7.8 (a) \( \dot{x}_3(t) = (\dot{x}_1(t), \dot{x}_2(t) \cos \theta, \dot{x}_2(t) \sin \theta) \)

\( \dot{\beta}(t) = (0, -x_2(t) \sin \theta, x_2(t) \cos \theta) \)

\( \dot{\beta}_0(t) \cdot \dot{\beta}(t) = 0 \)

(b) \( \dot{x}_3(t) = (\dot{x}_1(t), \dot{x}_2(t) \cos \theta, \dot{x}_2(t) \sin \theta) \)

*Sp* is \( \mathbb{R}^2 \), hard to write so must find another way

Notice that \( \dot{x}_3(t) \in *Sp*, \dot{\beta}(t) \in *Sp* by definition because \( \dot{x}_3(t), \dot{\beta}(t) \) are both on \( *S* 

by (a) \( \dot{x}_3(t) \perp \dot{\beta}(t) \). So \( \nabla \dot{x}_3(t), \dot{\beta}(t) \) form a basis of \( *Sp* \) \( (P = \dot{x}_3(t)) \)

So one only needs to check that \( \dot{x}_3(t) \) is orthonormal to \( \dot{x}_3(t), \dot{\beta}(t) \)

\( \dot{x}_3(t) \cdot \dot{x}_3(t) = \dot{x}_1(t) \dot{x}_1(t) + \dot{x}_2(t) \dot{x}_2(t) = 1 \). As \( \dot{x}_3(t) = (x_1(t), x_2(t)) \) has constant speed, by Ex 7.2 \( \dot{x}_3(t) \perp \dot{x}_3(t) \), \( \dot{x}_3(t) \perp \dot{x}_3(t) \) is easy to check

(c) \( \dot{\beta}_0(t) = (0, -x_2(t) \cos \theta, x_2(t) \sin \theta) \), obviously \( \dot{\beta}_0(t) \perp \dot{x}_3(t) \)

\( \dot{\beta}(t) \perp \dot{x}_3(t) \iff x_1(t) \dot{x}_2(t) = 0 \) \( \sin \theta \dot{x}_1(t) > 0 \) \( \dot{x}_2(t) = 0 \iff \left| x_1(t)/\dot{x}_1(t) \right| = 0 \)

7.9 First check \( \dot{x}(t) \) is a maximal geodesic with initial velocity \( \beta(t) = \dot{x}(t) \)

\( \dot{x}(t) = c \dot{x}(t) \). So \( \dot{x}(t) \mid t=0 = c \dot{x}(t) \mid t=0 = c \)

\( \dot{\beta}(t) = c^2 \dot{x}(t) \). As \( \dot{x} \) is geodesic, so \( \dot{x}(t) \in *S_\theta(t) \). So \( \dot{x}(t) \in *S_\theta(t) \)

So \( \dot{x}(t) \) is geodesic. It is easy. Since the geodesic with initial position and velocity given is unique, \( \dot{x}(t) \) is what the maximal geodesic in \( *S* \) with initial velocity \( c \dot{x}(t) \)

The domain \( I \) can be easily taken care of.

7.10 Define \( \gamma(t) = \beta(t+t_0) \), then \( \gamma(t) = \beta(t+t_0) = \beta(t) = \dot{\beta}(t) = \dot{\gamma}(t) = \dot{V}(t) \). So if \( \dot{V}(t) \) is geodesic, then by uniqueness theorem, \( \gamma(t) = \dot{x}(t) \), i.e. \( \beta(t+t_0) = \dot{x}(t) \), i.e. \( \dot{\beta}(t) = \dot{V}(t) \)

\( I \) is taken care of because \( \dot{x} \) is maximal.

7.11 Let \( \dot{V}(t) = \beta(t) \). \( \dot{V}(t) = \beta(t) = \dot{x}(t) = \dot{\beta}(t) = \dot{\gamma}(t) = \dot{x}(t) = \dot{\gamma}(t) = \dot{V}(t) \). So by Ex 7.10

\( \dot{V}(t) = \beta(t) \), i.e. \( \beta(t) = \dot{V}(t) \), i.e. \( \beta(t) = \dot{V}(t) \)

7.12 (a) complete by Example 3

(b) incomplete \( \alpha(t) = (1, 0, 0) \cos t + (0, 0, 0) \sin t \) is geodesic, but \( \alpha(t) \) is \( t \neq \frac{\pi}{2} + 2k\pi \), \( k \in \mathbb{Z} \)

(c) incomplete \( \alpha(t) = (0, 1, 1) + (0, 1, 1) t \), \( t \neq 1 \)

(d) complete by Example 2

(e) complete \( \alpha(t) = (0, 1, 0) \cos t + (1, 0, 0) \sin t \), \( t \neq \frac{\pi}{2} + 2k\pi \), \( k \in \mathbb{Z} \)