\[ \dot{x}(b) = \dot{x}(t) = 0 \] by (a). So now construct a continuous curve from \( P \) to \( Q \) in \( S(3) \):

\[
\begin{align*}
y(t) &= \left\{ \begin{array}{ll}
x_1(t) & t \in [0, t_1] \\
x_3(t-t_1+a) & t \in [t_1, t_1 + b-a] \\
x_2(t-b) & t \in [t_1 + b-a, t_1 + b-a + t_2]
\end{array} \right.
\end{align*}
\]

7.2 \[ ||\dot{x}(t)|| = \text{constant} \Rightarrow \dot{x}(t) \cdot \dot{x}(t) = ||\dot{x}(t)||^2 = 0, \text{ i.e. } \dot{x}(t) \perp \dot{x}(t) \]

7.3 Let \( S(t) = \int_0^t ||\dot{x}(t)|| dt \). As \( \dot{x}(t) \to 0 \), so \( S(t) \) monotone increasing so \( S(t) \) is invertible. Let \( h = S^{-1} \). \( h \) is onto by definition. \( h' = \frac{1}{\dot{x}(h(t))} ||\dot{x}(h(t))|| > 0 \)

\[ \beta = \dot{x}(h(t)) \cdot h'(t) = \dot{x}(h(t)) \cdot \frac{\dot{x}(h(t))}{||\dot{x}(h(t))||} \] so \( \beta \) is unit speed.

7.5 \( \dot{x}(t) \) for "if part" is by Example 2 in this chapter

"only if" \( \dot{x}(0) = (\cos b, \sin b, c) \), which has covered all possible points on cylinder \( \dot{x}(0) = (-10 \sin b, 10 \sin b, 0) \)

So \( \dot{x}(0) \) has covered all possible initial velocity in \( S_0(0) \).

As geodesic is uniquely determined by initial position and initial velocity, these are all possible geodesics on cylinder \( S \).

Another proof is by looking at \( (6) \) on page 41. \( N(x, y, z) = (x^2, y, 0) \)

7.6 "if part" is covered by Example 3 in this chapter

"only if" \( \dot{x}(0) = 0 \), \( \dot{x}(0) = a \dot{e}_2 \). Since \( e_2 \in S_{n-1} \), \( a \) allows all norm of velocity \( 0, \) allows all possible initial position. \( \dot{x}(0) \) allows all possible initial velocity due to uniqueness of geodesic by initial position and velocity, these are all possible geodesics on unit \( n \)-sphere.

7.7 "if part": \( \ddot{x}(h(t)) h(t) = \dot{x}(a+b) \cdot a \)

\[ \dot{x}(t) \perp S_{n+1}(b) \forall t. \] So \( \beta(t) \in S_{n+1}(a+b) \). So \( \beta(t) \) is geodesic "only if": \( \ddot{x}(t) = \dot{x}(h(t)) \cdot \dot{x}(h(t))^T + \dot{x}(h(t)) \cdot h'(t) \) if \( \beta \) is geodesic, \( \dot{x}(h(t)) \cdot S_{n+1}(h(t)) \) so \( S_{n+1}(h(t)) \) and \( \dot{x}(h(t)) \) are parallel.

\( \ddot{x}(h(t)) \) are parallel so we must require \( h'(t) = 0 \) (e.g. \( \dot{x}(t) = e^{x \cos t} \sin t \dot{x}(t) = e^{x \sin t} \dot{e}_x \sin t \) for \( \Theta_{x} = 0 \) \( S_{n+1} \) \( \dot{x} \) and \( \dot{x} \) are never parallel).

So \( h(t) = a+b \). We can't see why \( a > 0 \). Since \( a \neq 0 \) when \( a = 0 \), \( \beta \) is still geodesic.
7.8 (a) \( \dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t) \cos \theta, \dot{x}_2(t) \sin \theta) \)

\[ \dot{\beta}_1(t) = (0, -2x_2(t) \sin \theta, x_2(t) \cos \theta) \]

\[ \dot{x}(t) \cdot \dot{\beta}_i(t) = 0 \]

(b) \( \dot{\beta}_2(t) = (\dot{x}_1(t), \dot{x}_2(t) \cos \theta, \dot{x}_2(t) \sin \theta) \)

\( S^p_p N(\beta(t)) = \pm \beta(t) \), hard to write. So must find another way.

Notice that \( \dot{\beta}_1(t) \in \mathcal{S}_p \), \( \dot{\beta}_2(t) \in \mathcal{S}_p \) by definition because \( \dot{\beta}_1(t), \dot{\beta}_2(t) \) are both on \( S^p_p \).

by (a) \( \dot{x}(t) \perp \dot{\beta}_1(t), \dot{\beta}_2(t) \) form a basis of \( \mathcal{S}_p(p = \dot{\beta}_1(t)) \)

So one only needs to check that \( \dot{x}(t) \) is orthogonal to \( \dot{\beta}_1(t), \dot{\beta}_2(t) \)

\( \dot{\beta}_1(t) = \dot{x}(t) \perp \dot{x}(t) = -x_1(t) \dot{x}_1(t) - x_2(t) \dot{x}_2(t) \cos \theta \cos \theta + x_2(t) \dot{x}_2(t) \sin \theta \sin \theta \)

As \( \dot{x}(t) = (x_1(t), x_2(t)) \) has constant speed, by Ex 7.2, \( \dot{\beta}_1(t) \) and \( \dot{\beta}_2(t) \) is easy to check.

(c) \( \dot{\beta}_1(t) = (0, -x_2(t) \cos \theta, -x_2(t) \sin \theta) \), obviously \( \dot{\beta}_1(t) \perp \dot{\beta}_2(t) \)

\( \dot{\beta}_1(t) \perp \dot{\beta}_2(t) \) \( \iff \) \( x_1(t) \dot{x}_1(t) + x_2(t) \dot{x}_2(t) = 0 \) \( \sin \theta x_1(t) = 0 \) \( x_1(t) = 0 \) \( \iff \) \( \frac{x_1(t)}{x_2(t)} = 0 \)

7.9 First check \( \dot{x}(t) \) is a maximal geodesic with initial velocity \( \mathbf{v} \) : \( \beta(0) = x(0) \)

\( \dot{x}(t) = c \cdot \dot{x}(t) \). So \( \dot{x}(t) \big|_{t=0} = c \cdot \mathbf{v}(t) \big|_{t=0} = \mathbf{v} \).

\( \dot{\beta}(t) = c^2 \dot{x}(t) \). As \( x \) is geodesic, so \( \dot{x}(t) \in \mathcal{S}_p(t) \). So \( \dot{\beta}(t) \in \mathcal{S}_p(t) \)

So \( \dot{\beta}(t) \) is geodesic. I is easily since the geodesic with initial position and velocity given is unique, \( \beta(t) \) is what we the maximal geodesic in \( S \) with initial velocity \( \mathbf{v} \).

The domain \( I \) can be easily taken care of.

7.10 Define \( \gamma(t) = \beta(t + t_0) \), then \( \gamma(t) = \beta(t) = \mathbf{v}(t) = \mathbf{v} \). So if \( \gamma(t) \) is geodesic, then by uniqueness theorem, \( \gamma(t) = \dot{x}(t), \) i.e., \( \beta(t + t_0) = \dot{x}(t) \), i.e., \( \beta(t) = \dot{x}(t) \)

I is taken care of because \( x \) is maximal.

7.11 Let \( \gamma(t) = \beta(t) \), \( \gamma(t) = \beta(t) \), \( \gamma(t) = \dot{x}(t) \), \( \dot{\gamma}(t) = \dot{x}(t) \). So by Ex 7.10.

\( \gamma(t) = \beta(t) \) i.e., \( \beta(t) = \beta(t) \) i.e., \( \beta(t) = \beta(t) \)

7.12 (a) complete by Example 3.

(b) incomplete \( x(t) = (1, 0, -0) \cos t + (0, 0, 1) \sin t \) is geodesic but \( t = \frac{\pi}{2} \pm 2k\pi \) for \( k \in \mathbb{Z} \)

(c) incomplete \( x(t) = (0, 1, 0) \) is geodesic.

(d) complete by Example 2.

(e) in complete \( x(t) = (0, 0, 1) \) is geodesic.

(f) \( x(t) = (1, 0) \) is geodesic, \( t = \frac{\pi}{2} \pm 2k\pi \) for \( k \in \mathbb{Z} \)