varphi\colon H \to G$ be a homomorphism, where H is a finite induced subgraph of G. It suffices to show that for any vertex $v \in G$ there is an extension $\varphi\colon H \cup \{v\} \to G$. W.l.o.g. $v \notin H$. If v is not adjacent to any vertex of H, we can extend φ by mapping v to any vertex in G. Otherwise the vertex v is a cone of the set $N(v) \cap V(H)$, the intersection of its neighborhood with the graph H. By (3.) the set $\varphi(N(v) \cap V(H))$ has a cone c. Extending φ by mapping v to c yields a graph homomorphism.

(4. $\Rightarrow$ 1.) Since every monomorphism is a homomorphism this is trivial. $\square$

As observed in [2] every vertex in a connected infinite **HH** graph G has infinite degree. This was shown by proving that there are no finite maximal cliques. (Assume there is a finite maximal clique K_n, then no set of size n has a cone, this means there is no vertex with degree n or larger, so K_n is a component.) This statement can be strengthened further:

Lemma 4. *If a finite induced subgraph H of a connected countably infinite* **HH** *graph has a cone, then there is an infinite clique K_∞ in G, such that every vertex of K_∞ is adjacent to every vertex of H.*

Proof. It suffices to show that if c is a cone for $V(H)$ then $\{c\} \cup V(H)$ has a cone. By the remark above $\deg(c) = \infty$ and since H is finite, c has a neighbor $w \notin V(H)$. Then c is a cone for the subgraph K induced by the vertices $V(H) \cup \{w\}$. Consider the map $\varphi\colon V(H) \cup \{w\} \to V(H) \cup \{c\}$ that maps every vertex of H to itself and w to c. It preserves adjacencies since it fixes all vertices of H and since c is adjacent to all of $V(H)$, so it is a homomorphism. Since K has a cone, $\varphi(K) = V(H) \cup \{c\}$ has a cone. $\square$

We conclude that once a subset of a connected **MH** graph has a cone there is an infinite abundance of cones. In the light of this we can show that extensions of monomorphisms can be chosen monomorphic:

Theorem 5. *A countable connected graph is* **HH** *if and only if it is* **MM**.

Proof. Any **MM** graph is obviously an **MH** graph and therefore an **HH** graph. We show that any connected graph that satisfies condition (3.) from Theorem 3 is an **MM** graph. So let G be a graph that satisfies this condition, i.e. an **HH** graph. It suffices to show: Given any monomorphism $\varphi\colon H \to G$, where H is a finite induced subgraph of G and a vertex $v \in G$ with $v \notin V(H)$, there is an extension $\varphi\colon H \cup \{v\} \to G$ that is also a monomorphism. Let $S = N(v) \cap V(H)$ be the neighbors of v in H. W.l.o.g. $S \neq \{\}$. Then $\varphi(S)$ has a cone, by condition (3.) from Theorem 3, so by Lemma 4 it also has infinitely many cones. We extend φ by mapping v to a cone of $\varphi(S)$ which is not in the image of $\varphi(H)$. Then the extension of φ is a monomorphism. $\square$

Note by Theorem 3 a non-complete **MH** graph is connected if and only if non-edges (i.e. two vertices not adjacent) have cones. If non-edges do not have a cone, then any induced non-complete subgraph does not have a cone. With this one may deduce that a disconnected or finite graph is **MH** if and only if it is the disjoint union of complete graphs that all have the same size. A disconnected infinite graph is **MM** if and only if it is the disjoint union of infinite cliques. A finite graph is **MM** if and only if it is a complete graph or an empty graph. (These

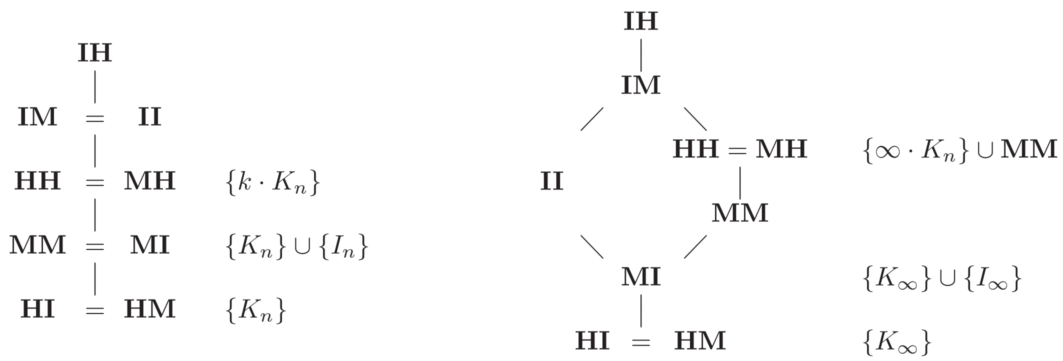

Figure 1: The classes of morphism-homogeneous structures in the category of finite (left) and infinite (right) graphs without loops.

results can also be found in [2].) As has been done in [1] for homomorphism-homogeneous partial orders the notion of homomorphism-homogeneity can be defined for any pair from {isomorphism, homomorphism, monomorphism}. For a given category the corresponding classes form a partial order which shows how the different classes relate to each other. Figure 1 shows this partial order in the category of graphs without loops for the finite and the infinite case. The fact that **IM** is a proper subset of **IH** is witnessed by the stargraphs $K_{n,1}$ and $K_{\infty,1}$ respectively. In the infinite case the infinite clique with one edge removed $K_\infty \setminus \{e\}$ shows that **II** is properly contained in **IM**. The other relations are trivial or follow from theorems within this section.

3 Graphs with loops

As just explained in the previous section there are no finite structurally interesting loopless graphs that are homomorphism-homogeneous. Contrary to the loopless case the class of homomorphism-homogeneous structures in the category of graphs with loops allowed contains a diversity of finite structures. For example any graph that contains a vertex with a loop that is connected to any other vertex (also called a universal vertex), will be homomorphism-homogeneous. Figure 2 shows an example of an **HH** that does not have such a vertex.

For graphs with loops there is an analogue of Theorem 3 that incorporates the fact that homomorphisms map vertices with loops to vertices with loops:

Theorem 6. *For a graph G (with loops allowed) the following statements are equivalent:*

1. *G is* **MH**.
2. *If $\varphi\colon H \to H'$ is a monomorphism, where H and H' are finite induced subgraphs of G, which is surjective onto the vertices of H', and if H has a cone, H' also has a cone. Furthermore if H has a cone with a loop, H' also has a cone with a loop.*
3. *If $\varphi\colon H \to H'$ is a homomorphism, where H and H' are finite induced subgraphs of G, which is surjective onto the vertices of H', then if H has a cone, H' also has a cone. Furthermore if H has a cone with a loop, H' also has a cone with a loop.*
4. *G is* **HH**.

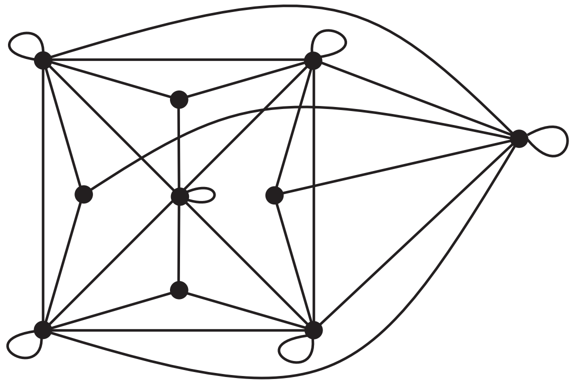

Figure 2: An **HH** graph of size 10 with independent set of size 4

Proof. The proof follows the lines of the proof of Theorem 3, with added distinctions whether a given cone-vertex has a loop or not. □

So also in the larger category of graphs with loops the classes **MH** and **HH** coincide. For finite **MM** graphs the situation is different when loops are allowed: A graph is **MM** if and only if it is an independent set or a clique where either all vertices have loops or no vertex has a loop: In a finite graph any map that maps a vertex without a loop to a vertex with loop cannot be extended to a monomorphism. The same applies to any map that maps two non-adjacent vertices to two adjacent vertices.

The finiteness opens the way for computational questions. In fact the class is so rich, that we get the following hardness result:

Theorem 7. *The problem of deciding if a graph with loops allowed is* **HH** *is co-NP complete.*

Proof. The problem is in co-NP: If a graph G is not **HH**, then there are two subgraphs H and H' of G such that H maps homomorphically onto the vertices of H' and in addition the vertices of H have a cone, but the vertices of H' do not. Provided with the map from H to H' one can verify this in polynomial time.

To show the hardness we reduce INDEPENDENT SET to our problem.

Given $G = (V, E)$ a finite graph without loops, we form G_k by the following construction: Join G with an independent set I_k of size k. For every set S of vertices in this join, that contains all vertices except exactly one vertex of I_k, add a cone-vertex with a loop adjacent to all vertices in S. Add all edges between these newly added loop vertices. A subgraph of this new graph G_k has a cone if and only if it does not contain all vertices of I_k. If so, then it also has a cone-vertex with a loop. Since any graph that maps onto the independent set I_k must itself contain an independent set of size k we conclude that the new graph is not **HH** if and only if the original graph G contains an independent set of size k. □

4 Examples of homomorphism-homogeneous graphs

The random graph is the unique countable graph R with the property that for any disjoint subsets $X, Y \subseteq R$ there is a vertex that is a cone for X and an anti-cone for Y. The random graph is homogeneous as well as homomorphism-homogeneous. Any graph that contains the random graph R as a spanning subgraph is **HH**, since any subset of it has a cone. It had been unknown whether there are other graphs, that do not contain R as spanning subgraph, but nevertheless are **HH**. We now give a simple construction for such graphs, it resembles the construction given in the co-NP hardness proof of Theorem 7, here beginning with the empty graph and replacing loop vertices with infinite cliques: Let I_n be an independent set of size $n > 1$. For any set S of vertices of size $n-1$ of the independent set add an infinite clique with vertices connected to S. Add all edges between these newly added cliques. In this graph a set has a cone if and only if it contains no independent set of size n (of which there is only one), and therefore Theorem 3 may be applied.

We conclude this section with the observation that some infinite **HH** graphs can be obtained by replacing loops in particular finite **HH** graph with infinite cliques. (This procedure will however not always yield an **HH** graph.) If this is done to the graph from Figure 2 for example, we obtain an **HH** graph without claws of size 3 which contains an independent set of size 4.

5 Epimorphisms

We now turn back to the loopless case and further extend the scope of our analysis. In this section we focus not only on monomorphisms and homomorphisms but also on epimorphisms.

Definition 8.

A graph G is an **ME** graph if every monomorphism of a finite subgraph H into G extends to an epimorphism onto G i.e. a homomorphism from G to G such that every vertex in G has a pre-image.

In the course of analyzing these graphs we will need two further definitions that allow for the extended epimorphisms to be partial maps: A graph G is an $\mathbf{ME}^{par}_{hom}$ *graph*, if every monomorphism of a finite subgraph H into G extends to a homomorphism from some subgraph of G that has the whole graph G as an image. A graph G is an $\mathbf{ME}^{par}_{mon}$ *graph*, if every monomorphism of a finite subgraph H into G extends to a monomorphism from some subgraph of G that has the whole graph G as an image.

In fact these two definitions involving partial morphisms describe the same class of graphs:

Lemma 9. *A graph G is $\mathbf{ME}^{par}_{hom}$ if and only if it is $\mathbf{ME}^{par}_{mon}$.*

Proof. Since every monomorphism is a homomorphism every $\mathbf{ME}^{par}_{mon}$ graph is an $\mathbf{ME}^{par}_{hom}$ graph. Conversely if a graph G is $\mathbf{ME}^{par}_{hom}$ and $\varphi\colon H \to G$ is a monomorphism, where H is a finite induced subgraph of G, then φ can be extended to a homomorphism, whose image is the whole graph G. For every vertex that is not in $\varphi(H)$ we choose a single pre-image and for every vertex in $\varphi(H)$, we chose the corresponding unique pre-image in H. The restriction of φ to the chosen vertices is the sought monomorphism, thus $G \in \mathbf{ME}^{par}_{mon}$ □

We next show that there is a connection between $\mathbf{ME}^{par}_{mon}$ and **MM**, namely they are complementary. In order to prove this we first need to prove the following simple lemma that states that if a map between the vertices of a graph induces a monomorphism, then the inverse of this map induces a monomorphism in the complement of the graph.

Lemma 10. *Let G be a graph and $\varphi\colon V' \to V''$ a bijection between sets of vertices V', V'' of G that induces a monomorphism from some subgraph of G. Then φ^{-1} induces a monomorphism from some subgraph in $\overline{G}$, the complement of G.*

Proof. It suffices to show that φ^{-1} preserves adjacencies in $\overline{G}$. Let $u, v \in V(\overline{G}) = V(G)$ with $(u, v) \in E(\overline{G})$. Then $(u, v) \notin E(G)$. Since φ is a monomorphism in G, we see that $(\varphi^{-1}(u), \varphi^{-1}(v)) \notin E(G)$ and therefore $(\varphi^{-1}(u), \varphi^{-1}(v)) \in E(\overline{G})$. □

Having proven this lemma we can move on to prove the aforementioned statement.

Theorem 11. *A graph G is $\mathbf{ME}^{par}_{mon}$ if and only if $\overline{G}$ is $\mathbf{MM}$. (I.e. $\mathbf{ME}^{par}_{mon} = \overline{\mathbf{MM}}$.)*

Proof. Let $G \in \mathbf{ME}^{par}_{mon}$ and let $\varphi\colon \overline{H} \to \overline{H'}$ be a monomorphism between finite induced subgraphs $\overline{H'}$ and $\overline{H}$ of $\overline{G}$. Lemma 10 shows that φ^{-1} is a finite monomorphism in G. Since $G \in \mathbf{ME}^{par}_{mon}$ there exists an extension $\widehat{\varphi}^{-1}$ of φ^{-1} that is a monomorphism from a (possibly infinite) subgraph of G whose image is the whole graph G. Using Lemma 10 again we obtain a map $\widehat{\varphi}\colon \overline{G} \to \overline{G}$, which is the monomorphic extension of φ, that was to be constructed.

The converse is proven similarly again by using Lemma 10 twice. We start with φ being an arbitrary monomorphism from a finite subgraph in G. We use Lemma 10 to show that φ^{-1} is a monomorphism in $\overline{G}$ and using the fact that G is $\mathbf{MM}$ extend φ^{-1} to $\widehat{\varphi}^{-1}$ a monomorphism that maps the whole graph $\overline{G}$ to itself. Then we use Lemma 10 again to show that $\widehat{\varphi}$ is the extension we needed. □

Additionally we may deduce that there is a correspondence between the ways monomorphisms extend in the two classes. In the view of this, we may translate the alternative definition for $\mathbf{MM}$ graphs from Theorem 3 into terms of $\mathbf{ME}^{par}_{mon}$ graphs:

A graph G is $\mathbf{ME}^{par}_{mon}$ if and only if the following holds: If $\varphi\colon H \to H'$ is a monomorphism, where H and H' are finite induced subgraphs of G, which is surjective onto the vertices of H', then if H' has an anti cone, H also has an anti cone.

We are now ready to relate the connected graphs in the class $\mathbf{ME}$ to the class $\mathbf{MM}$.

Theorem 12. *A connected graph is $\mathbf{ME}$ if and only if it is contained in $\mathbf{MM} \cap \overline{\mathbf{MM}}$*

Proof. Let G be an $\mathbf{ME}$ graph. For the first implication it suffices by Theorem 11 to show that G is in $\mathbf{MM} \cap \mathbf{ME}^{par}_{mon}$. From the definition it is straightforward to see that G is also an $\mathbf{ME}^{par}_{hom} (= \mathbf{ME}^{par}_{mon})$ graph. Since an extension to an epimorphism is in particular an extension to a homomorphism G is $\mathbf{MH}$. Since G is connected Theorem 5 implies that G is $\mathbf{MM}$.

Conversely we will show that if a countable graph G is in $\mathbf{MM} \cap \mathbf{ME}^{par}_{mon}$ then $G \in \mathbf{ME}$. Using the alternative definitions of $\mathbf{MM}$ and $\mathbf{ME}^{par}_{mon}$, we know that for every monomorphism $\varphi\colon H \to H'$, with H and H' finite induced subgraphs of G it follows that:

- For any vertex v of G there is an extension $\hat{\varphi}\colon H \cup \{v\} \to H' \cup \{u\}$, where $\hat{\varphi}(v) = u$.
- For any vertex u of G there is an extension $\hat{\varphi}\colon H \cup \{v\} \to H' \cup \{u\}$, where $\hat{\varphi}(v) = u$.

We will now show, that these two properties suffice to extend any monomorphism from a subgraph of G to a epimorphism. The construction is similar to the way uniqueness of the Rado graph [10, Theorem 0.1.5.] is usually shown.

Let $\varphi\colon H \to H'$ be a monomorphism, where H an H' are finite induced subgraphs of G. We enumerate the vertices of G and start with the smallest vertex $v \notin H$ not mapped so far. Since G is $\mathbf{MM}$ there exist a vertex $u \notin H'$ such that we can extend φ by setting $\varphi(v) = u$, preserving injectivity on the vertices. Continue with the smallest vertex $w \notin H'$ that does not have a pre-image yet. Since G is $\mathbf{ME}^{par}_{mon}$ there exists an unmapped vertex x, such that we can extend φ by setting $\varphi(x) = w$, again preserving injectivity on the vertices. Repeating this procedure and using the fact that G is countable we obtain an epimorphism from G to G. □

Note that for finite graphs, due to cardinality reasons, the classes $\mathbf{MM}$, $\mathbf{ME}$ and $\mathbf{ME}^{par}_{mon}$ coincide and the only graphs in them are cliques and independent sets. Also note that the infinite disconnected $\mathbf{ME}$ graphs for which the theorem does not hold are exactly the graphs in $\mathbf{MH}$ that are not in $\mathbf{MM}$, i.e. disjoint unions of finite cliques, and these are also exactly the graphs for which the epimorphism cannot be chosen monomorphic.

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