# Exact learning and inference for planar graphs

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# Ising model

- Particles modify their behaviour to conform with neighbours' behaviour
- What is the mean energy?
- Used in chemistry, physics, biology...
- More than 12,000 papers published!

# Ising problem

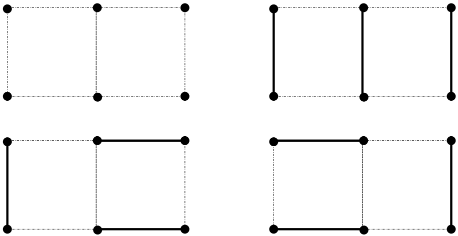
- Graph $G = (V, E)$
- Binary variables: $x_i \in \{-1, +1\}$
- No potential for disagreement edges: $\phi_{ij} = 0$ if $x_i \neq x_j$
- Model distribution

$$P(x) = \frac{1}{Z(\phi)} e^{\sum_{ij \in E} [x_i = x_j] \phi_{ij}} \text{, where}$$

$$Z(\phi) = \sum_x e^{\sum_{ij \in E} [x_i = x_j] \phi_{ij}} \text{ is the partition function}$$

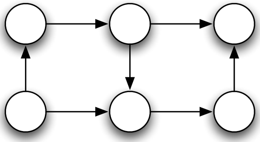
# Dimer problem

- How many perfect matchings does a graph have?
- **Perfect Matching**: A set of non-overlaping edges (dimers) that cover all vertices



# Counting Matchings

- Construct a **Pfaffian orientation**: each face (except possibly outer) has an odd number of edges oriented clockwise



- Construct a skew-symmetric matrix $K$ such that:

$$K_{ij} = \begin{cases} 1 & \text{if } i \rightarrow j \\ -1 & \text{if } i \leftarrow j \\ 0 & \text{otherwise} \end{cases}$$

- **Kasteleyn Theorem**:
  - Every planar graph has a Pfaffian orientation
  - Number of perfect matchings is $\mathrm{Pf}(K) = \sqrt{\det K}$

# The connection

- Let $G_\triangle$ be $G$ plane triangulated: each face becomes a triangle
- Let $G^*$ be the dual of graph $G_\triangle$: each face in $G_\triangle$ is a vertex in $G^*$
- Let $G_e^*$ be the expanded version of $G^*$: each vertex is replaced with 3 vertices in triangle
- **Connection**: There is a 1:1 correspondence between agreement edge sets in $G$ and perfect matchings in $G_e^*$

# Junction Tree

- State-of-the-art exact method for computing partition function, marginals and MAP assignment
- Graph is **a tree**: complexity polynomial in graph size
- Graph is **not a tree**:
  - Convert the graph into a tree of cliques
  - Complexity exponential in maximal clique size

# Planar graphs

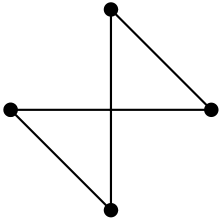
- Globerson & Jaakkola 2006 use previous results (Ising model) to compute the partition function exactly
- Restrictions:
  - Graph is planar (no crossing edges)
  - Binary-valued labels
  - Only edge potentials, no external field (node potentials)
- Complexity polynomial in graph size!

# My work

- Faster and simpler version of Globerson and Jaakkola algorithm
- No need to compute the dual $G^*$ and expanded version $G^*_e$
- Showed how to compute gradients and hence perform learning
- Applied to territory prediction in Go

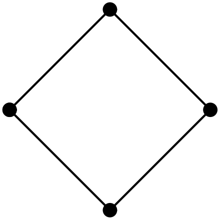
# Algorithm

- Original graph $G = (V, E)$



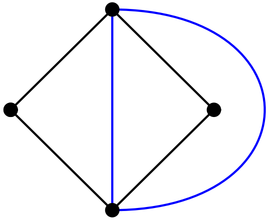
# Algorithm

- Obtain a planar embedding
- Using Boyer-Myrvold algorithm the complexity is $O(V + E)$



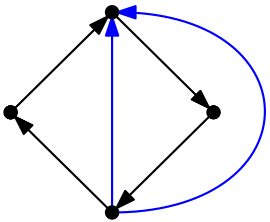
# Algorithm

- Plane triangulate the graph
- Using simple ear-clipping the complexity is $O(V^2)$



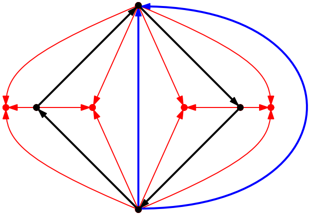
# Algorithm

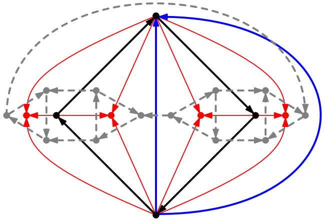
- Orient the edges such that each vertex has odd in-degree
- Equivalent to having a Pfaffian orientation in the dual graph
- Complexity is $O(E)$



# Algorithm

- Construct a skew-symmetric $2E \times 2E$ matrix K (for dual edges):
  - $K_{ij} = \pm e^{\phi}$ if $ij$ crosses **original** edge
  - $K_{ij} = \pm 1$ if $ij$ crosses added edge
- Complexity is $O(2E)$

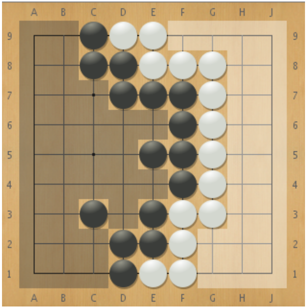


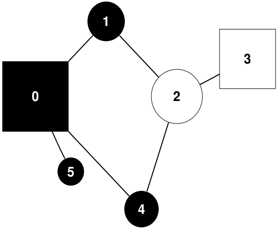


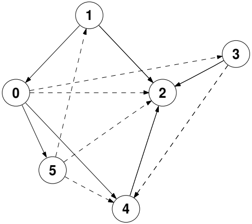
# Algorithm

- Compute partition function: $Z(\phi) = 2\sqrt{\det K}$
- Compute gradients: $\frac{\partial \ln Z(\phi)}{\partial \phi_k} = -[K^{-1} \odot K]_{2k-1,2k}$
- Computing inverse and determinant takes complexity $O(E^3)$

# Go application



# Results

- It works - correctly computes partition function and gradients
- Overall complexity is $O(E^3)$
- Performance

| Training Size | Error | | |
|---|---|---|---|
| | Block | Vertex | Game |
| 100 | $3.17 \pm 0.18$ | $2.32 \pm 0.06$ | $10.93 \pm 1.04$ |
| 1000 | $2.79 \pm 0.17$ | $1.94 \pm 0.05$ | $9.71 \pm 0.98$ |
| 2000 | $3.06 \pm 0.18$ | $2.17 \pm 0.05$ | $9.71 \pm 0.98$ |
| **5000** | $\mathbf{2.76 \pm 0.17}$ | $\mathbf{1.87 \pm 0.05}$ | $\mathbf{9.60 \pm 0.98}$ |
| 10000 | $3.20 \pm 0.18$ | $2.27 \pm 0.06$ | $10.04 \pm 1.00$ |

# Questions?

*”Some cause happiness wherever they GO, others whenever they GO”* - Oscar Wilde