22.1 Example 1: \[ | | \Psi(p) - \Psi(q) | | = | | p+q - (p+q) | | = | | 0 | | \]

Example 2: \[ | | \Psi(p) - \Psi(2) | | = | | AP - A 2 | | = | | (p-q) \mid \mid = | | p-q | | \overset{\alpha_t}{=} \left( \begin{array}{c}
\cos \theta \\
-\sin \theta 
\end{array} \right) \left( \begin{array}{c}
\cos \theta \\
\sin \theta 
\end{array} \right) \mid \mid = (x^2 + x^2)^{1/2} \]

Example 3: \[ | | \Psi(p) - \Psi(q) | | = | | p+q - (p+q) | | = | | p-q | | \overset{\alpha_t}{=} \left( \begin{array}{c}
\cos \theta \\
\sin \theta 
\end{array} \right) \left( \begin{array}{c}
\cos \theta \\
\sin \theta 
\end{array} \right) \mid \mid = (x^2 + x^2)^{1/2} \]

22.2 For \((x) \in V \times \mathbb{R}^{n+1}\), \(\Psi(\psi(x)) = \chi(x + a)^2 = \chi(x) + 2\psi(a) = \chi(z(x)) = \chi(x^2) = \chi(x) + \psi(a)\).

22.3(a) \(\Psi: \Psi: V \times W \Rightarrow V \times W \Rightarrow V \times V \Rightarrow (\Psi(X)) = (\Psi(X))(X)\) is orthonormal.

(b) If \(\{e_1, \ldots, e_n\}\) is orthonormal, then \(\hat{\Psi}(e_i)(\hat{\Psi}(e_j)) = e_i e_j = i^2 = \delta_{i,j}\)

(c) Let \(\Psi(e_i) = e_i / \sqrt{\Psi(e_i)}\), where \(\{e_1, \ldots, e_n\}\) is an orthonormal basis of \(V^{n+1}\).

22.4(a) By Ex 22.3(c), the matrix is orthonormal \(\Leftrightarrow\) orthonormal linear transformation.

(b) \(\text{rotation} \Rightarrow A = \left( \begin{array}{cc}
x \times y & y \\
x & x 
\end{array} \right) \Rightarrow A^T = A \Rightarrow A^T A = I \Rightarrow A = A^T \Rightarrow A \text{ is orthogonal}.\)
(c) For $A \perp e$, $\Psi(e) = v \cdot e = 0$, so $v \perp e$, so the matrix must be in the form of $\Lambda \left( \begin{array}{cc} \Theta & 0 \\ 0 & \Theta \end{array} \right)$. A orthonormal $\iff (x, y)$ orthonormal, $\langle x, y \rangle = \langle y, x \rangle = 1$.

So by the proof in (a), $A = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right)$.

2.9 $\Psi : (x, y, z) \mapsto (x, y, z) - (x, y, z) / (x^2 + y^2 + z^2)$.

Obviously, $\Psi((x, y, z))$ is on $x^2 + y^2 + z^2 = 2$, $\Psi((x, y, z)) = (x, y, z) / (x^2 + y^2 + z^2)$.

As $p \in E$, $\Psi(p) \perp \Psi(p)$, so $p \mod p$ is hyperplane.

(b) $\Psi$ is $E$, $\| \Psi(p) \| = \| \Psi(p) - \Psi(p) \| = \| p \|$. So $E \perp E$, so $E \leq E$.

(c) By (c), $p \perp \Psi(p)$. Obviously, $\Psi(p) = \frac{1}{2} (\Psi(p) + p) \in E$.

By (b) $\| \Psi(p) - p \| = \| \Psi(p) - \Psi(p) \|$, $\| p \| = \| \Psi(p) \|$. So the line segment $p \mapsto \Psi(p)$ intersects $E$ at $p$, $\perp p$. $\Psi(p) = p$ is fixed point of $\Psi^2$. Besides, $\Psi$ is $F \leq E$, and $\Psi$ is reflection through $E$.

It is obvious that $E \leq E$.

(d) Suppose $\Psi = \Psi_2 \circ \Psi_1$ is the unique decomposition of $\Psi$ into an orthogonal transformation $\Psi_1$ followed by translation $\Psi_2$. As $\Psi(0) = 0$, $\Psi_2$ is identity. So $\Psi_2(p) = \Psi_2(p) = \frac{1}{2}(\Psi_2(p) + p) \in E$.

We prove by induction that $E$ is the standard basis of $R^{n+1}$.

If $e_0 \in E(F)$, then denote $F_0 = \Psi_2 \circ \Psi_1$. $F_0$ is the set of fixed points of $\Psi_2 \circ \Psi_1$. By (c), $e_0 \in F_0$. If $e_1 \notin E(F)$, then $F_0 = \Lambda$. If $e_1 \in F_0$, then denote $F_1 = \Psi_2 \circ \Psi_1$. $F_1$ is the set of fixed points of $\Psi_2 \circ \Psi_1$.

The same procedure goes on, until $e_{n+1}$. Then $e_i \in F_{n+1}$, $i = 1 \ldots n+1$, $e_0 \in F_{n+1}$.

By (d), $\Psi_2(p_i) \in F_{n+1}$ whenever $p_i \in F_{n+1}$. So $F_{n+1} = R^{n+1}$. This means $\Psi_{n+1}$ is identity, i.e., there exists a $k \leq n+2$, and reflections $\Psi_i$ of $R^{n+1}$ such that $\Psi_k \circ \cdots \circ \Psi_0 = \Psi_{n+1}$. As reflections are all invertible and its inverse is itself, so $\Psi = \Psi_0 \circ \cdots \circ \Psi_k$. 

(a) The set of rigid motions of \( \mathbb{R}^n \) obviously forms a group under composition. It naturally satisfies the associative law, neutral element is identity transformation, inverse element exists because rigid motions map onto \( \mathbb{R}^n \) by Corollary 1. So it must be bijective. Its inverse is also a symmetry of \( S \). Thus the symmetries of \( S \) form a subgroup.

(b) For any symmetry \( \psi \), suppose \( \psi = \psi_1 \psi_2 \) is the unique decomposition of \( \psi \) into an orthogonal transformation \( \psi_1 \) followed by a translation \( \psi_2 \). By definition, for any \( p \in S^n \), \( \psi(p) = \psi_1(p) + \alpha \) (by definition). Let \( \psi_2 \) be translation by \( a \). As \( -p \in S^n \), \( \psi(-p) = \psi_1(p) + \alpha = -\psi_1(p) + \alpha \). Therefore, \( ||\psi_1(p) + a|| = 1 = ||\psi_1(p) - \alpha|| = 1 - \alpha \). So \( \alpha = \frac{1}{2} (\psi_1(p) + a)^2 - \frac{1}{2} (\psi_1(p) - a)^2 = 0 \). Thus \( \psi_2(p) = \alpha (\psi_1(p) + a) = ||a||^2 \). But as \( \psi_2 \) maps \( S^n \) onto \( S^n \), the only place there must be a \( p \in S^n \), s.t. \( \psi(p) = -a \). Then \( \psi(p) = -a = \alpha ||a||^2 \) unless \( \alpha = 0 \).

So if \( \psi \) is a symmetry of \( S^n \), then \( \psi \) must be an orthogonal transformation. Conversely, for any orthogonal transformation \( \psi \), if \( p \in S^n \), then \( ||\psi(p)|| = ||p|| = 1 \). By Corollary 1, \( \psi \) maps \( R^{n+1} \) onto \( R^{n+1} \) so for any \( q \in S^n \), there must be a \( p \in R^{n+1} \), s.t. \( \psi(p) = q \) then \( ||q|| = ||\psi(p)|| = ||p|| = 1 \), i.e., \( p \in S^n \).

Thus \( \psi \) maps \( S^n \) onto \( S^n \). Combining (a), we prove (b).

(c) Using notation as in (b), let \( \psi_t \) be a translation by \( (a_1, a_2, a_3) \), and \( \psi_t = (a_1, a_2, a_3) \).

Then for any \( p \in \mathbb{R}^3 \), \( \psi_t(p) = \psi_t(p) + a_1 \). Then \( \alpha_x(p) = (a_1, a_2, a_3) = a_1^2 \). \( \alpha_y(p) = (-a_2, a_1, a_3) = a_2^2 \). \( \alpha_z(p) = (a_3, a_1, a_2) = a_3^2 \).

If \( \psi \) maps \( C \) onto \( C \), then there must be a \( p_e \in C \), s.t. \( \alpha_x(p_e) = (1/2, 0, 0) \). Then \( \alpha_x(p_e) = \alpha_y(p_e) \). Hence, \( \alpha_z(p_e) = \alpha_z(p_e) \).

Assuming \( a > 0 \), so \( \alpha_z(p_e) = \alpha_z(p_e) a \). This equals 0 if \( \alpha = 0 \).

Now let the restrictions on \( \psi_t \). \( \psi_t(p) = (\alpha_x(p), \alpha_y(p), \alpha_z(p) + a_3) \).

Let the matrix of \( \psi_t \) wrt standard basis of \( \mathbb{R}^3 \) be \( A = \begin{pmatrix} \alpha_x(p) & \alpha_y(p) & \alpha_z(p) + a_3 \end{pmatrix} \).

(4) Let \( P = (p_1, p_2, p_3) \), then \( \psi_t(P) = \begin{pmatrix} \alpha_x(p) & \alpha_y(p) & \alpha_z(p) + a_3 \end{pmatrix}(P_1, P_2, P_3) \).

Since \( P \) can be in \( R^3 \), if \( \beta_{13}, \beta_{33} \neq 0 \), then the first two coordinates can go to infinity, so that restricted on a circle of radius \( a \). So \( \beta_{13} = \beta_{33} = 0 \).

Then there is a guarantee that \( \frac{1}{2} (\beta_{13} + \beta_{33})^2 + \frac{1}{2} (\beta_{13} + \beta_{33})^2 = a^2 \) so \( (\beta_{13}, \beta_{13}, \beta_{33}) \) is orthonormal by (4).

If \( \beta_{33} = 0 \), then \( \frac{1}{2} (\beta_{13} + \beta_{33})^2 \) can only be bounded because \( p_1, p_2 \) are bounded \( (p_1, p_2) = a \). So \( \beta_{33} = 0 \). This can also be seen by \( A \) being orthonormal and \( \beta_{13} = \beta_{33} = 0 \). But now \( \beta_{13} \) and \( \beta_{33} \) must be 0 because so far...
A is like \((\beta_1, \beta_2, 0)\). But as \((\beta_1) \perp (\beta_2)\), it is impossible for \((\beta_3)\) to be orthogonal to both \((\beta_1)\) and \((\beta_2)\), unless \((\beta_3) = 0\). Thus \(\beta_3 = \pm 1\). In sum

\[
A = \left(\begin{array}{ccc}
\alpha_1 & \beta_1 & 0 \\
\beta_1 & \alpha_2 & 0 \\
0 & 0 & \alpha_3
\end{array}\right).
\]

Finally the possible symmetric group of cylinder \(x^2 + y^2 = a^2\) in \(R^3\) is \(\Phi(p_1, p_2, p_3) = (p_1, p_1 + p_2, p_2, p_2 + p_3, p_3 + p_3)\), where \(p = 1\) or \(-1\), \(A \in R\), \((p_1, p_2, p_3)\) is orthonormal.

The discussion above has shown that the above conditions are both necessary and sufficient.

(d) Using same notation as in (c),

\[
\frac{1}{\alpha_1^2} (\alpha_1 (p_1 + a_1)^2 + \frac{1}{\alpha_2^2} (\beta_2 (p_2 + a_2)^2 + \frac{1}{\alpha_3^2} (\gamma_3 (p_3 + a_3)^2 = 1
\]

\[
\frac{1}{\alpha_1^2} (-p_1 (p_1 + a_1)^2 + \frac{1}{\alpha_2^2} (-p_2 (p_2 + a_2)^2 + \frac{1}{\alpha_3^2} (-p_3 (p_3 + a_3)^2 = 1,
\]

\[
\text{So} \quad \frac{\alpha_1^2}{\alpha_2^2} \Phi(p_1, p_2, p_3) + \frac{\alpha_2^2}{\alpha_3^2} \Phi(p_2, p_2, p_3) + \frac{\alpha_3^2}{\alpha_1^2} \Phi(p_3, p_3, p_3).
\]

As \(\Phi\) is onto, there must be a \(p_0\) on this ellipsoid \(S, s.t:\)

\[
\Phi(p_0) = (p_0, p_0 + a_1, p_2, p_2 + a_2, p_3, p_3 + a_3) = (-a_1, -a_2, -a_2, -a_3, -a_3, -a_3) / r
\]

where \(r = (a_1^2 + a_2^2 + a_3^2) / 2\). Assume now \(r \neq 0\).

Then \(a_1^2 \Phi(p_1, p_2, p_3) + a_2^2 \Phi(p_2, p_2, p_3) + a_3^2 \Phi(p_3, p_3, p_3) = -(\frac{a_1^2}{a_2^2} + \frac{a_2^2}{a_3^2} + \frac{a_3^2}{a_1^2}) < 0\), contradicting (x).

So we must have \(r = 0\), i.e. \(a_1 = a_2 = a_3 = 0\).

(iii) If \(a, b, c\) are distinct, then w.l.o.g. assume \(c < b < a\). Consider point \((0,0,0)\) on \(S, \Phi((0,0,0)) = (\alpha_1, \beta_1, \beta_2, \beta_3)\). If \(\Phi\) is on \(S\), then \(l = \frac{\alpha_1^2}{\alpha_1^2} + \frac{\beta_1^2}{\beta_1^2} + \frac{\beta_2^2}{\beta_2^2} = 1\). So the symmetry group of \(S\) is empty.

The equality holds iff \(\beta_1 = \beta_2 = 0\). So \(\beta_1, \beta_2 \neq 0\). Similarly \(\beta_1 = \beta_2 = 0\). So \(\alpha_1, \alpha_2, \alpha_3 \neq 0\).

So \(A\) is like \((\alpha_1^2, \beta_1, \beta_2, \beta_3)\). So \(A = (\alpha_1^2, \beta_1, \beta_2, \beta_3)\). So the symmetry group of \(S\) is \(\Phi(p_1, p_2, p_3) = (p_1, p_1 + p_2, p_2, p_2 + p_3, p_3 + p_3)\) where \(p = 1\) or \(-1\).

(i) If \(a + b = c\), then same logic as above. Otherwise consider point \((a, a, a)\)

\[
\Phi((a, a, a)) = (\alpha_1, \beta_1, \beta_2, \beta_3).\] If it is on \(S\), then

\[
l = \frac{\alpha_1^2}{\alpha_1^2} + \frac{\beta_1^2}{\beta_1^2} + \frac{\beta_2^2}{\beta_2^2} = 1\]

So still empty is the symmetry group of \(S\).

The equality holds iff \(\beta_1 = \beta_2 = 0\). So \(\beta_1, \beta_2 \neq 0\). Similarly \(\beta_1 = \beta_2 = 0\). So \(\beta_1, \beta_2 \neq 0\).

So \(A\) is like \((\alpha_1^2, \beta_1, \beta_2, \beta_3)\). So \(A = (\alpha_1^2, \beta_1, \beta_2, \beta_3)\). So the symmetry group of \(S\) is \(\Phi(p_1, p_2, p_3) = (p_1, p_1 + p_2, p_2, p_2 + p_3, p_3 + p_3)\) where \(p = 1\) or \(-1\).

(i) If \(a + b = c\), then as in (iii) we have \(\beta_1 = \beta_2 = 0\). Besides, as \((\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)\) is on \(S\), we have \(l = \frac{\beta_1^2}{\beta_1^2} + \frac{\beta_2^2}{\beta_2^2} + \frac{\beta_3^2}{\beta_3^2} = \frac{\beta_1^2 (\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)}{\beta_1^2 (\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)} = 1\). Equality holds iff \(\beta_1 = \beta_2 = 0\). Likewise \(\beta_3 = 0\). So \(A\) is like \((\alpha_1^2, \beta_1, \beta_2, \beta_3)\). \(A\) is orthonormal \(\Rightarrow \Phi(p_1, p_2, p_3)\) is orthonormal. Conversely \((\beta_1, \beta_2, \beta_3)\) being orthonormal is sufficient because \(\Phi(p_1, p_2, p_3) = (\pm p_1, \pm p_2, \pm p_3, \pm p_4, \pm p_5, \pm p_6)\) and \(l = \frac{\beta_1^2 (\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)}{\beta_1^2 (\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)} = \frac{1}{\beta_1^2} \left(\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6\right) = \frac{1}{\beta_1^2} (\beta_1^2 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6) = 1\). So \(\Phi(p_1, p_2, p_3)\) is \(S\) and obviously \((p_1, p_2, p_3) \rightarrow (\beta_1, \beta_2, \beta_3)\) is invertible and bijective from \(S\) itself. Thus the symmetry group of \(S\) is \(\Phi(p_1, p_2, p_3) = \left(\frac{1}{p_1, p_2, p_3}\right)\) where \((p_1, p_2, p_3)\) is orthonormal.