THE DIMENSION OF A GRAPH

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1. Introduction

FOLLOWING N. L. Biggs, G. R. Grimmett [2] defines a *d-dimensional lattice* to be a locally finite graph on which $\mathbb{Z}^d$ acts fixed-point freely and with a finite number of orbits. (This action is assumed to be faithful.) He conjectures that a graph cannot simultaneously be both a d-dimensional and a $(d+1)$-dimensional lattice. (This is Conjecture 1 of [2].)

We extend the notion of dimension in such a way that it applies to a wider class of graphs and becomes a graphical invariant. We are then able to show that if X is a connected graph on which $\mathbb{Z}^d$ acts faithfully and with a finite number of orbits, then X is d-dimensional in our sense. Consequently Grimmett's conjecture is correct.

A geometric interpretation of our definition of dimension is also provided.

Finally we provide a counterexample to Conjecture 2 of [2].

2. Preliminaries

Throughout this paper the word graph will mean a countable locally finite simple graph. If X is a graph we denote by $d_X(i,j)$ the distance between the vertices i and j in X. For $k \geq 0$, we define

$$S_k(i, X) = \{v \in V(X) \mid d_X(i, v) = k\},$$
$$B_k(i, X) = \{v \in V(X) \mid d_X(i, v) \leq k\}.$$

We set $s_k(i, X) = |S_k(i, X)|$, $b_k(i, X) = |B_k(i, X)|$. The argument X in these expressions will usually be omitted when it is clear from the context. The *distance generating function* of X with respect to the vertex i is

$$D(i, X) = \sum_{k=0}^{\infty} s_k(i, X)x^k.$$

We use C to denote the two-way infinite path. The *cartesian product* (see Definition 1.1 of [4]) of the graphs X and Y will be written as $X \times Y$ and the cartesian product of r copies of X will be written as $X^{(r)}$. Thus $C^{(r)}$ is the r-dimensional cubic lattice graph.

DEFINITION. 2.1. Let X be a connected graph and let ρ be a non-negative real number. If, for some vertex i in X, there are positive real numbers A_1 and A_2 such that

$$A_1 k^\rho \leqslant b_k(i, X) \leqslant A_2 k^\rho \qquad k = 0, 1, 2, \ldots$$

then we say that X has *dimension* ρ. Note that if $i, j \in V(X)$ and $d(i, j) = \Delta$ then

$$B_k(i) \subseteq B_{k+\Delta}(j) \subseteq B_{k+2\Delta}(i).$$

Consequently the dimension of X, if it exists, is independent of the vertex used in defining it.

Not every graph will have a dimension. In particular one can construct trees T with a vertex v such that $b_k(v, T)$ is an arbitrarily chosen strictly increasing function taking only positive integer values.

3. The main results

THEOREM 3.1. *If $G = \mathbf{Z}^d$ acts faithfully and with a finite number of orbits on the connected graph X then X has dimension d.*

Proof. We proceed in a number of steps.

(a) *G acts fixed-point freely on X.*

Let $\Omega_1, \ldots, \Omega_t$ be the orbits of G. Suppose $g \in G$ and that $v \in \Omega_1$ is a vertex fixed by g. Then if $h \in G$, $gh = hg$ and so

$$vhg = vgh = vh.$$

Hence g fixes vh for each element h in G and this implies that g fixes each vertex in Ω_1.

If g is non-trivial then it does not fix each vertex in X. By relabelling the orbits of G if necessary, we may assume there is an integer s, $1 < s \leqslant t$, such that the only vertices not fixed by g lie in orbits Ω_i, $s \leqslant i \leqslant t$. Since X is connected we can find adjacent vertices w, x in X such that g fixes w but not x. Assume, without loss of generality, that $x \in \Omega_s$.

Hence w is adjacent to each vertex in

$$\Phi = \{xg^m \mid m = 0, 1, 2, \ldots\}.$$

As X is locally finite, Φ is a finite set. Therefore there is an integer n such that g^n fixes a vertex in Φ and so, arguing as before, g^n fixes each vertex in Ω_s.

Proceeding by induction on s, we conclude that there is an integer, ν say, such that g^ν fixes each vertex in each orbit of G i.e. each vertex in X. Since G acts faithfully this implies that $g^\nu = e$, the identity element of G. Therefore g has finite order, which is impossible, because $G \cong \mathbf{Z}^d$. Consequently G must act fixed-point freely on X.

(b) *There is a Cayley graph $\mathbf{X}$ of G such that the dimension of $\mathbf{X}$, if it exists, equals that of X.*

As before we denote the orbits of G by $\Omega_1, \ldots, \Omega_t$. Since X is connected it follows from the proof of the contraction lemma in [1] that there is a connected graph W such that

$$|W \cap \Omega_i| = 1 \qquad (i = 1, \ldots, t).$$

Thus $|W| = t$. Since G acts fixed-point freely on X we see that if $g \in G \setminus e$ then $W \cap Wg = \varnothing$. We also have

$$V(X) = \bigcup_{g \in G} Wg.$$

Contracting each translate $Wg (g \in G)$ of W to a point gives rise to a locally finite graph $\mathbf{X}$ on which G acts regularly (the vertices in $\mathbf{X}$ corresponding to disjoint translates Wg and Wh of W are adjacent in $\mathbf{X}$ if some vertex in Wg is adjacent in X to some vertex in Wh). Since G acts regularly on $\mathbf{X}$, it is easy to show that $\mathbf{X}$ is a Cayley graph for G.

Assume now that $\mathbf{X}$ has dimension ρ. We show that in this case X too has dimension ρ. Let Δ be the diameter of W. If $i \in V(X)$ then denote by $\mathbf{i}$ the vertex in $\mathbf{X}$ onto which i is contracted. Then

$$b_k(i, X) \leq b_k(\mathbf{i}, \mathbf{X}) \leq b_{k(1+\Delta)}(i, X).$$

Further, there are real numbers A_1 and A_2 such that

$$A_1 k^\rho \leq b_k(\mathbf{i}, \mathbf{X}) \leq A_2 k^\rho.$$

Hence we have

$$A_1 \left(\frac{k}{1+\Delta}\right)^\rho \leq b_k(i, X) \leq A_2 k^\rho$$

and since $A_1(1+\Delta)^{-\rho}$ is constant, it follows that X has dimension ρ.

(c) *X has dimension d.*

Of course, we prove that $\mathbf{X}$ has dimension d, whence our conclusion follows using (b). Since $\mathbf{X}$ is a locally finite Cayley graph for $G = \mathbb{Z}^d$, we see by Lemma 1 of [3] that $\mathbf{X}$ has the same dimension as any other (locally finite) Cayley graph for G. Consequently $\mathbf{X}$ and $C^{(d)}$ have the same dimension. By Proposition 3.6 of [5] we see that $b_k(i, C^{(d)})$ is a polynomial of degree d in k. Hence $C^{(d)}$, as might be expected has dimension d and so $\mathbf{X}$ also has dimension d.

4. Imbedding d-dimensional graphs in euclidean space

We use E^d to denote d-dimensional Euclidean space with norm $\|\cdot\|$. An imbedding of the graph X in E^d is simply an injection of $V(X)$ into E^d. The image of i in $V(X)$ under a given imbedding will be denoted by $\hat{\imath}$.

We can always imbed X into E^d by choosing a vertex v in X, mapping it into a fixed point in E^d and then mapping vertices at distance r in X from v arbitrarily onto points at distance r from $\hat{v}$ in E^d. If, for some positive integer, d, X is d-dimensional then an imbedding of the type just described has two properties:

(a) for each vertex u in X and sequence $w_1, w_2, \ldots,$ of distinct vertices of X,

$$\frac{d(u, w_i)}{\|\hat{u} - \hat{w}_i\|} \to 1 \quad \text{as} \quad i \to \infty$$

(b) there are positive real constants C and D such that, if $c_r(p)$ denotes the number of points $\hat{u}(u \in V(X))$ lying in the ball of radius r about any point p in E^d then

$$C \leq \frac{c_r(p)}{r^d} \leq D \quad \text{(for all } r \text{ large enough).}$$

Conversely, if X is a graph with an imbedding into E^d satisfying (a) and (b), then X is d-dimensional. The proofs of these claims are quite straightforward and so are left for the reader.

5. On a second conjecture of G. R. Grimmett

Conjecture 2 of [2] asserts that if G_1 and G_2 are two copies of $\mathbb{Z}^d$ acting fixed-point freely and with a finite number of orbits on the d-dimensional lattice X then there is a third group G with the same properties containing G_1 and G_2 as subgroups. (If true, this would imply Conjecture 1.)

This conjecture is false. Let X be the graph $C \times K_2$. We identify $V(X)$ with $\mathbb{Z} \times \mathbb{Z}_2$. Then x has automorphisms g, a such that

$$g : (i, j) \to (i+1, j) \quad \text{and} \quad a : (i, j) \to (i, j+1).$$

Thus $a^2 = e$. It is easily verified that g and ga have infinite order and act fixed point freely with two orbits on X. However any subgroup of Aut (X) which contains g and ga, contains a. Hence X is the required counterexample.

It is easy to construct higher dimensional counterexamples of similar type.

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