22.1 Example 1: $\|\psi(p)-\psi(q)\| = \|p+a-(q+a)\| = \|p-q\|$

Example 2: $\|\psi(p)-\psi(q)\| = \|Ap-Aq\| = \|A(p-q)\| = \|p-q\|$, as $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$. $\|A\binom{x_1}{x_2}\| =$
$\|(\cos\theta\, x_1 - \sin\theta\, x_2, \sin\theta\, x_1 + \cos\theta\, x_2)\| = [(\cos\theta x_1 - \sin\theta x_2)^2 + (\sin\theta x_1 + \cos\theta x_2)]^{1/2} = (x_1^2+x_2^2)^{1/2}$

Example 3: $\|\psi(p)-\psi(q)\| = \|p+2(b-p\cdot a)\cdot a - q - 2(b-q\cdot a)a\| = \|p-q-2[(p-q)\cdot a]\cdot a\|$, let $x = p-q$
$= [(x-2(x\cdot a)\cdot a)^T(x-2(x\cdot a)\cdot a)]^{1/2} = [x^Tx + 4(x\cdot a)^2 - 4(x\cdot a)^2]^{1/2} = \|x\| = \|p-q\|$.

22.2. $\forall x \in R^{n+1}$, $\psi_1(\psi_2(x)) = \psi_1(x+a) \overset{\psi_1 \text{ is linear}}{=} \psi_1(x) + \psi_1(a) = \tilde{\psi}_2(\psi_1(x))$, $\tilde{\psi}_2(\hat{x}) = \hat{x} + \psi_1(a)$.

22.3(a) $\psi(v)\cdot\psi(w) = v\cdot w \Rightarrow \psi(v)\cdot\psi(v) = v\cdot v \Rightarrow \|\psi(v)\| = \|v\|$.
$\|\psi(v)\| = \|v\| \Rightarrow \psi(v)\cdot\psi(w) = \frac{1}{2}[\|\psi(v+w)\|^2 - \|\psi(v)\|^2 - \|\psi(w)\|^2] =$
$= \frac{1}{2}[\|v+w\|^2 - \|v\|^2 - \|w\|^2] = v\cdot w$

(b) ① $\forall$ orthonormal basis $\{e_1\cdots e_n\}$. let $v = \sum_{i=1}^n v_i e_i$, then if $\{\psi(e_1)\cdots\psi(e_{n+1})\}$ is orthonormal
we have $\|\psi(v)\| = \|\psi(\sum_{i=1}^n v_i e_i)\| = \|\sum_{i=1}^n v_i\,\psi(e_i)\| = \sqrt{\sum_{i=1}^n v_i^2} = \|v\|$

② By (a), If $\{e_1\cdots e_n\}$ is orthonormal, then $\psi(e_i)\cdot\psi(e_j) = e_i\cdot e_j = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$
so $\{\psi(e_1),\cdots,\psi(e_n)\}$ is orthonormal basis for R^{n+1}

(c) ⓪ Let $\psi(e_i) = \sum_{j=1}^n a_{ij} e_j$, where $\{e_1\cdots e_n\}$ is an orthonormal basis of R^{n+1}.
① If A is orthogonal, then letting $P = (\psi(e_1),\cdots,\psi(e_n)) = AQ$ where $Q = (e_1\cdots e_n)$,
we have $P^T\cdot P = Q^TA^TAQ = Q^TQ = I$, so $\psi(e_1)\cdots\psi(e_n)$ is orthonormal
By (b) we have ψ is orthogonal transformation.
② If ψ is orthogonal, then by (b) $P = (\psi(e_1)\cdots\psi(e_n))$ is also orthonormal
$I = P'P = AQQ^TA^T = AA^T$ so A is orthogonal.

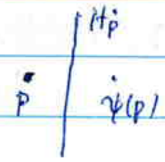
22.4 (a) By Ex 22.3(c). The matrix is orthonormal $\Leftrightarrow$ orthogonal linear transformation
So rotation $\Leftrightarrow \begin{vmatrix}x_1 & x_3\\ x_2 & x_4\end{vmatrix} = 1$ and $A^TA = I$ where $A = \begin{pmatrix}x_1 & x_2\\ x_2 & x_4\end{pmatrix}$.
$\Leftrightarrow x_1^2 + x_3^2 = 1$, $x_1x_2 + x_3x_4 = 0$, $x_2^2 + x_4^2 = 1$, $x_1x_4 - x_2x_3 = 1$ (*)
Let $x_1 = \cos\theta$, $x_3 = -\sin\theta$, $x_2 = \cos\varphi$, $x_4 = \sin\varphi$, we have
$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi = x_1x_2 + x_3x_4 = 0$
$\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi = -x_3x_2 + x_1x_4 = +1$
So $\theta+\varphi = 2k\pi + \frac{\pi}{2}$, $\sin\varphi = \cos\theta$, $\cos\varphi = \sin\theta$, so $A = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$
obviously, A in such a form must satisfy (*)

(b) $\forall$ eigenvalue and eigenvector α_i, λ_i: $\psi\alpha_i = \lambda_i\alpha_i$, then $\alpha_i^T\psi^T\psi\alpha_i = \alpha_i^T\lambda_i^2\alpha_i$
As $\psi^T\psi = I$ by Ex 22.3(c), $1 = \alpha_i^T\alpha_i = \lambda_i^2$. So $\lambda_i = \pm 1$. If all λ_i are -1
then $|\psi| = \prod_{i=1}^3 \lambda_i = -1$, violating definition of rotation. So $\exists_{\lambda_i=1}\alpha_i: \psi\alpha_i = \alpha_i$.

(c) For $\forall v \perp e_1$, $\psi(v)\cdot\psi(e_1) = v\cdot e_1 = 0$, So $v \perp e_i$, So the matrix must be in the form of $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_1 & x_3 \\ 0 & x_2 & x_4 \end{pmatrix}$. A orthonormal $\Leftrightarrow$ $\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$ orthonormal. $\begin{vmatrix} x_1 & x_3 \\ x_2 & x_4 \end{vmatrix} = |A| = 1$.
So by the proof in (a), $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$.

22.5 Map: $\forall (x_1, x_2)$ on $x_1 x_2 = 1$ to $\varphi(x_1, x_2) = \left(\frac{\sqrt{2}}{2}(x_1+x_2), \frac{\sqrt{2}}{2}(x_1-x_2)\right)$.
Obviously, $\varphi(x_1, x_2)$ is on $x_1^2 - x_2^2 = 2$. $\|\varphi(x_1,x_2) - \varphi(x_1', x_2')\| =$
$\left\|\left(\frac{\sqrt{2}}{2}(x_1+x_2-x_1'-x_2'), \frac{\sqrt{2}}{2}(x_1-x_2-x_1'+x_2')\right)\right\| = \sqrt{\frac{1}{2}(m+n)^2 + \frac{1}{2}(m-n)^2}$ $(m \triangleq x_1 - x_1',\ n \triangleq x_2 - x_2')$
$= \sqrt{m^2+n^2} = \|(x_1 - x_1', x_2 - x_2')\|$ So φ is rigid motion.

22.6 (a) $0 = \|x - \psi(p)\|^2 - \|x-p\|^2 = 2(p-\psi(p))\cdot x + \|\psi(p)\|^2 - \|p\|^2$ $(*)$



As $p \notin F$, so $p - \psi(p) \neq 0$. So H_p is hyperplane
(b) $\forall q \in F$, $\|q - \psi(p)\| = \|\psi(q) - \psi(p)\| = \|q-p\|$. So $q \in H_p$, so $F \subseteq H_p$.
(c) ~~By $(*)$ in (a), $p - \psi(p) \perp H_p$~~. Obviously, $q = \frac{1}{2}(\psi(p)+p) \in H_p$
By $(*)$ $q - p = \frac{1}{2}(\psi(p) - p) \perp H_p$, $q - \psi(p) = \frac{1}{2}(p - \psi(p)) \perp H_p$.
So the line segment $p \to \psi(p)$ intersects with H_p perpendicularly at q.
As $\|q-p\| = \|q - \psi(p)\|$, $\psi_p(\psi(p)) = p$ i.e. p is fixed point of $\psi_p \circ \psi$.
Besides, ~~$\forall$~~ as $F \subset H_p$ and $\psi(F) \subseteq F$ and ψ_p is reflection through H_p, it is obvious that $F \subseteq \{$fixed point of $\psi_p \circ \psi\}$
(d) Suppose $\psi = \psi_2 \circ \psi_1$ is the unique decomposition of ψ into an orthogonal transformation ψ_1 followed by translation ψ_2. As $\psi(0) = 0$, ψ_2 is identity. So
$$\psi\Big(\sum_{i=1}^{k} c_i p_i\Big) \underset{\text{as } \psi_2 = \text{identity}}{=} \psi_1\Big(\sum_{i=1}^{k} c_i p_i\Big) \underset{\text{linearity of } \psi_1}{=} \sum_{i=1}^{k} c_i \psi_1(p_i) \underset{\psi_2 \text{ is identity}}{=} \sum_{i=1}^{k} c_i \psi(p_i) \underset{\text{as } p_i \in F}{=} \sum_{i=1}^{k} c_i p_i, \text{ so } \sum_{i=1}^{k} c_i p_i \in F.$$
(e) Denote $\varphi_0 = \psi$, ~~$\varphi_i = \psi_{e_i} \circ \varphi_{i-1}$ for $i = 1, \cdots, n+1$ where e_i are standard bases of R^{n+1}~~
~~We prove by i~~ Let $e_1 \cdots e_{n+1}$ be the standard bases of R^{n+1}
If ~~all~~ $0 \notin F$, then denote $\varphi_0 = \psi_0 \circ \psi$, F_0 = the set of fixed points of $\psi_0 \circ \psi$. By (c) $0 \in F_0$. If $0 \in F$, then $\varphi_0 \triangleq \psi$, $F_0 = F$.
If $e_1 \notin F_0$, then denote $\varphi_1 = \psi_{e_1} \circ \varphi_0$, F_1 = the set of fixed points of φ_1.
By (c) $e_1 \in F_1$, $F_0 \subset F_1$ so $0 \in F_1$.
The same procedure goes on, until e_{n+1}. Then $e_i \in F_{n+1}$, $i = 1 \cdots n+1$, $0 \in F_{n+1}$.
By (d) $\sum_{i=1}^{n+1} c_i p_i \in F_{n+1}$ whenever $p_1 \cdots p_{n+1} \in F$, $c_i \in R$. So $F_{n+1} = R^{n+1}$. This means φ_{n+1} is identity, i.e. there exists a $k \leq n+2$, and reflections $\psi_1 \cdots \psi_k$ of R^{n+1} s.t. $\psi_k \circ \cdots \circ \psi_1 \circ \psi = I$. As reflections are all invertible and its inversion is itself, so $\psi = \psi_1^{-1} \circ \cdots \circ \psi_k^{-1} = \psi_1 \circ \cdots \circ \psi_k$.

22.7 (a) The set of rigid motions of R^{n+1} obviously forms a group under composition. It naturally satisfies associativity, neutral element is identity transformation, inverse element exists because rigid motions map onto R^{n+1} by ~~coll~~ corollary. Inverse is obviously rigid motion.

Rigid motion must be injective, as if $\psi(p)=\psi(q)$, then $\|p-q\| = \|\psi(p)-\psi(q)\| = 0$ so $p=q$.

Identity ~~belongs to~~ is a symmetry of S. For any symmetry of S ψ, as it maps onto S, it must be bijective. Its inverse is also a symmetry of S. Thus the symmetries of S form a subgroup.

(b) For any symmetry φ, suppose $\psi = \varphi_2 \circ \varphi_1$ is the unique decomposition of ψ into an orthogonal transformation φ_1 followed by a translation φ_2. By definition, for any $p \in S^n$, $\varphi(p) = \varphi_1(p) + a \in S^n$ (let φ_2 be translation by a). As $-p \in S^n$, $\varphi(-p) = \varphi_1(-p) + a = -\varphi_1(p) + a \in S^n$. So $\|\varphi_1(p)+a\| = 1 = \|-\varphi_1(p)+a\|$.

So $a \cdot \varphi_1(p) = \frac{1}{4}\left(\|\varphi_1(p)+a\|^2 - \|\varphi_1(p)-a\|^2\right) = 0$, so $a \cdot \varphi(p) = a(\varphi_1(p)+a) = \|a\|^2$, $\forall p \in S^n$.

But as ~~$\psi(p)$~~ φ maps onto S^n, ~~as the only possi~~ there must be a $p_0 \in S^n$, s.t. $\varphi(p_0) = -a/\|a\|$, then $a \cdot \varphi(p_0) = -\|a\| \neq \|a\|^2$ unless $a=0$.

So if φ is symmetry of S^n, then φ must be an orthogonal transformation. ①

Conversely, for any orthogonal transformation ψ, if $p \in S^n$, then $\|\psi(p)\| = \|p\| = 1$. So $\psi(p) \in S^n$. By Corollary, ψ maps R^{n+1} onto R^{n+1}, so for any $q \in S^n$, there must be a $p \in R^{n+1}$, s.t. $\psi(p) = q$. then $\|p\| = \|\psi(p)\| = \|q\| = 1$, i.e., $p \in S^n$. Thus ψ maps S^n onto S^n. Combining ①, we prove (b).

(c) Using ~~notation~~ as in (b), let φ_2 be ~~rotation~~ translation by (a_1, a_2, a_3), and $\varphi_1 = (\alpha_1, \alpha_2, \alpha_3)$.

Then for any $p \in$ this cylinder C, $\varphi(p) \in C$, i.e., $(\alpha_1(p)+a_1)^2 + (\alpha_2(p)+a_2)^2 = a^2$ ①

As $-p$ and $\varphi(-p) \in C$, $(-\alpha_1(p)+a_1)^2 + (-\alpha_2(p)+a_2)^2 = a^2$ ②. ①−②: $\alpha_1(p)\cdot a_1 + \alpha_2(p)\cdot a_2 = 0$

If φ maps C onto C, then there must be a $p_0 \in C$, s.t. $\begin{pmatrix}\alpha_1(p_0)+a_1 \\ \alpha_2(p_0)+a_2\end{pmatrix} = \begin{pmatrix}a_1 \\ a_2\end{pmatrix} \cdot \left(-a/(a_1^2+a_2^2)^{1/2}\right)$

then $\alpha_1(p_0)a_1 + \alpha_2(p_0)a_2 = \left[-\frac{a}{r}\begin{pmatrix}a_1 \\ a_2\end{pmatrix} - \begin{pmatrix}a_1 \\ a_2\end{pmatrix}\right] \cdot \begin{pmatrix}a_1 \\ a_2\end{pmatrix} = -ar - r^2$, where $r = \sqrt{a_1^2+a_2^2}$.

assuming $a>0$. So $\alpha_1(p_0)a_1 + \alpha_2(p_0)a_2 \leq 0$ and it equals 0 iff $r=0$ i.e. $a_1 = a_2 = 0$

Now look at restrictions on φ_1. $\varphi(p) = (\alpha_1(p), \alpha_2(p), \alpha_3(p)+a_3)$

By Ex 22.3 (c), A is orthonormal.

Let the matrix of φ_1 wrt standard basis of R^3 be $A = (\beta_1, \beta_2, \beta_3)^T$, $\forall p \in C$.

~~$\varphi(p)$~~ let $p = (p_1, p_2, p_3)$, then $\varphi(p) = \left(\sum_{k=1}^{3} \beta_{1k} p_k, \sum_{k=1}^{3} \beta_{2k} p_k, \sum_{k=1}^{3} \beta_{3k} p_k + a_3\right)$

Since p_3 can be in R, so if $\beta_{13}, \beta_{23} \neq 0$, then the first two coordinates can go to infinity, rather than restricted on a circle of radius a. So $\beta_{13} = \beta_{23} = 0$.

Then there is guarantee that $\left(\sum_{k=1}^{2} \beta_{1k} p_k\right)^2 + \left(\sum_{k=1}^{2} \beta_{2k} p_k\right)^2 = a^2$ as $\begin{pmatrix}\beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22}\end{pmatrix}$ is orthonormal ~~by Ex 22.3 (c)~~ and $\|p\| = a$. If $\beta_{33} = 0$, then $\sum_{k=1}^{2} \beta_{3k} p_k + a_3$ ~~can only~~ must be bounded because p_1, p_2 are bounded ($p_1^2 + p_2^2 = a^2$). So $\beta_{33} \neq 0$. [This can also be seen by A being orthonormal and $\beta_{13} = \beta_{23} = 0$]. But now β_{32} and β_{33} must be 0, because so far

A is like $\begin{pmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & \beta_{22} & 0 \\ \beta_{31} & \beta_{32} & \neq 0 \end{pmatrix}$. But as $\begin{pmatrix} \beta_{11} \\ \beta_{12} \end{pmatrix} \perp \begin{pmatrix} \beta_{21} \\ \beta_{22} \end{pmatrix}$, it is impossible for $\begin{pmatrix} \beta_{31} \\ \beta_{32} \end{pmatrix}$ to be orthogonal to both $\begin{pmatrix} \beta_{11} \\ \beta_{12} \end{pmatrix}$ and $\begin{pmatrix} \beta_{21} \\ \beta_{22} \end{pmatrix}$, unless $\begin{pmatrix} \beta_{31} \\ \beta_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $\beta_{33} = \pm 1$. In sum $A = \begin{pmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & \beta_{22} & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$. Finally the ~~possible~~ symmetric group of cylinder $x_1^2 + x_2^2 = a^2$ in R^3 is $\varphi(P_1, P_2, P_3) = (\beta_{11}P_1 + \beta_{12}P_2,\ \beta_{21}P_1 + \beta_{22}P_2,\ \nu P_3 + a_3)$, where $\nu = 1$ or -1, $a_3 \in R$, $\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$ is orthonormal.

The discussion above has shown that the above conditions are both necessary and sufficient.

(d) Using same notation as in (c), $\frac{1}{a^2}(\varphi_1(P) + a_1)^2 + \frac{1}{b^2}(\varphi_2(P) + a_2)^2 + \frac{1}{c^2}(\varphi_3(P) + a_3)^2 = 1$

$\frac{1}{a^2}(-\varphi_1(P) + a_1)^2 + \frac{1}{b^2}(-\varphi_2(P) + a_2)^2 + \frac{1}{c^2}(-\varphi_3(P) + a_3)^2 = 1$, So $\frac{a_1}{a^2}\varphi_1(P) + \frac{a_2}{b^2}\varphi_2(P) + \frac{a_3}{c^2}\varphi_3(P) = 0$ (*)

As φ is onto, there must be a P_0 on this ellipsoid S, s.t.

$\varphi(P_0) = (\varphi_1(P_0) + a_1,\ \varphi_2(P_0) + a_2,\ \varphi_3(P_0 + a_3)) = (-a_1 a, -a_2 b, -a_3 c)/r$

where $r = (a_1^2 a^2 + b^2 a_2^2 + c^2 a_3^2)^{1/2}$. Assume now $r \neq 0$.

Then $\frac{a_1}{a^2}\varphi_1(P) + \frac{a_2}{b^2}\varphi_2(P) + \frac{a_3}{c^2}\varphi_3(P) = -\frac{1}{r}\left(\frac{a_1^2}{a} + \frac{a_2^2}{b} + \frac{a_3^2}{c}\right) - \left(\frac{a_1^2}{a^2} + \frac{a_2^2}{b^2} + \frac{a_3^2}{c^2}\right) < 0$, contradicting (*)

So we must have $r = 0$, i.e. $a_1 = a_2 = a_3 = 0$.

(ii) If a, b, c are distinct, then w.l.o.g. assume $c < b$, $c < a$. Consider point $(0, 0, c)$ on S. $\varphi(0, 0, c) = (c\beta_{13}, c\beta_{23}, c\beta_{33})$, ~~which~~ If it is ~~should~~ ~~must~~ be on S, ~~thus~~ then $1 = \frac{c^2\beta_{13}^2}{a^2} + \frac{c^2\beta_{23}^2}{b^2} + \frac{c^2\beta_{33}^2}{c^2} \leq \frac{c^2}{c^2}(\beta_{13}^2 + \beta_{23}^2 + \beta_{33}^2) = 1$. ~~So the symmetry group of S is empty.~~

~~(i) If a≠b=c, then same logic as above. Otherwise consider point (a, 0, 0)~~ ~~$\varphi(a,0,0) = (a\beta_{11}, a\beta_{21}, a\beta_{31})$. If it is on S, then~~ ~~$1 = \frac{a^2\beta_{11}^2}{a^2} + \frac{1}{b^2}a^2\beta_{21}^2 + \frac{1}{c^2}a^2\beta_{31}^2 \geq \frac{a^2}{a^2}(\beta_{11}^2 + \beta_{21}^2 + \beta_{31}^2) = 1$. So still empty is the symmetry group of S.~~

The equality holds iff $\beta_{23} = \beta_{33} = 0$ So $\beta_{13} = \pm 1$. Similarly $\beta_{21} = \beta_{31} = 0$. $\beta_{11} = \pm 1$

So A is like $\begin{pmatrix} \pm 1 & \beta_{12} & 0 \\ 0 & \beta_{22} & 0 \\ 0 & \beta_{32} & \pm 1 \end{pmatrix}$, so $A = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$. So the symmetry group of S is $\varphi(P_1, P_2, P_3) = (\delta_1 P_1, \delta_2 P_2, \delta_3 P_3)$ where $\delta_i = \pm 1$, $i = 1, 2, 3$.

(i) If $b = c$, $a \neq b$, then as in (ii) we have $\beta_{21} = \beta_{31} = 0$. Besides, as $(\beta_{12}b, \beta_{22}b, \beta_{32}b)$ is on S, we have $1 = \frac{b^2\beta_{12}^2}{a^2} + \frac{b^2\beta_{22}^2}{b^2} + \frac{b^2\beta_{32}^2}{c^2} \geq \frac{b^2}{b^2}(\beta_{12}^2 + \beta_{22}^2 + \beta_{32}^2) = 1$

Equality hold iff $\beta_{12} = 0$. Likewise $\beta_{13} = 0$. So A is like $\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \beta_{22} & \beta_{23} \\ 0 & \beta_{32} & \beta_{33} \end{pmatrix}$.

A is orthonormal $\Rightarrow \begin{pmatrix} \beta_{22} & \beta_{23} \\ \beta_{32} & \beta_{33} \end{pmatrix}$ is orthonormal. Conversely $\begin{pmatrix} \beta_{22} & \beta_{23} \\ \beta_{32} & \beta_{33} \end{pmatrix}$ being orthonormal is sufficient because $\varphi(P_1, P_2, P_3) = (\pm P_1,\ \beta_{22}P_2 + \beta_{23}P_3,\ \beta_{32}P_2 + \beta_{33}P_3)$ and $\frac{1}{b^2}(\beta_{22}P_2 + \beta_{23}P_3)^2 + \frac{1}{c^2}(\beta_{32}P_2 + \beta_{33}P_3)^2 = \frac{1}{b^2}[P_2^2 + P_3^2]$, So $\varphi(P_1, P_2, P_3) \in S$, and obviously $(P_2, P_3)^T \to \begin{pmatrix} \beta_{22} & \beta_{23} \\ \beta_{32} & \beta_{33} \end{pmatrix}\begin{pmatrix} P_2 \\ P_3 \end{pmatrix}$ is invertible and bijective from ~~S to S~~ $P_2^2 + P_3^2 = b^2\left(1 - \frac{P_1^2}{a^2}\right)$ to itself. Thus the symmetry group of S is $\varphi(P_1, P_2, P_3) = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \beta_{22} & \beta_{23} \\ 0 & \beta_{32} & \beta_{33} \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$, where $\begin{pmatrix} \beta_{22} & \beta_{23} \\ \beta_{32} & \beta_{33} \end{pmatrix}$ is orthonormal