5.7 (a) (b) just write out (c) take \( u = (1,0,0), (0,1,0), (0,0,1) \) then get it.

5.8 (a) \( \text{consistent } \iff \text{det} \left( \frac{w}{n \cdot w} \right) > 0 \iff V \cdot (w \times n(p)) > 0 \iff N(p) \cdot (v \times w) > 0 \)

(b) Denote \( \widehat{x} = x/(1/11) \), consistent \( \iff \widehat{w} \cdot (N(p) \times \widehat{v}) > 0 \)

(Proof) As \( N(p) \cdot (v(x) \times \widehat{v}) = 0 \), so there must exist \( \theta \) such that \( \widehat{w} = c \widehat{v} \sin \theta + \cos \theta \cdot N(p) \times \widehat{v} \).

As \( |\widehat{w}| = 1 \), so there exists \( \theta \cdot s.t. \widehat{w} = c \widehat{v} \sin \theta + \cos \theta \cdot N(p) \times \widehat{v} \), where \( \widehat{v} = \left( \frac{w}{|w|} \right) \).

So \( \widehat{w} \cdot (N(p) \times \widehat{v}) = \sin \theta \). So \( \theta \in (0, \pi) \iff \widehat{w} \cdot (N(p) \times \widehat{v}) > 0 \iff (w, w) \) is consistent with \( N \).

5.9 (a) take \( u = (1,0,0), (0,1,0), (0,0,1) \) (b) just check.

5.10 (a) \( \text{det} \left( \frac{v_i}{w_i} \right) < 0 \iff \text{det} \left( \frac{w_i}{v_i} \right) > 0 \)

(b) \( \text{let } \mathbf{V} = \left( \frac{v_i}{w_i} \right), \mathbf{W} = \left( \frac{w_i}{v_i} \right) \)

\[ \mathbf{W} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \]

So \( \text{det} \left( \frac{w_i}{v_i} \right) = \text{det} \mathbf{A} \cdot \text{det} \left( \frac{v_i}{w_i} \right) \), thus consistency of \( W \) with \( N \) is identical to the consistency of \( V \) with \( N \) iff \( \text{det} \mathbf{A} > 0 \).

6.1 \( n=1 \) \( N(5) = \left\{ (0,1), (0,-1) \right\} \)

\( n=2 \) \( N(5) = \left\{ (1,0), (0,1) \right\} \)

6.3 \( n=1 \) \( N(5) = \left\{ (x, x^2) \mid x^4 + x^2 = 1 \right\} \)

\( n=2 \) \( N(5) = \left\{ (x, x^2, x^3) \mid x^2 + x^4 = 1 \right\} \)

6.4 \( n=1 \) \( N(5) = \left\{ (x, x^2, x^3) \mid x^2 + x^4 = 1, x < 0 \right\} \)

6.5 We only need to analyze \( n=1 \), the cases for \( n>2 \) can be determined by viewing as the surface of revolution obtained by rotating the curve for \( n=1 \) about the \( x_1 \)-axis.

For \( n=1 \) \( x_1^2 + x_2^2 = 1 \), like the right figure.

The spherical image is \( \left\{ (x_1, x_2) \in S^1 \mid x_1 \left( \frac{1}{\sqrt{1 + x_2^2}}, \frac{x_2}{\sqrt{1 + x_2^2}} \right) \right\} \).

For \( n>2 \) the spherical image is \( \left\{ (x_1, x_2) \in S^{n-1} \mid x_1 \left( \frac{1}{\sqrt{1 + x_2^2}}, \frac{x_2}{\sqrt{1 + x_2^2}} \right) \right\} \).

When \( a \to \infty \), it shrinks to a narrow band.

When \( a \to 0 \), it extends to the whole \( S^n \).
6.7. (a) Suppose the orientation at \( p \) is \( \text{N}(p) = \mathbb{N} \). Since \( \alpha(t) = p + ta \in \mathbb{S} \) for all \( t \in I \), so \( \alpha(0) = a \in \mathbb{S} \). So \( \alpha : \mathbb{N}(p) = 0 \) which is true for any \( p \in S \).

(b) If \( \alpha(t) \) is a curve, then \( \alpha(t) = p + ta \) is an integral curve of \( X \) and \( \alpha(0) = p \in S \). Then by the corollary to Theorem 1, Chapter 5, \( \alpha(t) \in S \) for all \( t \in I \) where \( I \) is the interval on which \( \alpha(t) \) is defined.

6.8. Suppose \( \mathbb{N}(S) = \mathbb{S} \). Let \( B \) be an open ball contained in \( U \) (\( S \) is a level set on \( U \)) and \( p \in \mathbb{S} \). Then, for \( X \) which satisfies \( (X - p) \cdot v = 0 \), we construct a constant vector field \( W(p) = (q, X - p) \), the restriction of \( W \) on \( U \) is a tangent vector field on \( S \). \( \alpha(t) = p + (X - p) \cdot t \) (\( \phi \rightarrow B \)), an integral curve of \( W \), such that \( \alpha(0) \in S \). Thus by the corollary to Theorem 1, Chapter 5, \( \alpha(t) \in S \), and specifically \( \alpha(0) = X \in \mathbb{S} \). Therefore, \( \{ x \in R^m : x \cdot v = 0 \} \) \( \cap \mathbb{S} = \mathbb{S} \).

Next, suppose \( \alpha : [a, b] \rightarrow S \) is continuous, parametrized curve and \( \alpha(t) \in \mathbb{S} \) for \( t \in [a, b] \).

If \( \alpha(t) \cdot v < \alpha(t) \cdot v \), then for any \( b \in (\alpha(t) \cdot v, \alpha(t) \cdot v) \), due to \( \alpha(t) \) being continuous, there exists \( t_1 \) such that \( \alpha(t_1) \cdot v = b \). Since \( \alpha(t_1) \in \mathbb{S} \), by above argument, we have \( \{ x \in R^m : x \cdot v = \alpha(t_1) \cdot v \} \) \( \cap \mathbb{S} = \mathbb{S} \). This is a contradiction.

Therefore, \( \{ x \in R^m : x \cdot v = \alpha(t_1) \cdot v \} \) \( \cap \mathbb{S} = \mathbb{S} \). But \( \mathbb{S} \) is an open set and therefore \( \mathbb{N}(S) = \mathbb{S} \) (because \( S \) contains an open set), contradicting with \( \mathbb{N}(S) = \mathbb{S} \). So \( \alpha(t) \cdot v \geq \alpha(t) \cdot v \). Likewise, \( \alpha(t_1) \cdot v \leq \alpha(t_1) \cdot v \).

So \( \alpha(t_1) \cdot v = \alpha(t_2) \cdot v \), i.e., \( \alpha(t) \cdot v = \alpha(t_1) \cdot v \). Since \( S \) is connected, \( \alpha \) takes any two points on \( S \) and \( \mathbb{S} \) can be connected by a continuous parametrized curve, therefore \( p \cdot v = \alpha(t) \cdot v \), i.e., all points in \( S \) lie on the same plane (or part of a plane).

6.9. (a) Let \( g(t) = f(\alpha(t)) \). So now we have \( g(t) = f(\alpha(t)) = \eta \cdot g(t) = c \). So \( g(t) = c \) for all \( t \in (t_1, t_2) \).

If \( g(\frac{t_1}{2}) \leq g(\frac{t_1}{2}) \), then there exists \( \frac{t_1}{2}, s. t. \) \( g(\frac{t_1}{2}) = g(\frac{t_1}{2}) \), where \( s \in (0, \frac{t_1}{2}) \). So \( g(t_1 + s) > 0 \), thus \( g(t_1 - s) > g(t_1) = c \). There exists \( \frac{t_1}{2}, s. t. \) \( g(\frac{t_1}{2}) = g(\frac{t_1}{2}) \), where \( s \in (0, \frac{t_1}{2}) \). So \( g(t_1 - s) > 0 \), thus \( g(t_1 - s) < g(t_1) = c \). Then for \( t \in (t_1, t_2) \),

\[ g(t) = \begin{cases} 
\eta & \text{if } t \leq t_1 - \frac{s}{2} \\
\eta - \frac{s}{2} & \text{if } t_1 - \frac{s}{2} < t < t_1 - \frac{s}{2} \\
\eta & \text{if } t \geq t_1 - \frac{s}{2} 
\end{cases} \]

\[ \text{s.t. } g(t) = c \text{, contradiction!} \]
(a) If \( \varepsilon(t_1) < 0, \varepsilon(t_2) < 0 \), same contradiction occurs. So \( \varepsilon(t_1) \varepsilon(t_2) < 0 \)

(b) If \( \varepsilon \) crosses \( S \) for an odd number of times \( t_1, \ldots, t_n \), then by (a) \( \varepsilon(t_i) \varepsilon(t_n) > 0 \). Without loss of generality, suppose \( \varepsilon(t_1) > 0, \varepsilon(t_n) > 0 \).

Since \( \varepsilon(t_i) = \frac{2}{\pi} \int_{S_i} \varepsilon \mathbf{n} \cdot \mathbf{u} dS \), \( \varepsilon(t_1) > 0 \) for all \( t < t_1 \); \( \varepsilon(t_1) > 0 \) for all \( t > t_n \).

However, as \( S \) is compact and \( \varepsilon \) goes to \(-\infty \) in both directions, we can find \( f : \mathbb{R}^n \to \mathbb{R} \) and consider \( \varepsilon(f) \). Since \( \varepsilon \) goes to \(-\infty \) in both directions,

there must be \( t_0, t_{n+1} \) with \( t_0 < t_1, t_{n+1} > t_n \), such that \( \varepsilon(t_0) \) and \( \varepsilon(t_{n+1}) \) \( \not\in S \).

As \( \varepsilon(f(\varepsilon(t_0))) < 0 \), \( f(\varepsilon(t_{n+1})) > 0 \) and \( f \) is continuous on \( S \), so \( f(\varepsilon(t_0)) < 0 \), \( f(\varepsilon(t_{n+1})) > 0 \) and \( f \) is continuous on \( S \).

As \( S \) is connected (see Ex. 5.1), there is a parametrized curve \( \beta(t) \in S \), s.t. \( \beta(t_0) = \varepsilon(t_0) \), \( \beta(t_{n+1}) = \varepsilon(t_{n+1}) \), \( \beta \) is continuous on \( S \),

there must be a \( t^3 \subset (t_0, t_{n+1}) \) st. \( f(\beta(t^3)) = 0 \).

But \( \beta(t^3) \in S \), so \( \beta(t^3) \not\in S \). This is contradiction!

6.10 (a) Suppose \( \beta(t) \). Since \( \beta(0) \in \sigma(S) \), there exists a continuous map \( \alpha : [0, +\infty) \to \mathbb{R}^n - S \)

s.t. \( \alpha(0) = \beta(0) \), \( \lim_{t \to +\infty} \alpha(t) = \infty \).

For \( t \beta(t) \). Construct curve \( y(t) = \left( \alpha(t-t_0), \alpha(t-t_0), \alpha(0) \right) \in \mathbb{R}^n - S \).

then \( y(t) \) is continuous from \( [0, +\infty) \to \mathbb{R}^n - S \). \( y(0) = \beta(0) \), \( y(\infty) = \alpha(0) \).

So to is arbitrary so \( T(t) \in \sigma(S) \) for all \( t \in [a, b] \).

(b) Open set. \( \mathbb{R}^n \setminus \sigma(S) \), as \( \mathbb{R}^n - S \) is open (due to \( S = \partial \mathbb{B}(p) \) is a sphere and by definition \( f \) is smooth). So there exists an \( \varepsilon \)-ball around \( p \), (p, e), such that \( \mathbb{R}^n \setminus \mathbb{B}(p, e) \) satisfy \( \not\in \mathbb{R}^n - S \).

We can easily construct a homotopy \( \mathbb{R}^n \setminus \mathbb{B}(p, e) \).

(3) Connected: \( \mathbb{R}^n \setminus \sigma(S) \). Suppose there is a \( \varepsilon \)-sphere \( S \setminus \sigma(S) \).

Suppose \( \varepsilon(t) \) is connected and \( \varepsilon(t) \in \mathbb{R}^n - S \), there's a curve \( \varepsilon(t) \) on \( S \), s.t. \( \varepsilon(t) = \alpha(t) \).
\[ x'_3(t) = x_2(t), \quad x'_3(t) \in C(5) \text{ by (a)}. \] So now construct a continuous curve from \( p \) to \( q \) in \( C(5) \):

\[
\begin{align*}
y(t) &= \begin{cases} 
x_1(t) & t \in [0, t_1] \\
x_3(t-t_1+a) & t \in [t_1, t_1+t_1-b-a] \\
x_2(t_1-t_1+1-b-a) & t \in [t_1+b-a, t_1+b-a+t_2) 
\end{cases}
\end{align*}
\]

7.2 \[ \| x'(t) \| = \text{constant at } x'(t_1) \cdot x'(t) = x'(t) \cdot x'(t) = 0, \text{ i.e. } x'(t) \perp x'(t) \]

7.3 Let \( S(t) = \int_0^t \| x'(s) \| ds \). As \( x'(t) \to 0 \), so \( S(t) \) monotonic increasing.

So \( S(t) \) is invertible. Let \( h = S^{-1} \). \( h \) is onto by definition.

\[
\beta = x(h(t)), h'(t) = \frac{x'(h(t))}{\| x'(h(t)) \|}
\]

so \( \beta \) is unit speed.

7.5 "if part" is by Example 2 in this chapter.

only if \( x(0) = (\text{ramp}, \text{rainbow}, \text{el}) \), which has covered all possible points on a cylinder.

\( x(0) = (-\cos b, \sin b, 0) \).

So \( x(0) \) has covered all possible initial velocity in \( Sx(0) \).

As geodesic is uniquely determined by initial position and initial velocity, these are all possible geodesics in cylinder \( S \).

Another proof is by looking at (6) on page 41. \( N(x, y, z) = (x, y, 0) \)

7.6 "if part" is covered by Example 3 in this chapter.

only if \( x(0) = e_1 \), \( x(0) = a \cdot e_2 \). Since \( E_2 \in X \), \( a \) allows all norm of velocity.

\( a \) allows all possible initial position, \( \beta(0) \) allows all possible initial velocity.

due to uniqueness of geodesic by initial position and velocity, these are all possible geodesics on unit \( n \)-sphere.

7.7 "if part": \( \beta(t) = x((h(t))h'(t)) = x((at+b) \cdot a) \)

\( x(0) \) is geodesic so

\( x(t) \in Sx(0) \forall t \). So \( \beta(t) \in Sx((at+b)) = Sx(0) \). So \( \beta \) is geodesic.

only if \( \beta(t) = x((h(t)) \cdot h'(t)) \) if \( x \) is geodesic. \( h'(t) \) is scalar.

\( x(t) \) are parallel.

so we must require \( h''(t) = 0 \) (E.g. \( x(t) = e_{0}t + e_{1} \sin t, x'(t) = e_{1} \cos t + e_{0} \sin t, \theta_{0,0} = 0 \)). \( x \) and \( x' \) are never parallel.

So \( h(t) = at+b \). We can't see why \( a > 0 \). Since \( x(t) \) when \( a = 0 \), \( \beta \) is still geodesic.