A FINITE FRAGMENT OF S3

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Abstract. It is shown that the pure (strict) implication fragment of the modal logic **S3** has finitely many non-equivalent formulae in one variable. The exact number of such formulae is not known. We show that this finiteness result is the best possible, since the analogous fragment of **S4**, and therefore of **S3**, in two variables has infinitely many non-equivalent formulae.

1. Introduction

Meyer [3] raises the question of the number of distinct (non-equivalent) formulae in one variable in the relevant logic $\mathbf{E}_{\rightarrow}$, while answering it for the stronger logic $\mathbf{R}_{\rightarrow}$. The answer in the latter case is 6, while in the former case it is still not known whether the number is finite or infinite.[1] Over the last four decades, several related results have appeared. $\mathbf{S4}_{\rightarrow}$, the pure (strict) implication fragment of the modal logic **S4**, has exactly 9 non-equivalent formulae in one variable [2], while the fragment of **E** with both implication and negation has infinitely many zero-variable formulae built up from the sentential constant f [4]. A little weaker than $\mathbf{E}_{\rightarrow}$ is $\mathbf{T}_{\rightarrow}$, for which the question is also open, and between $\mathbf{E}_{\rightarrow}$ and $\mathbf{S4}_{\rightarrow}$ lies the non-normal modal logic $\mathbf{S3}_{\rightarrow}$. The purpose of the present paper is to investigate the one-variable fragment of $\mathbf{S3}_{\rightarrow}$. We show that the free $\mathbf{S3}_{\rightarrow}$ algebra with one generator is finite, and thus that there are only finitely many non-equivalent formulae in $\mathbf{S3}_{\rightarrow}$. We then examine the two-variable fragment of **S4**, showing that it is infinite and therefore that finiteness does not extend beyond the one-variable case.

2. The problem

Consider propositional logics with just one connective, an impliction $\rightarrow$. The system $\mathbf{T}_{\rightarrow}$ of "ticket entailment", defined by Anderson and Belnap [1], has as rule of inference detachment, and as axioms all instances of the schemes:

a1. $A \rightarrow A$
a2. $(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$
a3. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
a4. $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

We consider several systems resulting by adding further axioms:

a5. $((A \rightarrow A) \rightarrow B) \rightarrow B$
a6. $(A \rightarrow B) \rightarrow (C \rightarrow C)$
a7. $A \rightarrow (B \rightarrow B)$
a8. $A \rightarrow ((A \rightarrow B) \rightarrow B)$
a9. $A \rightarrow (B \rightarrow A)$

[1] The latest count, produced by brute force enumeration, showing non-equivalence by testing in small models, stands at over 6 million.

To strengthen $\mathbf{T}_\to$ to $\mathbf{E}_\to$ add a5. To strengthen $\mathbf{E}_\to$ to $\mathbf{S3}_\to$ add a6 as well, and to get $\mathbf{S4}_\to$ replace this by the more general a7. $\mathbf{R}_\to$ is conveniently axiomatised as $\mathbf{T}_\to$ plus the axiom of "assertion" a8. Finally, intuitionist pure implication $\mathbf{J}_\to$, which is stronger than any of the above systems, is obtained by adding a9 to $\mathbf{T}_\to$.

It has long been known [1] that the logic of implicational formulae of **E** (that is, the logic whose "variables" are the implicational formulae of **E**) is just **R**. In particular, the **E** theorem

$$((A \to B) \to ((C \to D) \to E)) \to ((C \to D) \to ((A \to B) \to E))$$

says that in compound implications, antecedents which are themselves implications may be permuted. In the same way, and for the same reason, the logic of implicational formulae of **S4**, and indeed of **S3**, is exactly intuitionist logic.

We are concerned in this paper with the simplest natural fragment of these logics: that with only one atom p. In this fragment, of course, all formulae are implications except for p itself. The question we wish to address is the number of such formulae, up to provable equivalence, in particular in the logic $\mathbf{S3}_\to$. We switch freely between thinking of them as formulae in the logic and as elements in an $\mathbf{S3}_\to$ algebra, which we take to be an object $\langle \Sigma, \to \leq \rangle$ where $\langle \Sigma, \leq \rangle$ is a poset and $\langle \Sigma, \to \rangle$ is a groupoid with a distinguished element I, such that for all $a, b, c \in \Sigma$:

p1. $a \to b = I$ iff $a \leq b$
p2. $a \to b \leq (c \to a) \to (c \to b)$
p3. $a \to b \leq (b \to c) \to (a \to c)$
p4. $a \to (a \to b) \leq a \to b$
p5. $I \to a \leq a$
p6. $a \to b \leq I$

To correspond to the one-variable fragment of logic, of course, the algebra should have a single generator g. We note that the subalgebra obtained by simply omitting g is a Hilbert algebra – the implicational reduct of a Brouwerian algebra.

Meyer's question for $\mathbf{S3}_\to$ is whether the free $\mathbf{S3}_\to$ algebra with one generator is finite.

3. The one-variable fragment is finite

The situation in $\mathbf{S4}_\to$is simpler than that in $\mathbf{S3}_\to$. Every one-variable formula is equivalent to one of the following nine:

1. p	4. $1 \to 3$	7. $5 \to 1$
2. $1 \to 1$ i.e. $p \to p$	5. $4 \to 1$	8. $7 \to 4$
3. $2 \to 1$ i.e. $\Box p$	6. $1 \to 5$	9. $8 \to 6$

The implication matrix is:

$\to$	1	2	3	4	5	6	7	8	9
1	2	2	4	4	6	6	7	8	9
2	3	2	3	4	5	6	7	8	9
3	2	2	2	2	2	2	2	2	2
4	5	2	5	2	5	2	2	2	2
5	7	2	7	7	2	2	7	2	2
6	3	2	3	7	5	2	7	2	2
7	5	2	5	8	5	8	2	8	2
8	3	2	3	7	5	9	7	2	9
9	3	2	3	4	5	8	7	8	2

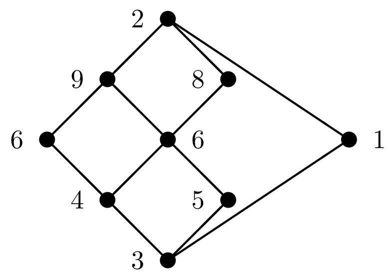

FIGURE 1. Hasse diagram of one-variable $\mathbf{S4}_{\rightarrow}$

An algebra such as this can be represented as a Kripke model quite easily: the worlds of the Kripke frame may be identified with the up-closed subsets of Σ, and for worlds X and Y, let $R(X, Y)$ mean that for all elements a and b, if $a \rightarrow b \in X$ and $a \in Y$ then $b \in Y$.

S3 frames are like those for **S4** except for the possible presence of non-normal worlds in which all implications are false. For the one-variable fragment, this means there are, up to equivalence, just two non-normal worlds: the one in which p is true and every other formula false, and the one in which all formulae are false. The latter cannot make a difference to the value of any formula at any other world, so it can be dropped without loss, leaving only the one non-normal world (at most) in any frame. Any world not reachable from the base world (where truth is evaluated) can also be dropped, of course. Obviously, then, by the transitivity of the accessibility relation, if the non-normal world is present at all, it is accessible to the base world of the frame.

Clearly, there are, in addition to p and $p \rightarrow p$, three types of formula in a one-variable implicational logic. A formula is either:

(1) $A \rightarrow p$, where A is itself an implication, or
(2) $p \rightarrow A$, where A is itself an implication, or
(3) $A \rightarrow B$, where both A and B are themselves implications.

In generating non-equivalent formulae of $\mathbf{S3}_{\rightarrow}$, formulae of type 1, with the exception of $\Box p$, can be ignored, since $A \rightarrow p$ is provably equivalent to $\Box(A \rightarrow p)$ (that is, $(p \rightarrow p) \rightarrow (A \rightarrow p)$) which, by permutation of antecedents, is equivalent to $A \rightarrow \Box p$, which is of type 3 and will occur in the enumeration of formulae even if $A \rightarrow p$ is dropped, because both its antecedent and its consequent are generated earlier than $A \rightarrow p$.

Formulae of type 2 do not generally have equivalents of type 3, but they do have the property that any two of them which are non-equivalent in $\mathbf{S3}_{\rightarrow}$ are also non-equivalent in $\mathbf{S4}_{\rightarrow}$. To see this, consider any two such formulae $p \rightarrow A$ and $p \rightarrow B$ and an **S3** frame which shows them to be distinct. There must be a world w in which one of them holds while the other does not. w must be a normal world, because the non-normal one makes neither of them true, so w might as well be the base world and the frame consist just of it and its descendents under R. The non-normal world cannot be accessible to w, because p holds in it while A and B do not, so neither $p \rightarrow A$ nor $p \rightarrow B$ can hold in any world that can "see" the non-normal one. Therefore there is no non-normal world in the frame, which is therefore an

S4 frame, showing that $p \to A$ and $p \to B$ are distinct in $\mathbf{S4}_{\to}$. By inspection of the matrix given above, we may note that $\mathbf{S4}_{\to}$ distinguishes only five such formulae.

The free $\mathbf{S3}_{\to}$ algebra with a single generator g therefore consists of g together with the Hilbert algebra generated by six elements: $I \to g$ and the five corresponding to the distinct type 2 formulae of $\mathbf{S4}_{\to}$. It is thus a finitely generated Hilbert algebra. But Hilbert algebras are locally finite [5], so this one is finite, and hence so is the one-variable fragment of $\mathbf{S3}_{\to}$.

4. The two-variable fragment is infinite

The above argument reducing the finiteness question for $\mathbf{S3}_{\to}$ to that for $\mathbf{S4}_{\to}$ does not depend on the number of variables: it is easily generalised to show that the n-variable fragment of $\mathbf{S3}_{\to}$ is finite iff the n-variable fragment of $\mathbf{S4}_{\to}$ is finite. This raises the hope that $\mathbf{S3}_{\to}$might be shown to be locally finite, by showing the local finiteness of $\mathbf{S4}_{\to}$. We now show that such a hope is vain, since the two-variable fragment of $\mathbf{S4}_{\to}$ is already infinite.

We construct an infinite **S4** algebra on two generators, making use of the fact that topological spaces are models of **S4**. Consider $X = (\mathbb{N}, \mathcal{T})$, with $\mathbb{N}$ being the natural numbers and $\mathcal{T}$ the family of down-closed subsets of $\mathbb{N}$: i.e. $U \in \mathcal{T}$ iff for all i and j, if $j \in U$ and $i < j$ then $i \in U$).

Lemma 1. *X is a topological space; in fact, an Alexandroff space.*

Proof. Clearly, $\emptyset$ and $\mathbb{N}$ belong to $\mathcal{T}$. Any union of down-closed sets of naturals is also down-closed and hence belongs to $\mathcal{T}$. Similarly, any intersection of members of $\mathcal{T}$ is again a member of $\mathcal{T}$. □

Where i is any natural number, let γ_i be the set $\{0, \dots, i\}$. Let α be the set of even numbers and let β be the set of odd numbers.

For any subsets S, T of $\mathbb{N}$ we define $S \to T$ to be $\mathit{int}\,(\overline{S} \cup T)$, where $\overline{S}$ stands for $\mathbb{N} \setminus S$ and $\mathit{int}\,U$ for $\{k : \gamma_k \subseteq U\}$. Since the Boolean algebra $\wp(\mathbb{N})$ with int as the modal (box) operator is an S4 algebra, the operation $\to$ is just the strict implication in this algebra. Note that $\overline{\alpha} = \beta$, $\overline{\beta} = \alpha$, $\mathit{int}\,\beta = \emptyset$ and $\mathit{int}\,\alpha = \{0\}$.

Lemma 2. *The following hold:*

1. $\alpha \to \beta = \emptyset$
2. $\beta \to \alpha = \gamma_0$
3. $\alpha \to \gamma_{2k} = \gamma_{2k+1}$, *for* $k \geq 0$
4. $\beta \to \gamma_{2k+1} = \gamma_{2k+2}$, *for* $k \geq 0$

Proof. Since $\overline{\alpha} = \beta$ as just noted, $\overline{\alpha} \cup \beta = \beta$, so $\alpha \to \beta = \mathit{int}\,\beta = \emptyset$, establishing (1).

Similarly, $\beta \to \alpha = \mathit{int}\,(\overline{\beta} \cup \alpha) = \mathit{int}\,\alpha = \{0\}$, establishing (2).

To prove (3) note that $\alpha \to \gamma_{2k} = \mathit{int}\,(\beta \cup \gamma_{2k})$. Now $2k+1 \in \beta$ but $2k+2 \notin \beta$, so $\mathit{int}\,(\beta \cup \gamma_{2k}) = \{0, \dots, 2k\} \cup \{2k+1\} = \gamma_{2k+1}$.

The proof of (4) is similar: $\beta \to \gamma_{2k+1} = \mathit{int}\,(\alpha \cup \gamma_{2k+1}) = \gamma_{2k+1} \cup \{2k+2\} = \gamma_{2k+2}$. □

Lemma 3. *The algebra generated by α and β is infinite.*

Proof. Each γ_k is generated from α and β. Since $(\gamma_k)_{k=0}^{\infty}$ is a strictly ascending chain of subsets of $\mathbb{N}$, the claim is proved. □

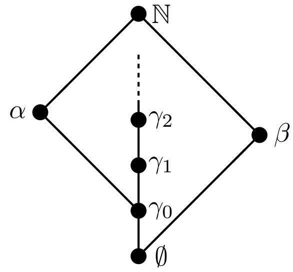

FIGURE 2. Hasse diagram of the infinite $\mathbf{S4}_{\rightarrow}$ model

The theorem that the two-variable fragment of $\mathbf{S4}_{\rightarrow}$ is infinite is immediate. As a corollary, note that since β is nothing but $\overline{\alpha}$, the one-variable fragment of **S4** in (strict) implication and (boolean) negation is also infinite.

5. Conclusion

It has been shown that $\mathbf{S3}_{\rightarrow}$ and $\mathbf{S4}_{\rightarrow}$ are finite in the one-variable fragment but infinite with two or more variables. The free $\mathbf{S3}_{\rightarrow}$ algebra with one generator, despite being finite, is still not known in much detail. We have computed an approximation to it by generating the $\mathbf{S3}_{\rightarrow}$ algebras (modulo isomorphism) with up to 8 elements, selecting those with one generator, selecting from those the subdirectly irreducible ones, and then finding the subalgebra of their direct product generated by the vector of their single generators. This algebra has 517 elements, meaning that there are at least 517 pairwise non-equivalent one-variable formulae in $\mathbf{S3}_{\rightarrow}$. We have no reason to think that these 517 exhaust the possibilities, but equally no direct evidence that they do not. The main remaining open question is to determine the number of non-equivalent formulae exactly: on this, we have nothing but the lower bound of 517 to offer.

References

[1] A. R. Anderson and N. D. Belnap. *Entailment: The Logic of Relevance and Necessity, Vol. 1.* Princeton University Press, 1975.

[2] M. Byrd. Single variable formulas in $S4_{\rightarrow}$. *Journal of Philosophical Logic*, 5:439–456, 1976.

[3] R. K. Meyer. RI – the bounds of finitude. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 16:385–387, 1970.

[4] J. Slaney. Sentential constants in systems near R. *Studia Logica*, 52:443–455, 1993.

[5] A. Urquhart. Implicational formulas in intuitionistic logic. *Journal of Symbolic Logic*, 39:661–664, 1974.