227. (a) The set of rigid motions of \( \mathbb{R}^n \) obviously forms a group under composition. It naturally satisfies associativity, neutral element is identity transformation, inverse element exists because rigid motions map onto \( \mathbb{R}^n \) by corollary. Inverse is obviously rigid motion.

Identity \( q = \mathbb{I} \) is a symmetry of \( S \). For any symmetry \( \psi \), as it maps onto \( S \), \( \psi \) must be bijective. Its inverse is also a symmetry of \( S \). Thus, the symmetries of \( S \) form a subgroup.

(b) For any symmetry \( \psi \), suppose \( \psi = \psi_1 \psi_2 \) is the unique decomposition of \( \psi \) into an orthogonal transformation \( \psi_1 \), followed by a translation \( \psi_2 \). By definition, for any \( p \in S^n \), \( \psi(p) = \psi_1(p) + t_2 \), let \( \psi_2 \) be translation by \( t \). Since \( \psi \) is invertible, \( \psi(p) = \psi_1(p) + t_2 \) for any \( p \in S^n \).

So \( \psi_1(p) = \psi^{-1}_1(\psi(p) - t_2) = \psi^{-1}_1(\psi(p) + t_2) \).

So \( \psi_1(p) = (\psi(p) + t_2) = \psi^{-1}_1(\psi(p) + t_2) \).

But as \( \psi \) maps onto \( S^n \), the only way there must be a \( p \in S^n \), s.t.

\[ \psi(p) = -t_2 \text{ or } \psi(p) = t_2 \text{ unless } t = 0. \]

So if \( \psi \) is symmetry of \( S^n \), then \( \psi \) must be an orthogonal transformation. Conversely, for any orthogonal transformation \( \psi \), if \( p \in S^n \), then \( \| \psi(p) \| = \| p \| \).

So \( \psi(p) \in S^n \). By Corollary, \( \psi \) maps \( \mathbb{R}^n \) onto \( \mathbb{R}^n \). So for any \( q \in S^n \), there must be a \( p \in \mathbb{R}^n \), s.t. \( \psi(p) = q \). Then \( \| p \| = \| \psi(p) \| = \| q \| \), i.e., \( p \in S^n \). Thus \( \psi \) maps \( S^n \) onto \( S^n \). Combining (1), we prove (b).

(c) Using notation as in (b), let \( \psi_1 \) be translation by \( (a_1, a_2, a_3) \), and \( \psi_2 = (\alpha_1, \alpha_2, \alpha_3). \)

Then for any \( p \in \mathbb{R}^n \), \( \psi_1(p) + \psi_2(p) = \psi_1(p) + (\alpha_1, \alpha_2, \alpha_3) \).

For any \( \psi(p) \in C \), i.e., \( (x_1(p) + \alpha_1)^2 + (x_2(p) + \alpha_2)^2 = a^2 \).

If \( \psi \) maps \( C \) onto \( C \), then there must be a \( p \) s.t. \( \psi_1(p) + \psi_2(p) = \psi_1(p) + (\alpha_1, \alpha_2, \alpha_3) = a \).

Thus \( a = (x_1(p) + \alpha_1)^2 + (x_2(p) + \alpha_2)^2 = a^2 \).

Assuming \( a > 0 \), so \( x_1(p)a_1 + x_2(p)a_2 \leq 0 \), and it equals 0 if \( p = 0 \).

Now look at restrictions on \( \psi_2 \). \( \psi_1(p) = (\alpha_1(p), \alpha_2(p), \alpha_3(p) + \alpha_3) \).

Let the matrix of \( \psi_1 \) wrt standard basis of \( \mathbb{R}^3 \) be \( A = [(\alpha_1, \alpha_2, \alpha_3)]^T \).

Since \( A \) can be in \( \mathbb{R} \), so if \( \beta_3 = 0 \), then the first two coordinates can go to infinity, rather than restricted on a circle of radius \( a \). So \( \beta_3 = 0 \).

Then there is guarantee that \( \frac{\beta_1}{\beta_3} = \frac{\beta_2}{\beta_3} = \frac{\alpha_1}{\alpha_3} \).

By Ex 227, (c) and \( \| p \| = a \). If \( \beta_3 = 0 \), then \( \frac{\beta_1}{\beta_3} \) must be bounded because \( p \) is bounded. \( (\beta_1^2 + \beta_2^2) = a^2 \).

So \( \beta_3 = 0 \). This can also be seen by \( A \) being orthonormal and \( \beta_3 = 0 \). But now \( \beta_3 = 0 \), because so far.