WAVES and IMPEDANCE

- Simple Wave Motion
- Kirchhoff's Current and Voltage Laws
- Impedance
- Skin effect

Maxwell's Equations: Integral Form

- **Gauss's law for the electric field.** Charge is the source of electric field:

$$\oint_A \mathbf{E}.\mathbf{dA} = \frac{q}{\epsilon_0}$$

- **Faraday's law.** A changing magnetic flux causes an electromotive force:

$$\oint_\gamma \mathbf{E}.\mathbf{dl} = -\int_A \frac{\partial \mathbf{B}}{\partial t}.\mathbf{dA}$$

- **Gauss's law for the magnetic field.** Magnetic fields are source free:

$$\oint_A \mathbf{B}.\mathbf{dA} = 0$$

- **Ampere's law:**

$$\oint_\gamma \mathbf{B}.\mathbf{dl} = \int_A \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right).\mathbf{dA}$$

Maxwell's Equations: Differential Form

- **Gauss's law for the electric field.**

$$\nabla . \mathbf{E} = \frac{\rho}{\epsilon_0}$$

- **Faraday's law.**

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- **Gauss's law for the magnetic field.**

$$\nabla . \mathbf{B} = 0$$

- **Ampere's law:**

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Sine Waves 1

- The solution of the wave equation for monochromatic waves.

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} : \quad Solution \quad \Psi\left(kx - \omega t\right)$$

- Sine waves

$$\Psi\left(kx - \omega t\right) = A \exp i(kx - \omega t)$$

where A is the amplitude.

- Substitute in the wave equation:

$$\omega^2 / k^2 = c^2$$

Sine Waves 2

- $\mathbf{k}$ is the *wave number* or *wave vector* or *propagation factor* and ω is the *radian frequency*.

$$k = \frac{2\pi}{\lambda}, \quad \omega = 2\pi f$$

- The exponential form represents sine wave propagation of unit amplitude and with a phase (radians) given by,

$$\phi = kx - \omega t$$

- Waves with $k > 0$, propagate toward positive x.
- Waves with $k < 0$, propagate toward negative x.

Sine Waves 3

- Waves experience a time delay as they propagate along the medium. The wave fields are said to be *retarded*
- Sinusoidal waves undergo a **phase lag** or **phase shift**
- In the complex model in dissipative media, the velocity can be complex.
- Expresses the fact that the wave is attenuated. (example: transmission lines).
- Only $\mathbf{k}$ and not ω is complex. Why?

$$k = k_o + i\gamma$$

- γ is the **attenuation factor** or **damping**.

Sine Waves 4

- Waves with opposite signs of k **interfere** and for **Standing waves**. If $k_1 = +k$ and $k_2 = -k$ then,

$$\Psi(x,t) = A_1 \exp i(kx - \omega t) + A_2 \exp i(-kx - \omega t)$$

- if $A_1 = A_2$ then

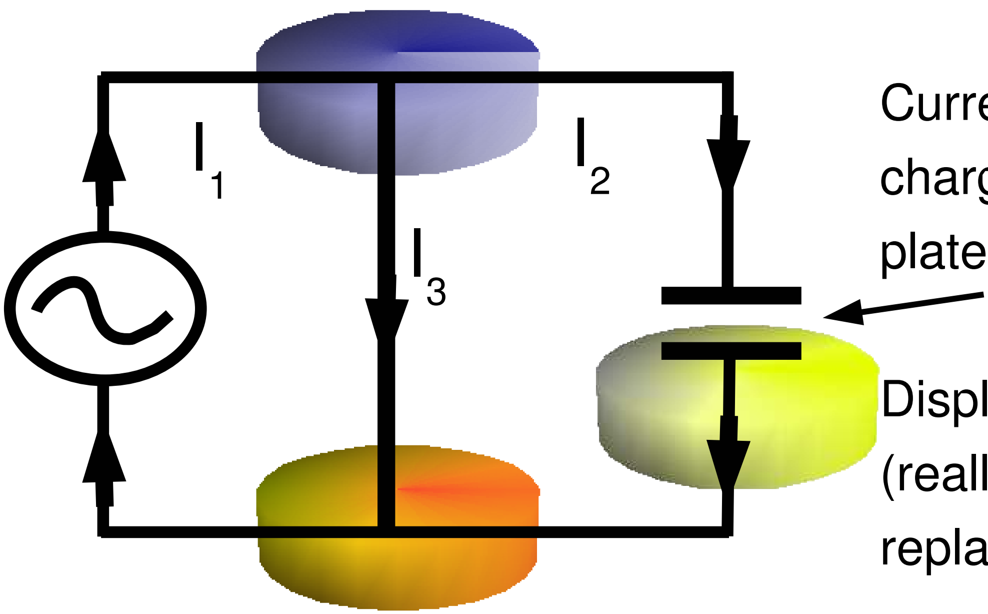
$$\Psi(x,t) = A_1 \left[\exp i(kx - \omega t) + \exp i(-kx - \omega t)\right] = 2A_1 \cos kx \exp(-i\omega t)$$

- Oscillates in time, but the spatial dependence is stationary.
- What if A_1 and A_2 are not equal?
- JAVA Applets..

Kirchhoff's Current Law 1

- Based on charge conservation in the short wavelength, zero time delay limit.

$$\oint_A \mathbf{j}.\mathbf{dA} = -\frac{\partial q}{\partial t}$$



Current 'disappears' into charge on the capacitor plates

Displacement current (really an E field) replaces the current

Kirchhoff's Current Law 2

- Let the surface include a junction as in the figure.
- The conservation of current on the surface just implies that the current entering a node is equal to that leaving a node.

$$\Sigma_k I_k = 0$$

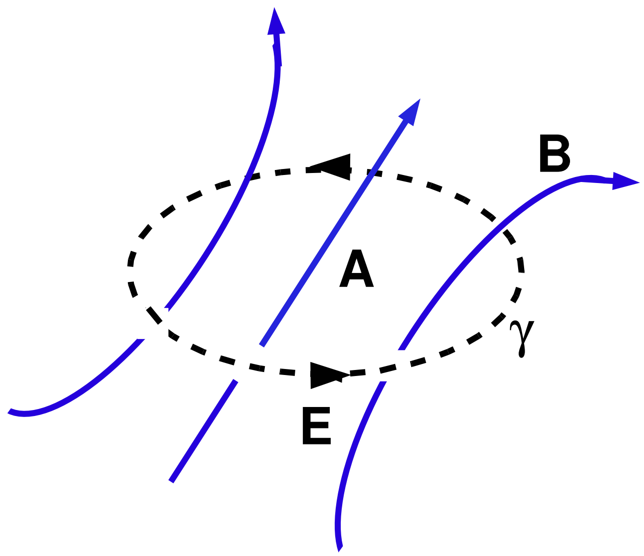
where $I_k = \int j_k dA$ and the current density of the kth branch is integrated over the cross-section of the wire.

- Through a capacitor the current is continued as a displacement current.

Kirchhoff's Voltage Law 1

- Kirchhoff's voltage law is based on **Faraday's law.**

$$\oint_{\gamma} \mathbf{E}.\mathbf{dl} = -\int_{A} \frac{\partial \mathbf{B}}{\partial t}.\mathbf{dA}$$



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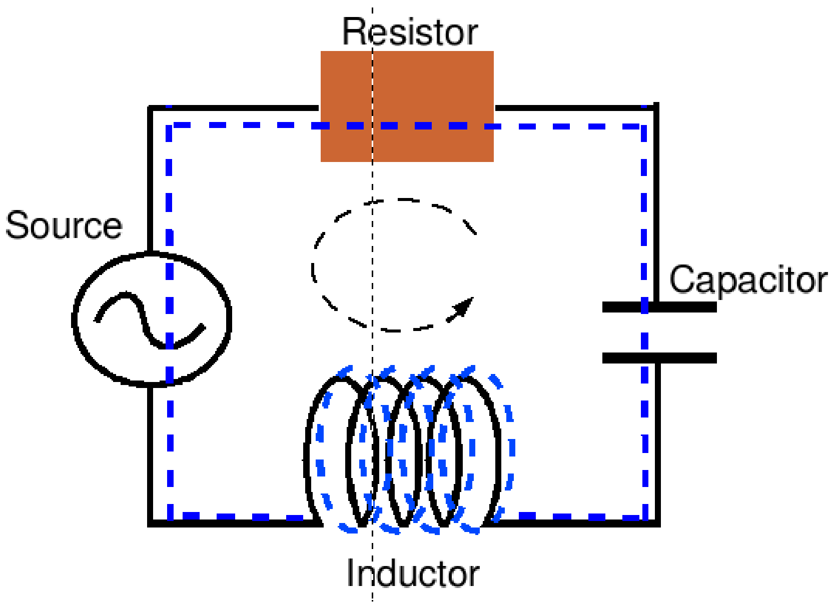
Kirchhoff's Voltage Law 2

- We study *small circuits* at **radiofrequencies**.
- In this case...
  - The current is taken to be the same in the entire circuit.
  - Retardation is to be neglected in the calculation of the fields.
- Call this limit **low frequencies** (following Ramo)



Digression: Impedance

- In order to apply Faraday's law we need to define the **physical boundaries** of the circuit elements that we wish to study.
- These circuit components are of course resistors, inductors, capacitors and transformers.
- Other components such as transmission lines, transmission line transformers, directional couplers and phase hybrids require at least indirectly some wave notions

Impedance 1: Wires (and Metals) at Low Frequencies

- Rule 1. Because identically: $\mathbf{j} = \sigma \mathbf{E}$, there is **no free inside a conductor**.
- Apply Current conservation, Ohm's law and the Gauss's law for the electric field in succession...

$$\oint_A \mathbf{j}.\mathbf{dA} = \oint_A \sigma \mathbf{E}.\mathbf{dA} = -\frac{\partial q}{\partial t} = -\oint_A \frac{1}{\epsilon_0}\frac{\partial \mathbf{E}}{\partial \mathbf{t}}.\mathbf{dA}$$

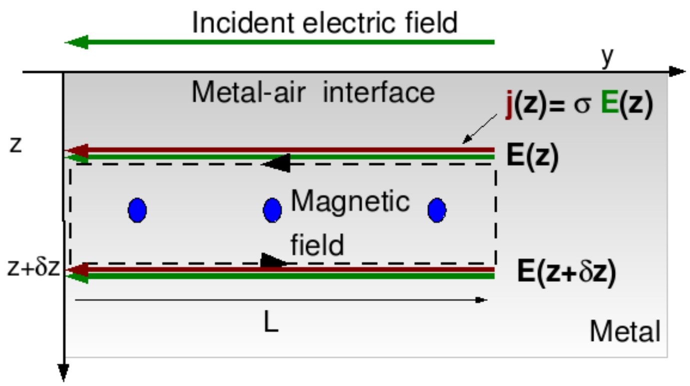
- This implies that ...

$$\frac{\partial \mathbf{E}}{\partial t} = -\frac{\sigma}{\epsilon_0}\mathbf{E}$$

- The solution to this equation is $\mathbf{E} = \mathbf{E_0} \exp\left(-\sigma/\epsilon_0 t\right)$
- For copper $\sigma/\epsilon_0 = 6.55 \times 10^{18} s^{-1}$. So that surplus charge must decay in about 10^{-19} seconds.
- **Under no conditions can charge appear inside a metal**

Impedance 2: Wires (and Metals) at Low Frequencies: Skin Effect

- Current **j** and therefore electric field however can (slightly) penetrate a metal.
- Consider the following diagram showing an electric field impinging on a metal slab...



$$(E_y(z+\delta z) - E_y(z))\, L = j\omega L \delta z B_x(z)$$

$$\frac{\partial E_y(z)}{\partial z} = j\omega B_x(z)$$

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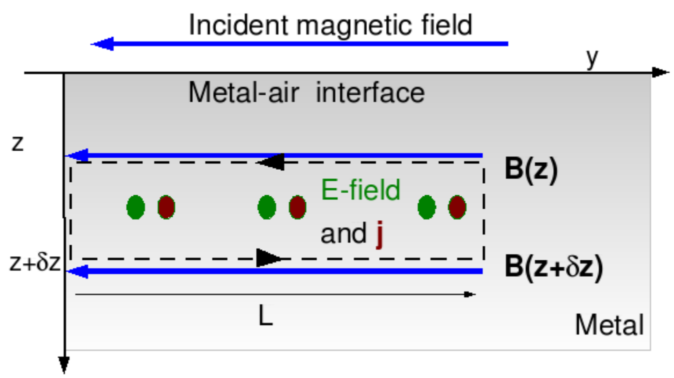
Impedance 3: Wires (and Metals) at Low Frequencies: Skin Effect

➤ Consider the following diagram showing an magnetic field impinging on a metal slab...



$$(B_x(z+\delta z) - B_x(z))\,L = L\delta z\left[\mu_0 j_y(z) - \frac{j\omega}{c^2}E_y(z)\right]$$

$$\frac{\partial B_x(z)}{\partial z} = \mu_0 j_y(z) - \frac{j\omega}{c^2}E_y(z)$$

Impedance 4: Wires (and Metals) at Low Frequencies: Skin Effect

- The current density can be replaced by the electric field using Ohm's law...

$$\frac{\partial E_y(z)}{\partial z} = j\omega B_x(z)$$

$$\frac{\partial B_x(z)}{\partial z} = \left[\mu_0\sigma - \frac{j\omega}{c^2}\right] E_y(z)$$

- Take $\partial/\partial z$ in the second equation and substitute in first to obtain...

$$\frac{\partial^2 E_y(z)}{\partial z^2} = \left[j\omega\sigma\mu_0 + k_0^2\right] E_y(z)$$

- where $k_0 = \omega/c$
- Compute the ratio $k_0^2/(\omega\sigma\mu_0) = 10^{-9}$ at 1 GHz.
- **Displacement current effects are negligible in good conductors.**

Impedance 4: Wires (and Metals) at Low Frequencies: Skin Effect

- The wave equation for metals simplifies to...

$$\frac{\partial^2 E_y(z)}{\partial z^2} = j\omega\sigma\mu_0 \, E_y(z)$$

- The solution...

$$E_y(z) = \exp\left(-\frac{1 + j}{\delta} z\right)$$

- where δ the **skin depth** is given by...

$$\delta = \sqrt{\frac{2}{\omega\sigma\mu_0}}$$

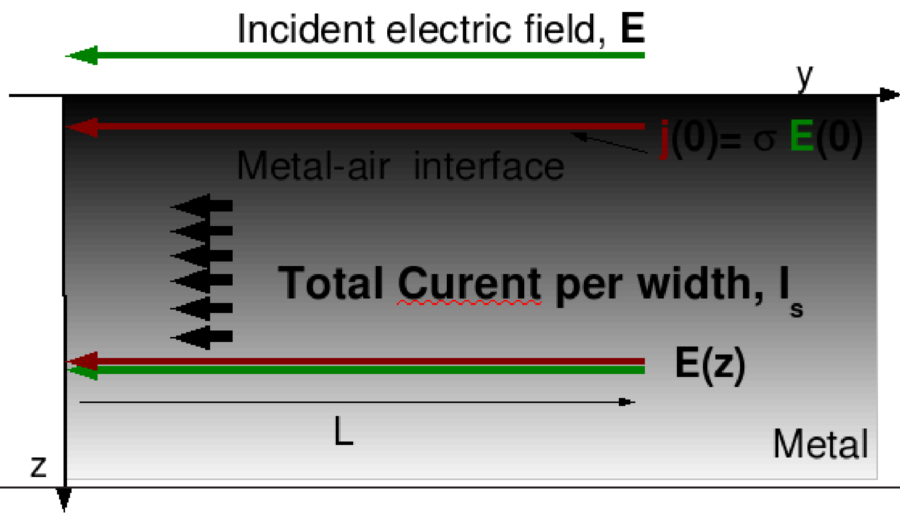
- Thus electromagnetic waves, **j, E, B, ...** only penetrate a distance δ into a metal. Check the magnitude of δ in lab and web exercises.

Impedance 5: Wires (and Metals) at Low Frequencies: Impedance per Square

- From the previous derivation of the skin effect we arrive at the definition of the surface impedance.

Define the current per unit width (x direction) as I_s, then

$$I_s = \sigma E_y(0) \int_0^\infty dz \; \exp\left(-\frac{1+j}{\delta} z\right) = \frac{\sigma\delta}{1+j} E_y(0)$$



Impedance 6: Wires (and Metals) at Low Frequencies: Impedance per Square

- Define the impedance per square as

$$Z_s = E_y(0)/I_s = \frac{1 + j}{\sigma\delta} = \sqrt{\frac{\pi\mu_0 f}{\sigma}}(1 + j)$$

- For a wire of radius, a, length L and circumference $2\pi a$, we obtain

$$Z = \frac{L}{2\pi a} Z_s$$