Let $\theta_0 = \arccos(-\varepsilon_2)$. As $\theta(u), \theta(x) \in (-\pi, 0)$, $|\theta(u) - \theta(x)| < \theta_0$.

So for all $t$, if $t \in [-\varepsilon_1, \varepsilon_1]$, $|\theta(u(t)) - \theta(x(t))| < 2\pi - \theta_0$

If $t_i, t_2, t_3 \in [0, \pi]$ such that $|\theta(u(t_i)) - \theta(x(t_i))| < 2\pi - \theta_0$

Then

$$|\theta(u(t_1)) - \theta(x(t_1))| < 2\pi - \theta_0$$

Note $\theta_0 \in (\pi, \pi/2]$

As $\theta(u(t)) - \theta(x(t))$ is continuous with respect to $t$, there must exist $t_3 \in (t_1, t_2)$

st. $|\theta(u(t_3)) - \theta(x(t_3))| < 2\pi - \theta_0$

if $\varepsilon_1$ is small enough and there is room enough. But $\cos(\theta(u(t_3)) - \theta(x(t_3))) < 0$, violating $(\ast\ast)$. Thus there is a $k \in \mathbb{Z}$

s.t. $u(t) \in [a, b]$. $|\theta(u(t)) - \theta(x(t))| < 2\pi - \theta_0$

But $|k(u(t)) - k(x(t))| = \frac{1}{2\pi} \left| (\theta(u(t)) - \theta(x(t))) \right|$

So $|k(u(t)) - k(x(t))| < \varepsilon_1$. $k(x(t))$ is a continuous function of $u$

Finally, as $k(u(t))$ can only assume integer value, $k(u(t)) = k(x(t))$

Note: $k(u(t))$ can be an any $\frac{a}{d}$ with $a, d \in \mathbb{Z}$ and $k(x(t)) = k(u(t))$

11.18 (a) We define $\chi(t) = (\alpha, \cos t, \sin t)$, i.e. $\chi(t) = (\alpha, \sin t, -t \cos t)$

Then following Example 2 on p. 75, $\int_{t_0}^{t_1} = \int_{t_0}^{t_1} \alpha \cos t dt = \frac{\alpha}{2\pi} t_1 - \frac{\alpha}{2\pi} t_0$

So the rotation index of $\chi$ is $n$.

(b) We follow the definitions of $\chi, \chi, \phi$ as in the book, but define to more formally.

Let $u \in \mathbb{R}^2$, $u \neq 0$ define $\theta(u) = \alpha(t) \cdot u$. Since $\alpha(t)$ is compact, $\theta$ must attain its minimum $\theta_0$, say, at $t_0$. By Lagrange multiplier theorem, $\theta(t_0) = 0$.

So $\theta'(t_0) = \alpha(t_0) \cdot u = 0$, i.e. $\chi(t_0) \perp u$. By definition, $\phi$ is continuous.

$k(\chi(t_0))$ is the rotation index of $\chi$, because $\chi(t) = \chi(t_0) \cdot u$. Since $\alpha(t)$ is compact, $\chi$ must attain its minimum $\alpha_0$, say, at $t_0$. By Lagrange multiplier theorem, $\alpha(t_0) = 0$.

As for $\chi(t_0)$, when $t \in (t_0, t_0 + t/2)$ $\chi(t) = \chi(t_0) + \chi(t_0 - t) = (\alpha(t_0 - t) - \alpha(t_0)) \cdot u + \alpha(t_0) \cdot u$

Now the $\eta$ is exact because $\chi(t_0) \cdot u \neq 0$ and we can set $\nu = -u$, and have $\nu \wedge \eta = 0$.

So $\int_{t_0}^{t} \eta = \oint_{t_0}^{t} \chi'(t) = \chi(t) - \chi(t_0)$, the $\chi(t_0)$, $\chi(t_0 + t/2)$

$\eta$ is exact on $V$. Hence $\chi(t_0) \cdot u \neq 0$, because $V \eta = h(t_0) = \chi(t_0) \cdot u$. $\chi(t_0) \cdot u > 0$.

Now $\eta = h(t) = \chi(t) \cdot u = 0$. $\chi(t_0) \cdot u = 0$.

So $\int_{t_0}^{t} \eta = \oint_{t_0}^{t} (\chi(t_0) \cdot u + \chi(t_0 + t/2))$

$\eta = \eta + \eta$ for $t \in (t_0 + t/2, t_0 + t)$, $\chi(t) = \chi(t_0 + t/2 + t) = (\alpha(t_0 + t/2 + t) - \alpha(t_0 + t/2))$.