So $V(\varphi) = \int_U \left(1+\sum_{i=1}^{n} a_i^2\right) \Big/ \left(1+\sum_{i=1}^{n} a_i^2\right)^{1/2} = \int_U \left(1+\sum_{i=1}^{n} (\partial g/\partial u_i)^2\right)^{1/2}$.

17.8 (a) Prove $J\varphi_n$ is not singular. We prove by induction. When $n=2$. $J_2 = \begin{pmatrix} \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 & 0 \\ \sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_2 \end{pmatrix}$

rank J_2 = rank $J_2 \cdot J_2^T$ = rank $\begin{pmatrix} \sin^2\theta_2 & 0 \\ 0 & 1 \end{pmatrix} = 2$, So J_2 is fully ranked. Suppose rank $J_{n-1} = n-1$.

Denote~~d~~ the $(i,j)^{th}$ component of J_n as $J_n^{i,j}$, $i=1\cdots n+1$, $j=1\cdots n$. then

$$J_n = \begin{pmatrix} J_{n-1}^{1,1}\sin\theta_n & \cdots & J_{n-1}^{1,n-1}\sin\theta_n & \varphi_{n-1}^{1}\cos\theta_n \\ & \ddots & \vdots & \vdots \\ & & J_{n-1}^{n,n-1}\sin\theta_n & \varphi_{n-1}^{n}\cos\theta_n \\ 0 & \cdots & 0 & -\sin\theta_n \end{pmatrix} = \begin{pmatrix} \sin\theta_n \cdot J_{n-1} & \varphi_{n-1}\cos\theta_n \\ 0 & -\sin\theta_n \end{pmatrix}$$

, so rank $J_n = n-1+1 = n$. i.e., φ_n is parametrized n-surface.

(b) ~~φ_n maps U one to one onto a subset of unit n-sphere S^n: $\{(a_1 \cdots a_{n+1}) \mid \sum_{i=1}^{n+1} a_i = 1$,~~

Let φ_n^i be the i^{th} component of φ_n. then $\sum_{i=1}^{n+1} {\varphi_n^i}^2 = 1$ So φ_n maps U to a subset of S^n.

We only need to prove one to one. If $\varphi_n(\theta_1 \cdots \theta_n) = \varphi_n(\hat{\theta}_1, \cdots, \hat{\theta}_n)$, then $\cos\theta_n = \cos\hat{\theta}_n$, As $\theta_n, \hat{\theta}_n \in (0,\pi)$. so $\theta_n = \hat{\theta}_n$. as $\sin\hat{\theta}_n \neq 0$ So $\varphi_{n-1}(\theta_1, \cdots, \theta_{n-1}) = \varphi_{n-1}(\hat{\theta}_1, \cdots, \hat{\theta}_{n-1})$. For the same reason, we have $\theta_{n-1} = \hat{\theta}_{n-1}, \cdots, \theta_2 = \hat{\theta}_2$.

Finally $(\sin\theta_1, \cos\theta_1) = (\sin\hat{\theta}_1, \cos\hat{\theta}_1)$. As $\theta_1, \hat{\theta}_1 \in (0, 2\pi)$, $\theta_1 = \hat{\theta}_1$. Thus φ_n is one to one

(c) ~~If $x \in S^n - \text{Image}\,\varphi_n$, then $\prod_{i=1}^{n} \sin\theta_i = 0$. This is because if $\prod_{i=1}^{n} \sin\theta_i \neq 0$,~~ ~~$\varphi(\theta_1 \cdots \theta_n)$~~ It is obvious that $\hat{\varphi}_n : U' \to R^{n+1}$ with $U' = \{(\theta_1 \cdots \theta_n) \in R^n \mid 0 \le \theta_1 < 2\pi,\ 0 \le \theta_i < \pi\ i \in [2,n]\}$ maps onto S^n. So if $x = \hat{\varphi}(\theta_1 \cdots \theta_n) \in S^n - \text{Image}\,\varphi$, then $(\theta_1 \cdots \theta_n) \in U' \setminus U$.

so $\prod_{i=1}^{n} \sin\theta_i = 0$ So $x_1 = 0$. Thus $S^n - \text{Image}\,\varphi$ is contained in the $(n-1)$-sphere $\{(x_1 \cdots x_{n+1}) \in S^n \mid x_1 = 0\}$. So $V(\varphi_n) = V(S^n)$

(d) $|\mathcal{E}_n| = \left| \begin{pmatrix} \sin\theta_n \cdot J_{n-1} & \varphi_{n-1}\cos\theta_n & \varphi_{n-1}\sin\theta_n \\ 0 & -\sin\theta_n & \cos\theta_n \end{pmatrix} \right| = \left| \begin{pmatrix} \sin\theta_n J_{n-1} & \varphi_{n-1}\cos\theta_n & \varphi_{n-1}\frac{1}{\sin\theta_n} \\ 0 & -\sin\theta_n & 0 \end{pmatrix} \right| = (\sin\theta_n)^n |J_{n-1}\ \varphi_{n-1}| = (\sin\theta_n)^n |\mathcal{E}_{n-1}|$

So $V(\varphi_n) = \int_0^\pi (\sin\theta_n)^n d\theta_n\, V(\varphi_{n-1})$ for $n \ge 3$. $V(\varphi_2) = 4\pi$.

(e) Note the fact: $I_n = \int_0^\pi (\sin\theta)^n d\theta = \frac{n-1}{n} \int_0^\pi (\sin\theta)^{n-2} d\theta = \frac{n-1}{n} I_{n-2}$ for $n \ge 2$.

$I_1 = 2$, $I_2 = \pi/2$, $I_0 = \pi$. $I_n = \frac{(n-1)!!}{n!!}\pi$ if n is even and $I_n = \frac{(n-1)!!}{n!!} 2$ if n is odd.

So $V(\varphi_n) = I_n \cdots I_3 \cdot V(\varphi_2) = 4 \prod_{k=1}^{n} I_k$. $(n \ge 2)$.

17.9 Denote $V_i = \frac{\partial\varphi}{\partial u_i}$ $(i=1,2)$ $N = V_1 \times V_2 / \|V_1 \times V_2\|$.

$A(\varphi) = \int_U \left| \begin{matrix} V_1 \\ V_2 \\ V_1 \times V_2 \end{matrix} \right| / \|V_1 \times V_2\| = \int_U (V_1 \times V_2) \cdot (V_1 \times V_2) / \|V_1 \times V_2\| = \int_U \|V_1 \times V_2\|$

17.10 (a) By Ex 14.9, W is normal vector field along φ. $\left| \begin{matrix} E_1(p) \\ \vdots \\ E_n(p) \\ W \end{matrix} \right| = \sum_{i=1}^{n} W_i^2 \ge 0$. So $W/\|W\|$ is the orientation vector field along φ

(b) $V(\varphi) = \int_U \left| \begin{matrix} E_1(p) \\ \vdots \\ E_n(p) \\ W/\|W\| \end{matrix} \right| = \int_U W \cdot W / \|W\| = \int_U \|W\|$

17.11 Let ~~$e = (e_1, \cdots, e_n)$, with~~ $e_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in R^n$, $A = (E_1^\varphi, \cdots, E_n^\varphi, N^\varphi)$, $B = (E_1^{\varphi\circ h}, \cdots, E_n^{\varphi\circ h}, N^{\varphi\circ h})$